PROBLEM SET 3 FOR MATH 71200 - SET THEORY AND LOGIC - LOGIC I
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For this problem set, work in the theory ZF^{--}_F.

Problem 1:
Let A be a class term. Show that there is a unique class $\mathsf{H}A$ that satisfies

$$\forall x \quad (x \in \mathsf{H}A \iff (x \in A \ \wedge \ \forall y \in x \quad y \in \mathsf{H}A)).$$

$\mathsf{H}A$ is called the class of *hereditarily* A sets.

Problem 2:
Now let T be the class of transitive sets. Show that the following classes are equal:

(1) $\{x \mid x \in \mathsf{T} \ \wedge \ \forall y \in x \quad y \in \mathsf{T}\}$,
(2) HT,
(3) On.

Problem 3:
Let $G : \mathrm{V} \longrightarrow \mathrm{V}$ and $x_0 \in \mathrm{V}$. Show that there is a unique function $F : \mathrm{On} \longrightarrow \mathrm{V}$ that satisfies:

(1) $F(0) = x_0$
(2) $F(s\alpha) = G(F(\alpha))$, for all $\alpha \in \mathrm{On}$
(3) $F(\lambda) = \cup F``\lambda$, if $\lambda > 0$ is not of the form $s\alpha$, for any α ("λ is a limit ordinal").

Problem 4:
For $\alpha \in \mathrm{On}$, let $F_\alpha : \mathrm{On} \longrightarrow \mathrm{On}$ be the unique function satisfying

(1) $F_\alpha(0) = \alpha$
(2) $F_\alpha(s\beta) = s(F_\alpha(\beta))$
(3) $F_\alpha(\lambda) = \cup\{F_\alpha(\beta) \mid \beta < \lambda\}$.

Write $\alpha + \beta := F_\alpha(\beta)$. Further, if $s_0 = \langle u_0, r_0\rangle$ and $s_1 = \langle u_1, r_1\rangle$ are well-orders, define $s_0 + s_1 := s = \langle u, r\rangle$ by

$$\begin{array}{rcl} u & = & (u_0 \times \{0\}) \cup (u_1 \times \{1\}), \text{ and} \\ \langle x, i\rangle r \langle y, j\rangle & \iff & ((i = j \wedge x r_i y) \vee (i = 0 \wedge j = 1)). \end{array}$$

Show that $s_0 + s_1$ is a well-order, and that if $\alpha = \mathrm{otp}(s_0)$ and $\beta = \mathrm{otp}(s_1)$, then (by induction on β)

$$\mathrm{otp}(s_0 + s_1) = \alpha + \beta.$$

Please submit your homework by email, as a pdf file created with LaTeX, by 2/24/2019.