COMPUTABILITY AND COMBINATORIAL ASPECTS OF MINIMAL PRIME IDEALS IN NOETHERIAN RINGS

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ABSTRACT. We begin the study of the Reverse Mathematics of the algebraic theorem PDL that says "If R is a commutative Noetherian ring, then R contains finitely many minimal prime ideals (i.e. R does not contain infinitely many primes)," which plays a key role in the proof of Noether's Primary Decomposition Theorem [Noe21]. In doing so, we introduce a combinatorial principle TAC which asserts the existence of infinite antichains in binary-branching trees with infinitely many splittings. After showing that TAC is distinct from the other combinatorial principles in the Reverse Mathematical "Zoo," we characterize TAC and WKL₀ + TAC over RCA₀ + B Σ_2 via algebraic principles related to PDL. Here RCA₀ denotes the Recursive Comprehension Axiom; WKL₀ denotes weak König's Lemma; and B Σ_2 denotes a bounding principle for Σ_2^0 -formulas that is equivalent to the Infinite Pigeonhole Principle over RCA₀.

1. INTRODUCTION

For over a century now, chain conditions have played a central role in the study of algebraic structures. In particular, the study of ascending and descending chain conditions on the ideals of rings yielded various foundational ring-theoretic results. Chain conditions can be thought of as a generalization of mathematical induction from the context of well-orderings to the context of partial orders. Like induction, chain conditions yield the powerful proof paradigm that says if a counterexample to a purported theorem exists, then a "minimal" counterexample exists.

Noether was one of the first to introduce and exploit ascending and descending chain conditions in rings [NS20, Noe21]; as a result the term *Noetherian* is synonymous with the ascending chain condition in many branches of mathematics. One of Noether's biggest ring-theoretic achievements is the Primary Decomposition Theorem of [Noe21], which says that in commutative rings with identity that satisfy the ascending chain condition (on their ideals), all ideals are the intersection of finitely many *primary ideals*¹, which generalize the prime ideals of Kummer and Dedekind². A key step in proving Noether's Primary Decomposition Theorem is the following lemma, which we refer to as Noether's Primary Decomposition Lemma, or PDL. See subsection 2.3 below for a brief introduction to Ring Theory, with references.

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¹Suppose that R is a commutative ring with identity. An ideal $I \subseteq R$ is said to be *primary* if for all natural numbers n and $x, y \in R$, either $x \in I$ or $y \in I$ whenever $xy^n \in I$.

²It is well-known that if $I \subseteq R$ is primary, then $\sqrt{I} \subset R$ is a prime ideal.

Definition 1.1. Let R be a commutative ring with identity, and let $P \subset R$ be a prime ideal. Then we say that P is <u>minimal</u> if for all R-prime ideals Q, we have that Q = P whenever $Q \subseteq P$.

Definition 1.2. Let R be a commutative ring with identity. We say that R is Noetherian if every infinite ascending chain of R-ideals eventually stabilizes.

PDL: (Noether's Primary Decomposition Lemma) Let R be a commutative Notherian ring with identity. Then R contains finitely³ many minimal prime ideals. Equivalently, if Rcontains infinitely many minimal prime ideals, then R also contains an infinite strictly ascending chain of ideals.

1.1. Summary of Results. We begin an investigation into the logical strength of PDL, introduced in subsection 2.4 below, in the contexts of Computable and Reverse Mathematics. Previously, we investigated ART, i.e. the statement that says "every Artinian⁴ ring is Noetherian" [Con10, Con19] in the context of Reverse Mathematics, and showed that (over the Recursive Comprehension Axiom, RCA₀, defined later on) ART is equivalent to weak König's Lemma via a Computable Structure Theorem for Local Artinian Rings [Con19, Theorem 7.1, Corollary 7.3]. Our investigation here shows that PDL implies a combinatorial principle, which we call the Tree-Antichain Theorem (and denote by TAC), that asserts the existence of infinite antichains within computably enumerable (i.e. Σ_1^0 -definable) binary branching trees⁵ with infinitely many splittings (see Definition 3.2 below for more details). Additionally, we show that PDL follows from the conjunction of Ramsey's Theorem for Pairs (RT₂²) and weak König's Lemma (WKL₀), introduced in subsection 2.4 below.

Most of our results in this initial investigation involve the Computability Theory and Reverse Mathematics of TAC, which we will formally introduce in Section 3 below. In Section 4 we show that TAC follows from:

- the combinatorial principle ADS that says "every infinite linear order contains an infinite chain," as well as
- the principle 2-MLR that says "for every set of natural numbers X, there exists a set fo natural numbers Y that is 2-random relative to X."

The implication described in the second item above assumes more induction than the first. Additionally, we will show that TAC is not implied by weak König's Lemma, WKL_0 . More

- the trees utilized in [GM17] are of a higher complexity, namely Σ_2^0 -definable, while the trees that we utilize here are Σ_1^0 -definable; and
- the rings studied in [GM17] are all integral domains, while none of the rings that we consider here belong to this class.

³There are two logical interpretations of finiteness here, corresponding to negations of Aristotle's potential and actual infinities. However, in Theorem 5.11 below, which builds upon Proposition 5.4, we show that if Ris a computable Noetherian ring with no uniformly computable infinite ascending chains of ideals, then there is a uniformly computable enumeration of infinitely many prime ideals in R. Also, our lower bounds are obtained via computable rings containing an infinite uniformly computable enumeration of minimal prime ideals, and thus it follows that our results are independent of our interpretation of (potential vs actual) infinity. We will omit our interpretation of infinity going forward, to highlight the more interesting and consequential aspects of our analysis.

⁴A commutative ring with identity is called *Artinian* if it satisfies the descending chain condition (on its ideals); i.e. if every infinite descending chain of R-ideals eventually stabilizes.

 $^{{}^{5}\}text{A}$ tree is a set of finite binary strings closed under prefix, while an antichain is a prefix-free set of such strings. Similar trees were used by Greenberg and Melnikov [GM17] to analyze the computability of divisibility and factorizations in computable integral domains. Two obvious differences (one logical, one algebraic) between our work here and the study [GM17] are:

information, including references to the definition of 2-randomness, as well as induction schemes, are contained in our introduction to Reverse Mathematics in Section 2.4 below.

In Section 5 we develop some of the theory of Noetherian rings in the context of Reverse Mathematics, and end by showing that $\mathsf{RT}_2^2 + \mathsf{WKL}_0$ implies PDL. In Section 6 we examine the following two weaker versions of PDL, each of which applies to a restricted class of Noetherian rings.

RPDL: Let R be a commutative Notherian ring with identity, such that every prime ideal of R is both minimal and maximal⁶. Then R contains finitely many minimal prime ideals; i.e. R does not contain infinitely many minimal primes. Or, equivalently, if R contains infinitely many minimal prime ideals, then R also contains an infinite strictly ascending chain of ideals.

Definition 1.3. Let R be a commutative ring with identity, and let $P, Q \subset R$ be distinct prime ideals. Basic Algebra (see [DF99] for more details) says:

P and Q are coprime whenever R = P + Q, i.e. there exist elements x ∈ P, y ∈ Q such that 1_R = x + y, and in this case we also have that
P ∩ Q = PQ.

Additionally, we say that P and Q are <u>uniformly coprime</u> if for all $w \in P \cap Q$ there exist elements $x = x_w \in P$, $y = y_w \in Q$, and $\overline{a = a_w, b = b_w \in R^7}$ such that

$$ax + by = 1_R$$
 and $w = x_w y_w$.

URPDL: Let R be a commutative Notherian ring with identity, such that every pair of minimal prime ideals are uniformly coprime. Then R contains finitely many minimal prime ideals; i.e. R does not contain infinitely many minimal primes. Or, equivalently, if R contains infinitely many minimal prime ideals, then R also contains an infinite strictly ascending chain of ideals.

Clearly, RPDL implies URPDL.

We introduce RPDL and URPDL as potential algebraic characterizations of TAC and $\mathsf{WKL}_0+\mathsf{TAC}$ in the context of Reverse Mathematics. In Section 6 we show that:

- TAC implies URPDL, while
- $WKL_0 + TAC$ implies RPDL,

in the context of Reverse Mathematics. In Section 7 we show that

• URPDL (and hence RPDL) implies TAC,

in the context of Reverse Mathematics and the Infinite Pigeonhole Principle; see Subsection 2.4 for more information. We also introduce a principle WTAC such that:

- TAC implies WTAC,
- URPDL implies WTAC, and
- \bullet WKL_0 does not imply WTAC

in the context of Reverse Mathematics with or without the Infinite Pigeonhole Principle. These results yield an algebraic characterization of TAC via URPDL, modulo the Infinite Pigeonhole Principle.

Finally, in Section 8 we will introduce an algebraic principle related to ART (described above), denoted NMMA, that we will show is equivalent to $WKL_0 + TAC$ in the context of Reverse Mathematics modulo the Infinite Pigeonhole Principle.

⁶An ideal $M \subset R$ is maximal if for each ideal $X \subset R$ such that $M \subseteq X$, we have that X = M.

⁷For our purposes in this article we could just as well always take $a = b = 1_R$.

1.2. The Computability of Combinatorial Principles. Many contributions to Reverse Mathematics, such as this article, depend upon relevant contributions to Computable Structure Theory. Most of this article is devoted to analyzing the computability and definability complexity of infinite antichains in computably enumerable trees with infinitely many splittings, i.e. computability-theoretic aspects of the Tree Antichain Theorem TAC. The computability of Partial Orders such as trees and their combinatorial substructures has received much attention since the turn of the century, including [GMS13, DHLS03, JKL⁺09, CDSS12, HS07]. We will utilize some important and relevant results contained therein for our purposes. For more information on Computable Structure Theory, including an introduction, see [AK00, Mon21, Mon].

2. Background

2.1. Trees in Baire space and Cantor space. All of the sets and structures that we consider will be countable, coded as subsets of the natural numbers. We use \subset to denote strict inclusion of sets, and \subseteq to denote non-strict inclusion. Let $\omega = \{0, 1, 2, \ldots\}$ denote the standard natural numbers, and $\omega^+ = \{1, 2, 3, \ldots\}$ denote the positive standard natural numbers. Let $2^{<\omega}$ denote the set of finite binary sequences ordered by extension, and let $\omega^{<\omega} (\supset 2^{<\omega})$, denote the set of finite sequences of natural numbers, partially ordered by extension. Many constructions and definitions in the context of $2^{<\omega}$ can be easily adapted to $\omega^{<\omega}$. We will explicitly write our finite sequences in $2^{<\omega}$ and $\omega^{<\omega}$ using angled brackets, like so

 $\langle a_0, a_1, a_2, \dots, a_n \rangle \in 2^{<\omega}, \ n \in \omega, \ a_i \in \{0, 1\}, \ 0 \le i \le n.$

For all $\sigma \in 2^{<\omega}$ and $k \in \{0,1\}$, $\sigma k \in 2^{<\omega}$ denotes the 1-bit concatenation (extension) of σ with (by) k. For any natural number $\ell \in \omega$, let $2^{=\ell} \subset 2^{<\omega}$ denote those finite binary sequences of natural numbers of length ℓ . Let \emptyset denote the root of $2^{<\omega}$, and for all $\sigma \in 2^{<\omega}$, let $|\sigma| \in \omega$ denote the length of σ . For any $\sigma, \tau \in 2^{<\omega}$, we write $\tau \subseteq \sigma$ to denote the fact that τ is a prefix of σ ; we write $\tau \subset \sigma$ to denote the fact that τ is a proper prefix of σ . Note that \subseteq yields a natural partial ordering on $2^{<\omega}$. A set $X \subseteq 2^{<\omega}$ is prefix-free if for any $\rho, \tau \in X, \rho$ is not a prefix of τ . We say that $T \subseteq 2^{<\omega}$ is a tree if for all $\sigma \in T$ and $\tau \subseteq \sigma$ we have that $\tau \in T$. Let 2^{ω} denote the set of infinite binary sequences and ω^{ω} denote the set of infinite sequences of natural numbers. Many definitions and constructions in the context of 2^{ω} can be easily adapted to ω^{ω} . We write $\sigma \subseteq f, \sigma \in 2^{<\omega}$, let $[T] \subseteq 2^{\omega}$ denote the set of infinite binary sequences the set of infinite point in that σ is a finite initial segment of f. For any given tree $T \subseteq 2^{<\omega}$, let $[T] \subseteq 2^{\omega}$ denote the set of infinite point point in the set of infinite point poi

$$f \upharpoonright n = \langle f(0), f(1), f(2), \cdots, f(n-1) \rangle \in T$$

where $f(k) \in \{0,1\} = 2$ denotes the k^{th} bit of f. If $T \subseteq 2^{<\omega}$ is a tree, we say that a given $\sigma \in T$ is (T-) extendable whenever there exists $f \in [T] \subseteq 2^{\omega}$ such that $\sigma \subset f$. We say that the tree $T \subset 2^{<\omega}$ is extendable whenever T every $\sigma \in T$ is extendable. For any given $\sigma \in 2^{<\omega}$, let

$$[\sigma] = \{ f \in 2^{\omega} : f \supset \sigma \}$$

and for any subset $A \subseteq 2^{<\omega}$ let

$$[A] = \bigcup_{\sigma \in A} [\sigma] \subseteq 2^{\omega}.$$

If $T \subseteq 2^{<\omega}$ is a tree, $\lambda \in T$, but $\lambda 0, \lambda 1 \notin T$, (i.e. if $\lambda \in T$ has no *T*-extensions) then we say that λ is a (T-) leaf.

Definition 2.1. Let $\sigma, \tau \in 2^{<\omega}, \sigma \neq \tau, |\sigma| \leq |\tau|$. Then we say that

- " τ extends σ " if $\tau \supset \sigma$.
- Otherwise there exists $1 \le k_0 \le |\sigma|$ such that $\sigma(k_0) \ne \tau(k_0)$, in this case we say that: - " τ is to the left of σ " (or " σ is to the right of τ ") whenever $\tau(k_0) < \sigma(k_0)$, and
 - " τ is to the right of σ " (or " σ is to the left of τ ") whenever $\tau(k_0) > \sigma(k_0)$.

Given a tree $T \subseteq 2^{<\omega}$, it is possible to construct a corresponding linear order

$$T_{KB} = (T, <_{KB})$$

such that for any given σ, τ as in Definition 2.1 above we have that

- $\tau <_{KB} \sigma$ if either τ extends σ or τ is to the right of σ , and
- $\tau >_{KB} \sigma$ if τ is to the left of σ .

This well-known construction is called the *Kleene-Brower linearization of* T.

Definition 2.2. Let $T \subseteq 2^{<\omega}$ be an infinite tree, and let $H \subseteq \omega, c \in \{0,1\}$. Now, for each $h \in H$ let

•
$$T_h = \{ \sigma \in T : |\sigma| = h \} \subseteq 2^{=h},$$

- $T_{h,c} = \{\sigma(1-c) : \sigma \in T\} \subseteq 2^{=h+1}, and$ $T_{h,c,0} = \{\tau \in 2^{<\omega} : (\exists \sigma \in T_{h,c})[\tau \supseteq \sigma]\} \subseteq 2^{<\omega}.$

Finally, set

$$T_{H,c,0} = \bigcup_{h \in H} T_{h,c} \subseteq 2^{<\omega},$$

and

$$T_{H,c} = T \setminus T_{H,c,0}$$

It follows that $T_{H,c} \subseteq T \subseteq 2^{<\omega}$ is a tree. Moreover, if $T_{H,c}$ is infinite for some $c \in \{0,1\}$, then we say that $H \subseteq \omega$ is homogeneous for T.

Definition 2.3. An enumeration of a countable set X is a listing of its elements, $X = \{x_i : x_i \}$ $i \in \omega$, without repetitions. A <u>subenumeration</u> of X is a listing of the elements of X in which repetitions are allowed.

2.2. Computability Theory. Our computability-theoretic notation is standard and follows that of [Soa16]. A computable nondecreasing unbounded function $h: \omega \to \omega$ is called an order function. We say that a function $f: \omega \to \omega$ is computably approximable or *limit computable* whenever there exists a computable function $q: \omega \times \omega \to \omega$ such that $f(x) = \lim_{s} q(x,s)$ exists for all $x \in \omega$. Moreover, we say that the computable approximation g obeys the order function h, if for every $x \in \omega$ we have that

$$|\{s: g(x,s) \neq g(x,s+1)\}| \le h(x),$$

where $|A| \in \omega$ denotes the size of the finite set $A \subset \omega$. Furthermore, we say that $X \subset \omega$ is h-c.e. whenever the characteristic function of X is limit computable via some $g: \omega \times \omega \to \omega$ that obeys h. Let $\{\varphi_e : e \in \omega\}$ be a fixed uniformly computable enumeration of the partial computable functions, $\varphi_e : \omega \to \omega$, and let $\{\Phi_e : e \in \omega\}$ denote a fixed uniformly computable enumeration of the oracle computable functionals, i.e. $\Phi_e: 2^{\omega} \times \omega \to \omega$. Recall that $\varphi(x) \downarrow$ denotes that the partial computable function φ eventually halts on input $x \in \omega$, and that $\varphi_s(x) \downarrow (\varphi_{e,s}(x) \downarrow)$ says that the (e^{th}) partial computable function halts on input $x \in \omega$ in at most $s \in \omega$ steps. Similar definitions apply for $\Phi_e^{\alpha}(x) \downarrow$ and $\Phi_{e,s}^{\alpha}(x) \downarrow$, $e, x, s \in \omega$, $\alpha \in 2^{<\omega}$. A partial computable function $\varphi: \omega \to \omega$, is said to be *total* whenever $\varphi(x) \downarrow$ for all $x \in \omega$. We say that $X \subset \mathbb{N}$ is of hyperimmune degree whenever there is some function $f: \omega \to \omega$, $f \leq_T X$, i.e. X computes f, such that for every $e \in \omega$ either there exist infinitely many natural numbers $x \in \omega$, such that $f(x) > \varphi_e(x)$ whenever $\varphi_e : \omega \to \omega$ is total. Finally, we say that $X \subset \omega$ is of DNR degree (diagonally nonrecursive degree) whenever there is some $g \leq_T X, g: \omega \to \omega$, such that for all $e \in \omega, g(e) \neq \Phi_e(e)$ whenever $\Phi_e(e) \downarrow$. Moreover, if $\emptyset' \subset \omega$ denotes Turing's Halting Set and $g(e) \neq \Phi_e^{\emptyset'}(e)$, whenever $\Phi_e^{\emptyset'}(e) \downarrow, e \in \omega$, then we say that X is 2-DNR. Finally, if in addition we have that g(e) < h(e), for some nondecreasing function h and every $e \in \omega$, then we will say that X is 2-h-DNR.

Definition 2.4. We say that $X \subseteq \omega$ is of <u>PA</u> Turing degree if for every infinite computable binary branching tree $T \subseteq 2^{<\omega}$, T computes an infinite path $f \in [T] \neq \emptyset$.

It is well-known that X is of PA Turing degree if and only if X computes a complete and consistent extension of Peano Arithmetic.

Definition 2.5. Let $A, B \subseteq \omega$ be disjoint. We say that $X \subseteq \omega$ is an (A, B)-separator whenever $A \subseteq X$ and $X \cap B = \emptyset$.

Remark 2.6. It is well-known that $X \subseteq \omega$ is of PA Turing degree if and only if for any two disjoint computably enumerable sets⁸ A, $B \subseteq \omega$, X computes an (A, B)-separator.

2.2.1. The Finite Injury Priority Method. Our proof of Theorem 4.11 employs the Finite Injury Priority Method. Roughly speaking, this proof technique shows how one can satisfy a countable sequence of "requirements," R_e , $e \in \mathbb{N}$, the satisfaction of which culminates in the proof of the main theorem. This method assigns, to each requirement R_e , a priority $e \in \omega$ such that lower indices have higher priorities. Additionally, this method assigns to each R_e , $e \in \omega$, a "proof module" M_e whose aim is to satisfy R_e in isolation. Finally, the key component of this methodology says that higher priority modules M_e can "disrupt" or "reset" lower priority modules, but not vice versa; i.e. lower priority proof modules can only act in ways that respect the (usually finite) boundaries set by higher priority modules. Consequently, M_e can only be disrupted (reset) by the finitely many $M_{e'}$, $0 \le e' < e$; this is where the name "finite injury" is derived. From the point of view of Reverse Mathematics, the validation of proofs employing the Finite Injury methodolgy usually requires a certain level of induction known as Σ_2 -Bounding, denoted $\mathsf{B}\Sigma_2$, which we will define in the following subsection and essentially says that "a finite union of finite sets is finite." For more information on the Finite Injury Priority Method, consult [Soa16, Chapter 7].

2.3. **Ring Theory.** All of the rings R we consider here will be countable, commutative, and possess additive and multiplicative identity elements, 0_R and 1_R , respectively. Recall that a ring is *Noetherian* whenever it satisfies the ascending chain condition on its ideals. A ring is said to be *Artinian* whenever it satisfies the descending chain condition on its ideals. A prime ideal is *minimal* whenever it does not properly contain any prime ideals. Let R be a ring and $x_1, x_2, \ldots, x_N \in R$, $N \in \omega$. We will write

$$X = \langle x_1, x_2, \dots, x_N \rangle_R \subseteq R$$

to denote the *R*-span of (*R*-ideal generated by) x_1, x_2, \ldots, x_N .

The following construction is well-known.

Definition 2.7. Let R be a ring, and let $U \subset R \setminus \{0_R\}$ be a multiplicative(ly closed) subset of R. Then

$$R[U^{-1}] = \left\{ \frac{r}{u} : r \in R, \ u \in U \right\}$$

is the localization of R at U.

Moreover, it is well-known that if U contains no zero divisors, then the function $\varphi : R \to R[U^{-1}]$ given by

$$\varphi(x) = \frac{x}{1_R} \in R[U^{-1}], \ x \in R,$$

⁸Recall that a set $A \subseteq \omega$ is *computably enumerable* if it is the range of a computable function $f : \omega \to \omega$. More information can be found in [Soa16].

is an injective ring homomorphism.

For more information on basic Algebra, consult [Lan93, Eis95, DF99, Mat04].

2.3.1. Computable Ring Theory.

Definition 2.8. A computable ring R is a computable subset of ω , $R_0 \subseteq \omega$, equipped with:

- computable functions $+_R, \cdot_R : R \to R$, and
- a computable reflexive, symmetric, and transitive relation $=_R$,

such that $R = (R_0, =_R, +_R, \cdot_R)$ satisfies the ring axioms.

We say that an abstract ring R has a computable presentation or is computably presentable whenever R is isomorphic to a computable ring.

The following are consequences of the previous definition.

- If R is a computable ring and $I \subseteq R$ is a computable R-ideal, then R/I is also a computable ring (i.e. R/I has a computable presentation).
- If R is a computable ring containing $x_1, x_2, \ldots, x_N, N \in \omega$, then $X = \langle x_1, x_2, \ldots, x_N \rangle_R \subseteq R$ is Σ_1^0 -definable, but not always computable.
- If R is a computable ring and $U \subset R \setminus \{0_R\}$ is a computable multiplicative subset of R containing no zero divisors, then $R[U^{-1}]$ has a computable presentation via the equivalence relation

$$\frac{r_1}{u_1} \sim_{R[U^{-1}]} \frac{r_2}{u_2}$$
 if and only if $r_1 u_2 =_R r_2 u_1$.

Definition 2.9. Let R be a computable ring. We say that R satisfies computable-ACC whenever R satisfies the ascending chain condition for uniformly computable chains of ideals. In other words, R does not contain an infinite uniformly computable strictly ascending chain of ideals.

In the context of Reverse Mathematics (introduced in the following subsection), computable-ACC corresponds to the Noetherian property over RCA_0 .

Definition 2.10. Let R be a computable ring. We say that an ideal $X \subset R$ is PA-maximal if there is a set $A \subseteq \omega$ of PA Turing degree that does not compute any ideal $Y \subset R$ such that $X \subset Y$.

Remark 2.11. It follows from [Sim09, Theorem IV.6.4] that, if $X \subset R$ is PA-maximal as in the previous definition, and $z \in R \setminus X$, then there exists $a = a_z \in R$ and $x = x_z \in X$ such that

 $x + az = 1_R.$

Corollary 2.12. If R is a computable ring and $X \subset R$ is a PA-maximal ideal, then X is a maximal R-ideal.

The next two definitions are made over RCA_0 .

Definition 2.13 (RCA₀). Let R be a ring, and let $P \subset R$ be a prime ideal. We say that P is <u>minimal</u> if for all prime ideals $Q \subset R$ such that $Q \subseteq P$, we have that Q = P.

Definition 2.14 (RCA₀). Let R be a ring, and let $M \subset R$ be an ideal. We say that M is <u>maximal</u> if for every ideal X such that $M \subseteq X \subset R$, we have that M = X.

2.4. **Reverse Mathematics.** Reverse Mathematics is the subfield of Computability Theory and Proof Theory that aims to classify mathematical theorems in the context of Second-Order Arithmetic and countable structures, via their effective content. More specifically, in Reverse Mathematics one works over a weak base theory known as the Recursive Comprehension Axiom RCA₀ that, in the context of ω -models and full induction⁹, says:

- \emptyset exists,
- whenever $X \subseteq \mathbb{N}$ exists and $Y \leq_T X$, then Y also exists, and
- whenever $X, Y \subseteq \mathbb{N}$ exist, then

$$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\} \subseteq \mathbb{N}$$

exists.

Here \mathbb{N} denotes the (possibly nonstandard) first-order part of a model of Second-Order Arithmetic. The theorems typically analyzed in this context assert the existence of certain sets within structures. Roughly speaking, to show that theorem T_1 implies another theorem T_2 over RCA_0 it suffices to show that, given an instance of T_2 -i.e. a computable structure Xthat satisfies the hypotheses of T_2 -one can use finitely many iterations of solution sets for instances of T_1 (beginning with a computable T_1 -instance) to compute a solution set for X. For more details consult [Sim09].

We now state some theorems of Second-Order Arithmetic that will be relevant throughout the rest of our article. References (in order of appearance) for each of these theorems in the context of Reverse Mathematics are [Sim09, HS07, HSS09, NS20]. More information on Martin-Löf Randomness can be found in [DH10, Chapter 6]. A set $X \subseteq \mathbb{N}$ is MLR (Martin-Löf Random) if it passes every Martin-Löf test. Furthermore, X is 2-MLR if it passes every Martin-Löf test relative to Turing's Halting Set \emptyset' .

WKL₀: (Weak König's Lemma) Every infinite tree $T \subseteq 2^{<\mathbb{N}}$ contains an infinite path/chain.

- CAC: (Chain-Antichain Theorem) Every infinite partial order contains either an infinite chain, or an infinite antichain.
- ADS: (Ascending-Descending Chain Theorem) Every infinite linear order contains either an infinite ascending chain, or an infinite descending chain.

It is well-known [HS07, LST13], [Pat16, Corollary A.2.10] that CAC is strictly stronger than ADS over RCA₀, even in the context of ω -models.

HYP : For every set $X \subseteq \mathbb{N}$ there is a set of pairs $Y \subseteq \mathbb{N} \times \mathbb{N}$ such that Y is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is hyperimmune relative to X.

The following Ramsey-type König's Lemma was first introduced by Flood in [Flo12], and also studied by Bienvenu, Patey, and Shafer in [BPS17].

- 2-RWKL : For every infinite limit computable tree $T \subseteq 2^{<\mathbb{N}}$ there exists a *T*-homogeneous set $H \subseteq \mathbb{N}$ and $c \in \{0, 1\}$ such that $T_{H,c} \subseteq T$ (as described in Definition 2.2 above) is infinite.
- 2−MLR: For every set $X \subseteq \mathbb{N}$ there is a set $Y \subset \mathbb{N}$ such that Y is 2−MLR relative to X (i.e. Y is MLR relative to the Turing jump of X, denoted $X' \subset \mathbb{N}$).

It is known that 2-MLR implies 2-RWKL over RCA_0 , from which it follows that every instance T has measure one many (Turing oracle) solutions in Cantor space.

An introduction to diagonally nonrecursive functions in the context of Reverse Mathematics can be found in [NS20, Section 7]. Fix a nondecreasing function $h : \mathbb{N} \to \mathbb{N}$.

h−2−DNR: For every set $X \subseteq \mathbb{N}$ there is a set $Y \subseteq \mathbb{N} \times \mathbb{N}$ that is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is h - 2 - DNR relative to X, i.e. f_Y is h - DNR

⁹More information on induction schemes in the context of Second-Order Arithmetic follows.

relative to the Turing jump of X, denoted $X' \subset \mathbb{N}$.

2−DNR : For every set $X \subseteq \mathbb{N}$ there is a set $Y \subseteq \mathbb{N} \times \mathbb{N}$ that is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is 2−DNR relative to X, i.e. f_Y is DNR relative to X'.

An intermediate principle that is trivially implied by h-2-DNR, for any fixed $h : \mathbb{N} \to \mathbb{N}$, and implies 2-DNR, over RCA_0 , is the following.

 $\mathsf{O}-\mathsf{2}-\mathsf{DNR}$: For each set $X \subseteq \mathbb{N}$ there exists an order function $h_X : \mathbb{N} \to \mathbb{N}$ and a set $Y \subseteq \mathbb{N} \times \mathbb{N}$ that is $h_X - 2 - \mathsf{DNR}$ relative to X.

More information on DNR Turing degrees and their relationship to Martin-Löf randomness can be found in [HS07, Nie09]. A well-known but unpublished result of J. Miller shows that 2-DNR is equivalent to the Rainbow Ramsey Theorem for Pairs RRT_2^2 over RCA_0 ; see [NS20, Theorem 7.4] and the following paragraph for more details.

2.4.1. First-Order Reverse Mathematics. We assume that the reader is familiar with the hierarchy of arithmetical formulas; for more information on this topic we invite the reader to consult either [Soa16, Chapter 4] or [AK00, Chapter 2]. Now, RCA₀ includes a restricted induction scheme that only applies to Σ_1^0 formulas where a computable predicate is preceded only by existential quantifiers. Aside from asserting the existence of certain sets, theorems of Second-Order Arithmetic may also have First-Order (i.e. arithmetical, or number-theoretic) consequences and thus may require additional induction schemes (beyond Σ_1^0 -induction) in their proofs. For example, it is well-known that CAC cannot be proved from Σ_1^0 -induction alone. Rather, the first-order part of CAC includes the following bounding principle for Σ_2^0 -formulas that implies, but is not equivalent to Σ_1^0 -induction, and will play a role in some of our proofs below.

B Σ_2 : Let $\psi(x)$ be a Σ_2^0 -formula. Then, for any given $n \in \mathbb{N}$, if there exist $x_1, x_2, \ldots, x_n \in \mathbb{N}$ such that $\psi(x_i)$ holds for $1 \leq i \leq n$, then there exists $N \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in \mathbb{N}$ such that $\psi(y_i)$ holds for $1 \leq i \leq n$ and $\max\{y_i : 1 \leq i \leq n\} < N$.

One can easily generalize $\mathsf{B}\Sigma_2$ to $\mathsf{B}\Sigma_n$, the corresponding bounding principle for Σ_n^0 -formulas.

An ω -model is a model of Second-Order Arithmetic whose first-order part is the standard natural numbers $\omega = \{0, 1, 2, ...\}$ and therefore satisfies induction for all formulas. It is useful to keep in mind that, in the context of Reverse Mathematics, to show that a theorem T_1 *does not* imply another theorem T_2 it suffices to produce an ω -model of RCA₀ in which T_1 holds but T_2 does not.

Remark 2.15. It is well-known that $I\Sigma_n$ (i.e. the Σ_n^0 -induction scheme) is equivalent to each of the following:

- the Π_n^0 -induction scheme,
- the Well-Ordering Principle for Σ_n^0/Π_n^0 -definable sets¹⁰, and
- the comprehension scheme for bounded Σ_n^0/Π_n^0 definable sets¹¹.

Also, $\mathsf{B}\Sigma_2$ is equivalent to both:

- the Infinite Pigeonhole Principle¹², and
- the statement that says "a finite union of finite sets is finite."

¹¹For every Σ_n^0/Π_n^0 – predicate $\varphi(x)$ and natural number N, the bounded set of natural numbers

$$\{x : x \le N \& \varphi(x)\}$$

exists.

¹²For any natural number N, if the natural numbers are partitioned into N-many classes, then one of the classes in the partition is infinite.

¹⁰For every Σ_n^0/Π_n^0 -predicate $\varphi(x)$, either no natural number x satisfies φ , or else there is a minimal natural number x_0 such that $\varphi(x_0)$ holds.

Moreover, it is well-known that for all $n \in \omega$ the strict implications $|\Sigma_{n+1} \to B\Sigma_{n+1} \to |\Sigma_n exist.$

2.4.2. Our Presentation. We will usually prove theorems in the context of Reverse Mathematics (i.e. over RCA_0) by first proving their corresponding effective versions in the context of ω -models. Afterwards, we will state the (more general) reverse mathematical theorem over RCA_0 as a corollary to the effective version immediately preceding it. For us, translating effective proofs in the context of ω -models to their corresponding corollaries over RCA_0 amounts to:

- interpreting computability as Δ_1^0 -definability,
- interpreting computable enumerability as Σ_1^0 -definability, and finally (and usually most importantly)
- verifying that the first-order part of the argument can be carried out via the fragment of First-Order Arithmetic corresponding to Σ_1^0 -induction, and nothing more.

For example, it is well-known and not difficult to show that every infinite computably enumerable set contains an infinite computable subset (enumerated in strictly increasing order). In the context of Reverse Mathematics, this leads to the following fact.

Proposition 2.16 (RCA₀). Every infinite Σ_1^0 -definable set contains an infinite subset. In other words, if $\varphi(x)$ is a Σ_1^0 -fomula such that

$$(\forall n)(\exists x_n)[x \ge n \& \varphi(x_n)]$$

then there exists $A \subseteq \mathbb{N}$ such that

$$(\forall n)(\exists x_n)[x_n \in A \& x_n \ge n \& \varphi(x_n)]$$

The issue here is that, in the absence of Arithmetic Comprehension (ACA₀), the set $\{x : \varphi(x)\} \subseteq \omega$ may not exist. However, given the hypotheses, over RCA₀ the infinite set $A \subseteq \mathbb{N}$ of Proposition 2.16 always does.

We will explicitly mention the level (i.e. strength) of the induction or bounding principle being utilized in a proof of an effective theorem. Afterwards, we will include the strongest such principle in the hypotheses of the corresponding reverse mathematical corollary. In the end the only two subsystems of First-Order Arithmetic that we will end up using beyond $I\Sigma_1$ are:

- $\mathsf{B}\Sigma_2$ (mentioned above) and
- $I\Sigma_2$ (the induction scheme for Σ_2^0 -formulas).

Recall that $I\Sigma_2 \to B\Sigma_2 \to I\Sigma_1$, but the arrows are both irreversible.

Reverse mathematicians usually use $\omega = \{0, 1, 2, ...\}$ to denote the standard natural numbers, and \mathbb{N} to denote a possibly nonstandard model of arithmetic that may not satisfy full induction. In keeping with this convention, our effective theorems will mention ω , while their corresponding (reverse mathematical) corollaries will mention \mathbb{N} .

3. The Tree Antichain Theorem

We now introduce our main combinatorial principle, namely the Tree Antichain Theorem. In the next section we will examine its reverse mathematical strength, and subsequent sections reveal its relationship to the theory of Noetherian Rings.

Definition 3.1 (RCA₀). We say that a tree $T \subseteq 2^{<\mathbb{N}}$ is <u>completely branching</u> if for all $\sigma \in T$ such that $\sigma k \in T$ for some $k \in \{0, 1\}$ we have that $\sigma(\overline{1-k}) \in T$ as well. In other words, every node $\sigma \in T$ is either a leaf, or else $\{\sigma 0, \sigma 1\} \subset T$.

Additionally, for any given infinite completely branching tree $T \subseteq 2^{<\mathbb{N}}$, we say that $\{T_s : s \in \mathbb{N}\}$ is an <u>enumeration</u> of T whenever:

- $T_0 = \{\emptyset\} \subset 2^{<\mathbb{N}};$
- for each s > 0, $s \in \mathbb{N}$, there exists a unique T_{s-1} -leaf λ such that $T_s = \{\lambda 0, \lambda 1\} \cup T_{s-1}$; and
- $T = \bigcup_{s \in \mathbb{N}} T_s$.

It follows that T is Σ_1^0 -definable (i.e. computably enumerable) if and only if there exists an effective enumeration of T, i.e. a computable function that performs the enumeration of T described in the items above.

Definition 3.2 (RCA₀). Let TAC be the theorem that says "every infinite Σ_1^0 -definable completely branching tree $T \subseteq 2^{<\mathbb{N}}$, contains an infinite $(2^{<\mathbb{N}}-)$ antichain."

Remark 3.3. TAC is not a standard theorem of mathematics that one would expect to find featured in a textbook, and so requires a proof. To see why TAC holds in Second-Order Arithmetic, let $\varphi(x,t)$ be a Σ_1^0 -formula such that

$$T = \{x : (\exists t)[\varphi(x,t)]\} \subseteq 2^{<\mathbb{N}}$$

is an infinite completely branching tree. Now, via Arithmetic Comprehension (ACA₀), T exists and corresponds to an infinite partial order via $2^{<\mathbb{N}}$. Furthermore, by the Chain-Antichain Principle (CAC), T either contains an infinite chain or an infinite antichain. If T contains an infinite antichain then we are done, otherwise $T \subseteq 2^{<\mathbb{N}}$ contains an infinite chain/path $f \in [T] \subseteq 2^{\mathbb{N}}$.

For each $k \in \omega$ let $\sigma_k = f \upharpoonright k \in T \subseteq 2^{<\mathbb{N}}$ denote the unique initial segment of f of length k. Since T is completely branching, we have that both $\sigma_k 0, \sigma_k 1 \in T$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\tau_k = \sigma_k j \in T$ for the unique $j \in \{0, 1\}$ such that $\tau_k \neq \sigma_{k+1} \subset f \in 2^{\mathbb{N}}$, i.e. $\tau_k \nsubseteq f$. It follows that $\{\tau_k : k \in \mathbb{N}\} \subseteq T \subseteq 2^{<\mathbb{N}}$ is an infinite antichain, as required.

It is not difficult to eliminate our use of ACA₀ in the argument above, and produce an effective proof of TAC via RCA₀ + CAC. More specifically, a Σ_1^0 formula such as φ above easily corresponds to an effective enumeration $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ as in Definition 3.1. Now we can construct a computable partial order on ω by associating

- $0 \in \mathbb{N}$ with $\emptyset \in T$, and
- $2s + 1, 2s + 2 \in \mathbb{N}, s \in \mathbb{N}, with the pair of nodes \sigma 0, \sigma 1 \in T_{s+1} \setminus T_s.$

By Σ_1^0 -induction this process yields a computable partial order \mathcal{P} with domain \mathbb{N} and computable isomorphism $F: \omega \to T$. Furthermore, CAC says that \mathcal{P} contains either an infinite chain or an infinite antichain, which (via F) corresponds to a Σ_1^0 -definable subset of T. Now, Proposition 2.16 above says that (over RCA₀) every infinite Σ_1^0 -definable set contains an infinite subset, thus yielding an infinite T-antichain (via our comments above).

Building on our elementary analysis here, in the next section we will provide a similar proof of TAC via the Kleene-Brower construction $T_{KB} = (T, <_{KB})$ described above (Definition 2.1 and the following sentence) and the Ascending-Descending Chain Principle for infinite linear orders (ADS).

Definition 3.4 (An alternate characterization of TAC over RCA_0). The following equivalent version of TAC does not mention Σ_1^0 -definability or enumerations.

 TAC_1 : Let $T \subseteq 2^{<\mathbb{N}}$ be an extendable tree containing infinitely many splittings, i.e. infinitely many $\sigma \in T$ such that $\sigma 0, \sigma 1 \in T$. Then T contains an infinite $(2^{<\mathbb{N}}-)$ antichain.

The next two propositions show how to use solutions to TAC_1 to derive solutions to TAC, and vice versa in the context of ω -models. They also verify that each argument can be carried out via Σ_1^0 -induction, implying Corollary 3.7 below which says that TAC_1 is equivalent to TAC over RCA₀, and exemplifies our presentation of Reverse Mathematics, as described in Subsection 2.4.2 above. **Proposition 3.5.** Let $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ be the enumeration of an infinite Σ_1^0 completely branching tree as described in Definition 3.1 above. Then there is a computable extendable tree $T_1 \subseteq 2^{<\omega}$ with infinitely many splittings such that any T_1 -antichain computes a T-antichain.

Proof. Let $T = \bigcup_{s \in \omega} T_s \subseteq 2^{\omega}$ be as in the statement of the current proposition. We enumerate an infinite computable extendible tree $T_1 = \bigcup_{s \in \omega} T_{s,1} \subseteq 2^{<\omega}$ with infinitely many splittings, and a corresponding one-to-one $2^{<\omega}$ -strict-extension-preserving map $F = \bigcup_{s \in \omega} F_s : T \to T_1$ such that the range of F is the set of completely branching nodes of T_1 , uniformly in stages $s \in \omega$, as follows:

- $T_{0,1} = \{\emptyset\} = T_0 \subseteq 2^{<\omega}, F_0(\emptyset) = \emptyset;$
- at stage $s + 1 \in \omega$, $s \in \omega$, assume that every $T_{s,1}$ -leaf $\lambda \in T_{s,1}$ has a unique prefix of the form $F_s(\rho)$, for some T_s -leaf ρ . Now, let $\rho_0 \in T_s$ be the unique T_s -leaf such that $T_{s+1} \setminus T_s = \{\rho_0 0, \rho_0 1\}$, and (by Σ_1^0 -induction) let $\lambda_0 \in T_{s,1}$ be the unique maximal $T_{s,1}$ -extension of $F(\rho_0)$; it follows that λ_0 is a $T_{s,1}$ -leaf. We define $T_{s+1,1} \supset T_{s,1}$ and $F_{s+1} \supset F_s$ via:

$$- T_{s+1,1} = T_{s,1} \cup \{\lambda_0 0, \lambda_0 1\} \cup \{\lambda 0 : \lambda \neq \lambda_0 \text{ a } T_{s,1} - \text{leaf}\}, \text{ and} - F_{s+1}(\rho_0 i) = \lambda_0 i, i \in \{0, 1\}.$$

By our construction (and Σ_1^0 -induction) it follows that every $T_{s+1,1}$ -leaf ρ has a unique prefix of the form $F_{s+1}(\lambda) \subseteq \rho$ for some T_{s+1} -leaf λ , from which it follows that our constructions of $T_1 = \bigcup_{s \in \omega} T_{s,1} \subseteq 2^{<\omega}$ and $F = \bigcup_{s \in \omega} F_s : T \to T_1$ are well-defined. The reader can also verify that:

- $T_1 \subseteq 2^{<\omega}$ is an infinite computable extendible tree containing infinitely many splittings, and
- $F: T \to T_1$ is one-to-one, computable, and $2^{<\omega}$ -extension-preserving.

Now, given any infinite antichain

$$A = \{a_k : k \in \omega\} \subset T_1$$

by our construction of T_1 above it follows that for each $k \in \omega$ there is a unique maximal initial segment of $a_k \in T_1$, $\alpha_k \subseteq a_k$, that completely branches in T_1 , i.e. $\{\alpha_k 0, \alpha_k 1\} \subset T_1$, and moreover $F(\beta_k i) = \alpha_k i$ for a unique $\beta_k \in T$ and all $i \in \{0, 1\}$. For each $k \in \omega$ let $i_k \in \{0, 1\}$ be such that

$$F(\beta_k i_k) = \alpha_k i_k \subseteq a_k \in T_1$$

Finally, since F is $2^{<\omega}$ -extension-preserving we have that

$$B = \{\beta_k i_k : k \in \omega\} \subset T$$

is an infinite T-antichain.

Proposition 3.6. Let $T \subseteq 2^{<\omega}$ be a computable extendable tree with infinitely many splittings. Then there is an infinite Σ_1^0 completely branching tree $T_0 \subseteq 2^{<\omega}$ such that every infinite T_0 -antichain computes an infinite T_1 -antichain.

Proof. Let $T \subseteq 2^{<\omega}$ be as in the statement of the current proposition. Let

$$R = \{\rho_k : k \in \omega\} \subseteq T$$

be the length-lexicographic enumeration of the infinitely many splittings of T, i.e. R is an infinite subset of T such that for all $k \in \omega$ we have that $\rho_k 0, \rho_k 1 \in T$. Let

$$R_1 = \{ \rho_k i : k \in \omega, \ \rho_k \in R, \ i \in \{0, 1\} \} \subset T$$

There is a partial computable function $F_1 : R_1 \to R$ such that for all $\rho_k i \in R_1, k \in \omega$, $\rho_k \in R, i \in \{0, 1\},$

$$F_1(\rho_k i) = \rho \in R,$$

where ρ is the unique node of least length in R extending $\rho_k i$, if such a ρ exists. Construct an infinite Σ_1^0 completely branching tree $T_0 \subseteq 2^{<\omega}$ along with a partial computable function $F_0: T_0 \to R$, in stages $s \in \omega^+$, via R and F_1 as follows:

- at stage s = 1 set $T_1 = \{\emptyset, 0, 1\} \subset 2^{<\omega}$ and define $F_0(\emptyset) = \rho_0 \in R$, and
- at stage s > 1, $s \in \omega$, assume we are (uniformly computably) given a finite tree $T_{s-1} \subset 2^{<\omega}$ and $F_s : T_s \to R$ such that:
 - the domain of $F_s: T_s \to R$ at the current stage $s \in \omega$ is exactly (the uniformly computable tree of) all non-leaf nodes of T_s , and

- the range of F_0 at the current stage $s \in \omega$ is exactly $\{\rho_k : 0 \leq k \leq s - 1\} \subset R$. By our construction of R it follows that at each stage s > 1, $s \in \omega$, there is a unique T_s -leaf λ such that

$$F_1(\lambda) = \rho_s$$

Furthermore, if $i \in \{0, 1\}$ is the final bit of λ then we have that

 $F_0(\lambda^-)i \subset \rho_s.$

In this case we set $F_0(\lambda) = \rho_s$ and enumerate the splitting above λ , namely $\lambda 0$ and $\lambda 1$ in T_s . Finally, we proceed to the next stage s + 1, where our induction hypotheses about the domain and range of F_0 (given above) remain valid. This completes our construction of $T = \bigcup_{s \in \omega} T_s$.

By our constructions of $T_0 = \bigcup_{s>0} T_s \subseteq 2^{<\omega}$, $F_0 : T_0 \to R \subseteq T$, and Σ_1^0 -induction, it follows that the domain of F_0 is exactly the set of non-leaves of T_0 , which is computably enumerable. Let

$$A = \{\alpha_k : k \in \omega\} \subseteq T_0$$

be an infinite T_0 -antichain. Then, either A contains an infinite subset of the domain of F_0 , or not. If A contains an infinite subset of the domain of F_0 , then via Proposition 2.16 above it follows that there is an infinite A-computable subset of A in the domain of F_0 . By our construction of $F_0: T_0 \to T$ above, it follows that F_0 preserves the extension relation on nodes (binary strings), from which it follows that

$$F_0(A) = \{F_0(\alpha) : \alpha \in A\} \subseteq T$$

is a T-antichain. On the other hand, if A does not contain an infinite subset of the domain of F_0 then it follows that cofinitely many elements of A are T-leaves for which F_1 is undefined. In this case, let $k_0 \in \omega$ be such that

$$A_{k_0} = \{\alpha_k : k \ge k_0\} \subseteq A \subseteq T$$

contains only T-leaves. Let $I_{k_0} = \{i_k : k \ge k_0\}$ be the corresponding enumeration of the last bits of A_{k_0} . It follows that

- α_k^- is not a T_0 -leaf and thus is in the domain of F_1 , but
- $F_0(\alpha_k)$ is undefined.

Furthermore, $F_0(\alpha_k^-) \in R \subseteq T$ defines a T-splitting for all $k \geq k_0, k \in \omega$, and since α_k is not in the domain of F_0 for all $k \geq k_0, F(\alpha_k^-)i_k \in T_1$ has no splittings extending it, from which it finally follows that

$$B = \{F(\alpha_k^-)i_k : k \ge k_0, \ k \in \omega\}$$

is an infinite T_1 -antichain.

Corollary 3.7 (RCA_0). TAC is equivalent to TAC_1 .

Proof. The previous two propositions.

3.1. TAC in the context of infinite computable completely branching trees. One might ask about the reverse mathematical strength of TAC in the context of computable (rather than computably enumerable) trees $T \subseteq 2^{<\omega}$. In this case one can argue that every infinite computable completely branching tree $T \subseteq 2^{<\omega}$ has an infinite computable antichain $A \subseteq T$, from which it follows that TAC with computable trees is provable in RCA₀. Let $T \subseteq 2^{<\omega}$ be an infinite computable completely branching tree. Then, either:

- there exists $\sigma \in T$ such that $\sigma 0^{\infty} \in [T] \subseteq 2^{\omega}$, or else
- no such σ exists.

If such a $\sigma \in T$ exists then

$$A = \{\sigma 0^k 1 : k \in \omega\} \subset T$$

is an infinite computable T-antichain. Otherwise, since T is computable, for any given $\sigma \in T$ there exists $k_{\sigma} \in \omega$, computable uniformly in σ , such that

$$\lambda_{\sigma} = \sigma 0^{k_{\sigma}} \in T$$
 is a T - leaf.

It follows that the set $A = \{\lambda_{\sigma} : \sigma \in T\}$ is an infinite computable T-antichain. A similar argument shows that every infinite Σ_1^0 completely branching tree has an infinite antichain that is computable via Turing's Halting Set \emptyset' (i.e. an infinite Δ_2^0 antichain).

3.2. TAC is not equivalent to any "known" subsystem of Second-Order Arithmetic. One consequence of our analysis of TAC says that TAC is not equivalent to any other subsystem of Second-Order Arithmetic that has thus far been studied and is included in the "Reverse Mathematical Zoo"¹³ that has been developed and promulgated by Dzhafarov and others (it suffices to do so in the context of ω -models). Now, to see why this is the case, via the diagram given in https://production.wordpress.uconn.edu/mathrmzoo/wp-content/uploads/sites/841/2014/09/diagram_oi.pdf, which we will refer to as simply diagram_oi.pdf, the reader can verify that, in the context of ω -models and RCA₀, the strongest consequences of ADS are

- AMT + COH, whose strongest consequences, in turn, are:
 - AMT, and
 - -COH;

and

• SADS, whose strongest consequence is AMT.

Here

- COH denotes the Cohesive Principle (see [CJS01, HS07] for more details),
- AMT denotes the Atomic Model Theorem (see [HSS09] for more details), and
- SADS denotes the Stable Chain-Antichain Principle for (stable) infinite linear orders (see [HS07]) for more details.

From our description of diagram_oi.pdf above, it follows that every consequence of ADS (over RCA_0 and in the context of ω -models) either:

- implies one of COH, AMT, or
- is implied by one of COH, AMT.

Now, when taken together

- [Pat16, Theorem 9.1.2],
- the paragraph following [Pat16, Definition 7.3.2] (which generalizes a result in [Wan13]), and
- Corollary 4.8 below

¹³See rmzoo.math.uconn.edu for more details, and rmzoo.math.uconn.edu/diagrams for visualizations of the Reverse Mathematical Zoo.

imply that TAC does not prove either COH or AMT, since (as we shall see in the next section) TAC follows from 2-MLR, but our references to [Pat16] above say that 2-MLR does not imply either COH or AMT. On the other hand, the conservativity considerations of [HS07, Corollary 2.21] and [HSS09, Corollary 3.15] say that neither COH nor AMT imply TAC. Therefore, TAC cannot be equivalent to any principle mentioned in diagram_oi.pdf.

4. The Reverse Mathematics of the Tree-Antichain Theorem

4.1. Upper bounds on TAC. We establish the weakness of TAC in several respects, in the context of Reverse Mathematics.

Theorem 4.1. Let $T \subseteq 2^{<\omega}$ be an infinite Σ_1^0 completely branching tree. Then there is a computable infinite linear order \mathcal{L} such that any infinite ascending or descending chain in \mathcal{L} computes a T-antichain.

Proof. Given an infinite Σ_1^0 completely branching tree $T \subseteq 2^{<\omega}$, let $T_{KB} = (T, <_{KB})$ be the (Σ_1^0) Kleene-Brower linearization of T (viewed as a partial order) as described in the previous section. We can construct a computable linear order $\mathcal{L} \cong T_{KB}$ with domain ω as we did for partial orders in Remark 3.3 above. Now, via ADS, let $\mathcal{C} \subseteq \mathcal{L}$ be an infinite (ascending/descending) \mathcal{L} -chain. By construction, \mathcal{C} corresponds to an infinite Σ_1^0 (ascending/descending) T_{KB} -chain, C_0 . Proposition 2.16 yields an infinite computable $C \subseteq C_0 \subseteq T_{KB}$, and any such C is an infinite computable (ascending/descending) T_{KB} -chain.

Let $C = \{c_k : k \in \omega\} \subseteq T_{KB}$ be a T_{KB} -ordered enumeration of C (in either increasing or decreasing order). If C is an ascending T_{KB} -chain, then by construction of $T_{KB} = (T, <_{KB})$ it follows that the set

$$\{c_k : k \in \omega, c_{k+1} \text{ is left of } c_k\}$$

is an infinite T-antichain. On the other hand, if C is a descending T_{KB} -chain, then there are two subcases to consider. First, suppose that there exists $k_0 \in \omega$ such that for all $k \geq k_0$ we have that $c_k \subset c_{k+1}$ (here \subset denotes the extension relation on $2^{<\omega}$); in this case we can construct an infinite T-antichain as we did in Remark 3.3 above. Otherwise the (computable) set

$$\{c_k : k \in \omega, c_{k+1} \text{ is right of } c_k\}$$

is an infinite T-anchain, as required by TAC.

Corollary 4.2 (RCA_0). ADS *implies* TAC.

The following theorem will be useful in showing that 2-MLR implies TAC below; it is essentially due to Kučera and we refer the reader to [NS20, Section 7] and [BPS17, Theorem 2.8] for more details on the following theorem and subsequent corollary.

Theorem 4.3. Let $h(x) = 2^x$, $x \in \omega$. Then every 2-MLR set $X \subseteq \omega$ computes an h-2-DNR function f.

Corollary 4.4 (RCA₀ + $I\Sigma_2$). 2-MLR implies h-2-DNR for $h(x) = 2^x$, $x \in \mathbb{N}$.

Theorem 4.5. Let $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ be an enumeration of an infinite Σ_1^0 completely branching tree, and let $h : \omega \to \omega$ be a computable order function and $f : \omega \to \omega$ h-2-DNR. Then f computes an infinite T-antichain.

Proof. Let $T \subseteq 2^{<\omega}$, $h, f : \omega \to \omega$ satisfy the hypotheses of the current theorem; we will construct an infinite Σ_1^0 (i.e. computably enumerable) finitely branching tree

$$\mathcal{A} = \bigcup_{s \in \omega} \mathcal{A}_s \subseteq \omega^{<\omega}, \ \mathcal{A}_s \subseteq \mathcal{A}_{s+1} \text{ finite},$$

along with a partial computable oracle reduction

$$\Psi : \mathcal{A} \times \omega \to T, \ \Psi = \bigcup_{s \in \omega} \Psi_s, \ \Psi_s \subseteq \Psi_{s+1}, \ \Psi_s : \mathcal{A}_s \to T,$$

such that for every h - 2-DNR function $f : \omega \to \omega$ computes some $g \in [\mathcal{A}] \subseteq \omega^{\omega}$ such that

$$A = \{\Psi^g(k) \downarrow = a_k : k \in \omega^+\} \subseteq T$$

is an infinite T-antichain, as required by TAC.

We now describe the basic module M_{λ} , λ an \mathcal{A}_s -leaf, $|\lambda| = \ell$, for constructing $\mathcal{A}_{s+1} \supset \mathcal{A}_s$, $s \in \omega$ and corresponding $\Psi_{s+1} \supset \Psi_s$. For each \mathcal{A}_s -leaf λ , the module M_{λ} uniformly attempts to uniformly and computably enumerate λi , $0 \leq i \leq n_{\lambda}$ into \mathcal{A}_{s+1} , for some $n_{\lambda} \in \omega^+$, and define $\Psi_{s+1}^{\lambda i}(\ell+1)$ for each $0 \leq i \leq n_{\lambda}$. Now, suppose (by Σ_1^0 -induction) that

$$A_{\lambda} = \{\Psi^{\lambda}(k) : 1 \le k \le |\lambda|\}$$

is an \mathcal{A} -antichain. First, M_{λ} seeks infinitely many numbers $e_{\lambda,i}, t_{\lambda,i} \in \omega, i \in \omega$, such that for each *i* the (finite) tree

$$T_{t_{\lambda_i}} \subset T \subseteq 2^{<\omega}$$

has at least $h(e_{\lambda,i})$ -many leaves that are each (individually) incomparable with all $a \in A_{\lambda}^{14}$. When M_{λ} finds such numbers $e_{\lambda,i}, t_{\lambda,i}$, for some $i \in \omega$, then it effectively enumerates/indexes the corresponding $(A_{\lambda}$ -incomparable) $T_{t_{\lambda,i}}$ -leaves

$$\rho_{0,i}, \rho_{1,i}, \dots, \rho_{n_{\lambda,i},i} \in T_{t_{\lambda,i}}, \ n_{\lambda,i} = h(e_{\lambda,i}) \in \omega^+.$$

If there exist infinitely many candidate pairs $\{(e_{\lambda,i}, t_{\lambda,i}) : i \in \omega\} \subseteq \omega \times \omega$, for some \mathcal{A}_s -leaf λ , then by the Recursion Theorem [Soa16, Theorem 2.2.1], there exists $i_0 \in \omega$ such that

$$\Phi_{e_{\lambda}}^{\varphi}(e_{\lambda}) = n \in \omega, \ 0 \le n \le n_{\lambda} = n_{\lambda,i_0}, \ n_{\lambda} > 0, \ e_{\lambda} = e_{\lambda,i_0} \in \omega_{\lambda,i_0} \in \omega_{\lambda,i$$

is the index of the unique extendable leaf $\rho_n = \rho_{n,i_0} \in T_{t_{\lambda}}$, if such a leaf exists, and is undefined otherwise, for some corresponding $t_{\lambda} = t_{\lambda,i_0}$, $i_0 \in \omega$, in the previous paragraph. Moreover, by the uniformity and effectiveness of the Recursion Theorem it follows that $e_{\lambda}, t_{\lambda} \in \omega$ are uniformly computable in λ , assuming that infinitely many $e_{\lambda,i}, t_{\lambda,i}$ exist. Now, once we have (uniformly and computably) selected $n_{\lambda} = n_{\lambda,i_0} \in \omega$, $n_{\lambda} > 1$, via the Recursion Theorem, we enumerate the nodes

$$\lambda 0, \lambda 1, \ldots, \lambda n_{\lambda} \in \mathcal{A}_{s+1}.$$

Finally, set

$$\Psi^{\lambda i}(\ell+1) = \Psi^{\lambda i}(|\lambda|+1) = \rho_i \in T, \ i = 0, 1, 2, \dots, n_\lambda \in \omega$$

By our construction of \mathcal{A}_{s+1} and Ψ_{s+1} it follows that

$$A_{\lambda i} = \{\Psi_{s+1}^{\lambda i}(k) : 1 \le k \le \ell + 1\} \subset T$$

is a T-antichain of size $\ell + 1$, for all $0 \le i \le n_{\lambda}$. This completes our construction of the Σ_1^0 tree $\mathcal{A} \subset \omega^{<\omega}$. The rest of our proof verifies that:

• \mathcal{A} is infinite, and

a

• any h-2-DNR function $f: \omega \to \omega$ computes an infinite path $g \in [\mathcal{A}] \subset \omega^{\omega}$.

¹⁴In the following paragraphs we will argue that infinitely many $e_{\lambda,i}, t_{\lambda,i} \in \omega$, $i \in \omega$, always exist for at least one \mathcal{A}_s -leaf λ that can be uniformly and effectively computed by an h - 2-DNR function f.

By our construction of Ψ and g it follows that

$$A = \{\Psi^g(x) : x \in \omega\}$$

codes an infinite T-antichain, as required.

We claim that for all $s \in \omega$ there exists an \mathcal{A}_s -leaf $\lambda_s \in \omega^{<\omega}$ with infinitely many corresponding $e_{\lambda,i}, t_{\lambda,i} \in \omega$ as we hypothesized in the previous paragraph. Moreover, we aim to show that (via $|\Sigma_2\rangle$) λ_s is uniformly computable in s, relative to any h-2-DNR oracle $f: \omega \to \omega$. To do so, however, we require an important (Π_2^0) definition.

Definition 4.6. Let $\lambda \in \mathcal{A} \subset \omega^{<\omega}$. By our construction of \mathcal{A} it follows that $\Psi^{\lambda}(x) \downarrow$, $x \in \omega$, for all $1 \leq x \leq |\lambda|$ and moreover

$$A_{\lambda} = \{\Psi^{\lambda}(x) : 1 \le x \le |\lambda|, \ x \in \omega\} \subset T$$

is a finite T-antichain. We say that λ is <u>T-antichain-extendible</u> if for every $n \in \omega$ there is a set $B \subseteq T$, |B| = n, such that $A_{\lambda} \cup B$ is a T-antichain.

Above, we argued (via Σ_1^0 -induction) that $\lambda \in \mathcal{A}$ is *T*-antichain-extendable whenever the module M_{λ} described above enumerates infinitely many $e_{\lambda,i}, t_{\lambda,i} \in \omega, i \in \omega$. It is clear that the converse also holds. Now, suppose that $\lambda \in \mathcal{A}$ is *T*-antichain-extendable and let $e_{\lambda}, t_{\lambda}, n_{\lambda} \in \omega, n_{\lambda} > 1$, and

$$\rho_0, \rho_1, \dots, \rho_{n_\lambda} \in T_{t_\lambda} \subseteq 2^{<\omega}$$

be obtained from the infinite sequence of $e_{\lambda,i}, t_{\lambda,i} \in \omega$, $i \in \omega$, (via some $i = i_0 \in \omega$ obtained via the Recursion Theorem) in our construction of M_{λ} above. Then, since each of the ρ_k , $0 \leq k \leq n_{\lambda}$, are leaves of $T_{t_{\lambda}}$, it follows that they are mutually incomparable and furthermore (by construction)

$$A_{\lambda} \cup \{\rho_k : 0 \le k \le n_{\lambda}\} \subseteq T_{t_{\lambda}}$$

is a $T_{t_{\lambda}}$ -antichain. Now, since $T \subseteq 2^{<\omega}$ contains no cycles, it follows that there can be at most one ρ_i , $0 \leq i \leq n_{\lambda}$, $i \in \omega$, such that all but finitely many $\sigma \in T$ extend ρ_i , and moreover this ρ is Σ_2^0 , uniformly in λ . In other words, at most one ρ_i , $0 \leq i \leq n_{\lambda}$, is not T-antichain-extendable, from which it follows that that some ρ_i , $0 \leq i \leq n_{\lambda}$ is T-antichain-extendable, since $n_{\lambda} > 1$. Furthermore, by our use of the Recursion Theorem above in producing $i_0 \in \omega$ for M_{λ} and our comments here, it follows that any h-2-DNR function $f: \omega \to \omega$ can compute a T-antichain-extendable ρ_i , $0 \leq i \leq n_{\lambda}$, uniformly in $\lambda \in \mathcal{A}$. We have now verified the induction step for an instance of Π_2^0 -induction relative to any h-2-DNR function $f: \omega \to \omega$ that (given the trivial base case¹⁵) produces a total Turing reduction

 $g \leq_T f$

where $g \in [\mathcal{A}] \subseteq \omega^{\omega}$ and

$$\{\Psi^g(x): x \in \omega\} \subseteq T$$

is an infinite T-antichain.

Corollary 4.7 (RCA₀ + $I\Sigma_2$). O-2-DNR *implies* TAC.

Corollary 4.8 (RCA₀ + $I\Sigma_2$). 2–MLR *implies* TAC.

Proof. Corollary 4.4 and Theorem 4.5 above.

Finally, we come to a natural uniform proof of TAC via 2–RWKL.

¹⁵The base case says that $\emptyset \in T$ is *T*-antichain-extendible, which is obviously true since *T* is assumed to be infinite and completely branching.

Theorem 4.9. Let $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ be an infinite Σ_1^0 completely branching tree, and (via 2-RWKL) let $H \subseteq \omega$ be such that H is homogeneous for T (via some $c \in \{0, 1\}$). Then H computes an infinite T-antichain.

Proof. By Definition 2.2 above, let $c \in \{0, 1\} \subset \omega$ be such that $T_{H,c} \subseteq T \subseteq 2^{<\omega}$ is infinite, and for each $\ell \in H$, let $\sigma_{\ell} \in T$ be a node of length ℓ . Then, by construction of H and the fact that T is completely branching it follows that

$$\{\sigma_{\ell}(1-c): \ell \in H\} \subset T$$

is an infinite T-antichain.

Corollary 4.10 (RCA₀). 2–RWKL *implies* TAC.

4.2. Coding into TAC. The main goal of this subsection is to establish the non-trivial strength of TAC by showing that TAC does not follow from WKL_0 , even in the context of ω -models.

Theorem 4.11. For any order function $h : \omega \to \omega$ there exists an infinite Σ_1^0 completely branching tree $T \subseteq 2^{<\omega}$ such that every infinite T-antichain, $A_T \subset T$, is not the image of an h-c.e. function.

Proof. Let $h : \omega \to \omega^+$ be a computable order function. We will employ a finite injury argument (see Section 2.2.1 above for an introduction and references) that computably enumerates an infinite (Σ_1^0) completely branching tree $T \subseteq 2^{<\omega}$, $T = \bigcup_{s \in \omega} T_s$, such that

$$T_s = \{F_s(x) : 0 \le x \le 2s\}, \ s \in \omega,$$

is an enumeration of T via a computable function F (as outlined in the previous section). Moreover, our main goal here is to enumerate T (i.e. construct F) to ensure that for every strictly increasing h-c.e. function $f: \omega \to \omega$ we have that

$$A = A_f = \{F(f(x)) : x \in \omega\} \subseteq T,$$

is not a T-antichain.

To achieve our goal, for now assume that there exists a uniform computable approximation $f(e, x, s) : \omega^3 \to \omega$ to every strictly increasing h-c.e. function $f_e : \omega \to \omega$ such that:

(a) for every $e, x \in \omega$ we have that:

-f(e, x, s) = x whenever $0 \le s \le x$, and

 $-f(e, x, s) \leq s$, for all $e, x, s \in \omega, s \geq x$.

- (b) for every $s \in \omega$ there is at most one pair of numbers $e, x \in \omega$ such that $f(e, x, s) \neq f(e, x, s + 1)$ and $0 \leq x \leq s$.
- (c) For every $e \in \omega$, $f_e(x) = \lim_{s \to \infty} f(e, x, s)$ is strictly increasing in $x \in \omega$.

The first subitem in (a) above implies that $s \ge x$ whenever $f(e, x, s) \ne f(e, x, s + 1)$. Moreover, items (a) and (b) above imply the existence of an infinite computable sequence of stages

$$0 = s_0 < s_1 < s_2 < s_3 < \dots < s_k < s_{k+1} < \dots, \ k \in \omega,$$

such that for each $k \in \omega$ there exists a unique pair of numbers $e_k, x_k \in \omega, 0 \leq x_k \leq s_k$, such that

$$- f(e_k, x_k, s_k) \neq f(e_k, x_k, s_k - 1) \text{ and} - f(e_k, x_k, s_k), f(e_k, x_k, s_k - 1) \leq s_k.$$

We will construct $f : \omega^3 \to \omega$ immediately following the current proof; for now we assume that f exists. We will use f to satisfy the following requirement for each $e \in \omega$:

 R_e : There exists $x_e, t_e \in \omega$ such that for all $t \geq t_e, t \in \omega$, we have that

$$F(f_e(x_e)) \subset F(t) \in \mathcal{I}$$

It follows that σ_e cannot be included in any infinite antichain of T and so the image of $F \circ f_e : \omega \to T$ is not an infinite T-antichain.

We now construct $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ in stages $s \in \omega$ via

$$F = \bigcup_{s \in \omega} F_s, \ F_s : \{0, 1, 2, \dots, 2s\} \to T_s, \ s \in \omega,$$

as above. At superstage $k = 0^{16}$ and stage s = 0 we set

- $T_0 = \{\emptyset\} \subseteq 2^{<\omega},$
- $F_0(0) = \emptyset \in T_0$,
- $x_{0,0} = 0$, and finally we
- set $\varepsilon_0 = 0$, and reset all requirements $R_e, e \in \omega^+$.

At the next superstage k > 0, assume that we are given

- a number $\varepsilon = \varepsilon_{k-1} \in \omega$ such that the requirements that are currently reset (at the start of superstage k) are exactly those requirements R_e such that $e > \varepsilon$;
- a finite Σ_1^0 completely branching enumeration up to stage $s_{k-1} \in \omega$, $T_{s_{k-1}} = \bigcup_{t=1}^{s_{k-1}} T_t \subseteq 2^{<\omega}$;
- a finite sequence of bijections $F_t : \{0, 1, 2, \dots, 2t\} \to T_t, 0 \leq t \leq s_{k-1}$, such that $F_{t-1} \subset F_t$ for all $0 < t \leq s_{k-1}$; and
- a nonempty finite sequence of superstages

$$0 = k_0 < k_1 < \dots < k_{\varepsilon} \le k_{s-1}$$

and corresponding

$$0 = 2k_0 = x_0 = x_{0,k-1} < 2k_1 = x_1 = x_{1,k-1} < 2k_2 = x_2 = x_{2,k-1} < \dots < 2k_{\varepsilon} = x_{\varepsilon} = x_{\varepsilon,k-1} \le 2s_{k-1}$$

such that

$$\sigma_0 = \sigma_{0,k-1} \subset \sigma_1 = \sigma_{1,k-1} \subset \cdots \subset \sigma_{\varepsilon-1} = \sigma_{\varepsilon-1,k-1},$$

where

$$\sigma_e = F_{s_k-1}(f(e, x_e, s_{k-1})) \in T_{s_{k-1}} \subseteq 2^{<\omega}, \ 0 \le e < \varepsilon.$$

Let $e_k, z_k \in \omega, 0 \leq z_k \leq s_k$, be such that

$$f(e_k, z_k, s_k) \neq f(e_k, z_k, s_k - 1)$$
, and $0 \le f(e_k, z_k, s_k) \le s_k \le 2s_k$.

There are two cases to consider. The first case says that $z_k = x_{e_0}$ for some $0 \le e_0 \le \varepsilon$, and in this case (by our construction of $\sigma_{e_0} = \sigma_{e_0,k-1} \in T_{s_{k-1}}$ above) we have essentially¹⁷ witnessed a change in σ_{e_0} at the current superstage k. Let $\lambda = \lambda_k \in T_{s_{k-1}} \subseteq 2^{<\omega}$ be the leftmost $T_{s_{k-1}}$ -leaf extending σ_{e_0} . We proceed to the next superstage k+1 with the following definitions/constructions:

- $\varepsilon_k = e_0$, and reset all requirements R_e , $e > e_0$;
- $x_{e,k} = x_{e,k-1}, \ 0 \le e \le e_0;$

¹⁶Our main goal here is to produce an enumeration of an infinite Σ_1^0 completely branching tree $T = \bigcup_{s \in \omega} T_s$ in stages s, as in Definition 3.1 above. Our enumeration is via "standard" stages $s \in \omega$, and so it follows (essentially by our construction of $f : \omega^3 \to \omega$) that $k \in \omega$ must refer to "superstages" (i.e. an infinite subset of stages).

¹⁷We will eventually set $\sigma_{e_0,k} = F_{s_k}(f(e_0, x_{e_0}, s_k)) \neq F_{s_{k-1}}(f(e_0, x_{e_0}, s_k)) = \sigma_{e_0, x_{e_0}, s_{k-1}}$, and although $f(e_0, x_{e_0}, s_k) \leq s_k$ may not be in the domain of $F_{s_{k-1}}$, our construction will ensure that (by the end of the current superstage k) it is in the domain of F_{s_k} .

• $-T_{\ell} = T_{\ell-1} \cup \{\lambda 0^{\ell-s_{k-1}}, \lambda 0^{\ell-s_{k-1}-1}1\}, \text{ and}$ $-F_{\ell}(2\ell-1) = \lambda 0^{\ell-s_{k-1}}, F_{\ell}(2\ell) = \lambda 0^{\ell-s_{k-1}-1}1,$ for $\ell = s_{k-1} + 1, s_{k-1} + 2, \dots, s_k;$

It now follows that $f(e_0, x_{e_0}, s_k) \leq s_k$ is in the domain of the bijection $F_{s_k} : \{0, 1, 2, \dots, s_k\} \rightarrow T_{s_k}$. Finally, note that

$$\sigma_{e_0,s_k} = F_{s_k}(f(e_0, x_{e_0}, s_k)) \supset \lambda \supseteq F_{s_k}(f(e_0 - 1, x_{e_0 - 1}, s_k)) = \sigma_{e_0 - 1, s_k}$$

The second case of our construction says that $z_k \neq x_{e_0}$, for any $0 \leq e_0 \leq \varepsilon = \varepsilon_{k-1}$. In this case we do not need to act to satisfy any requirement R_e , $0 \leq e \leq \varepsilon$, and so our main objective now is to focus on the least currently unsatisfied requirement $R_{\varepsilon+1}$ and construct $x_{\varepsilon+1,s_k} \in \omega$ for its satisfaction. Let $\lambda = \lambda_k \in T_{s_{k-1}} \subseteq 2^{<\omega}$ denote the leftmost $T_{s_{k-1}}$ -leaf extending σ_{ε} , and then proceed as follows:

- (2a) set $\varepsilon_k = \varepsilon_{k-1} + 1 = \varepsilon + 1 \in \omega$ and $R_{\varepsilon+1}$ is no longer reset for now (but may be reset again at some later superstage k' > k);
- (2b) $x_{e,k} = x_{e,k-1}, 0 \le e \le \varepsilon;$
- (2c) Construct $T_{\ell} \supset T_{s_{k-1}}$ and $F_{\ell} \supset F_{s_{k-1}}$, exactly as in item (1c) above (with our current $\lambda \in T_{s_{k-1}}$), for all $\ell = s_{k-1} + 1, s_{k-1} + 2, \dots, s_k$;
- (2d) set $x_{\varepsilon+1,s_k} = 2s_k$.

By our construction of $T_{s_k} \subseteq 2^{<\omega}$ and $F_{s_k} : \{0, 1, 2, \dots, 2s_k = x_{\varepsilon+1, s_k}\} \to T_{s_k}$. It follows that $x_{\varepsilon+1, s_k}$ is in the domain of F_{s_k} and $F_{s_k}(x_{\varepsilon+1, s_k}) \in T_{s_k}$, and by our induction hypotheses and our definition of $\lambda \in T_{s_{k-1}}$ above it follows that

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{\varepsilon} \subseteq \lambda \subset \sigma_{\varepsilon+1},$$

where

$$\begin{aligned} \sigma_{\varepsilon+1} &= \sigma_{\varepsilon+1,s_k} = F_{s_k}(f(\varepsilon+1, x_{\varepsilon+1,s_k}, s_k)) = \\ &= F(f(\varepsilon+1, 2s_k, s_k)) = F(2s_k) = \lambda 0^{s_k - s_{k-1} - 1} 1 \in T_{s_k}. \end{aligned}$$

Proposition 4.12.

$$\lim_{k \to \infty} \varepsilon_k = \infty.$$

Proof. It is not difficult to verify that $\varepsilon_k \in \omega$ for every $k \in \omega$, and so to show that $\lim_k \varepsilon_k = \infty$ it suffices to show that $\lim_k \varepsilon_k$ is not finite. Suppose for a contradiction that $\lim_k \varepsilon_k = \varepsilon \in \omega$. Then there exists $k_0 \in \omega$ such that

- $\varepsilon_k \geq \varepsilon$ for all $k \geq k_0$, and
- there exist infinitely many $k \ge k_0$ such that $\varepsilon_k = \varepsilon$.

It follows that the requirement R_{ε} is never again reset on or after stage k_0 , since we only reset requirements of index strictly greater than ε_k at superstage k (and this only happens via Case 1-we do not reset any new requirements via Case 2). Now, by our construction above it follows that $x_{\varepsilon,k} = x_{\varepsilon,k_0} = x_{\varepsilon} \in \omega$ for all $k \ge k_0$.

Recall that in the second case of our construction above (at superstage $k \in \omega, k > 0$,) we set $\varepsilon_k = \varepsilon_{k-1} + 1$; therefore, the only way that $\varepsilon_k = \varepsilon$ when $k > k_0$ is if:

- $\varepsilon_{k-1} > \varepsilon$,
- we land in Case 1 at superstage k 1 of the construction, and
- $f(\varepsilon, x, s_{k-1}) \neq f(\varepsilon, x, s_k).$

It follows that there are infinitely many superstages $k \ge k_0$, $k \in \omega$ such that our uniform approximation to $f_{\varepsilon}(x)$ changes, contradicting our assumption that

$$|\{s: f(\varepsilon, x, s) \neq f(\varepsilon, x, s+1)\}| \le h(x).$$

Therefore, we must have that

$$\lim_{k \to \infty} \varepsilon_k = \infty.$$

The following corollary itemizes several immediate consequences of the previous lemma, all of which are easily verified (in order). The end result is that every requirement $R_e, e \in \omega$, is eventually satisfied.

Corollary 4.13. For each $e \in \omega$ there exists a superstage $k_e \in \omega$ such that

- $\varepsilon_{k_e} = e;$
- all requirements $R_{e'}$, e' > e, are reset at the end of stage s_{k_e} ; and
- for all $k > k_e$ we have that $\varepsilon_k > e$ and thus k_e is the last supporting at which we act (via our actions at stage s = 0 or else Case 1 above) to satisfy any $R_{e'}$, $e' \leq e$, $e' \in \omega$.

Consequently, it follows that:

- $x_e = x_{e,k} = x_{e,k_e}$, for all $e \in \omega$ and $k \ge k_e$;
- $\sigma_e = \sigma_{e,k} = F(f(e, x_e, s)), \text{ for all } e \in \omega \text{ and } s \geq s_{k_e};$
- at superstage $k \ge k_e$ we have that $\varepsilon_k > e$ and consequently

$$\sigma_0 = \sigma_{0,k} \subset \sigma_1 = \sigma_{1,k} \subset \cdots \subset \sigma_e = \sigma_{e,k} \subset \cdots \subset \sigma_{\varepsilon_k} = \sigma_{\varepsilon_k,k} \in T_{s_k}.$$

By our construction above (both Case 1 and Case 2), and Σ_1^0 -induction on $k \ge k_e$, it follows that $\tau \supset \sigma_e$ for every $\tau \in T_{s_k} \setminus T_{s_{k-1}}$, from which it follows that all but finitely many nodes in $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ extend σ_e . Thus, R_e is satisfied with $t_e = s_{k_e}$.

Let $A_T \subseteq T \subseteq 2^{<\omega}$ be an infinite *T*-antichain, and let

$$A = F^{-1}(A_T) = \{a \in \omega : F(a) \in A_T\} \leq_T A_T$$

be the pullback of A_T via (the total computable function) $F = \bigcup_{s \in \omega} F_s : \omega \to T$, and let

$$A = \{a_k : k \in \omega\}, \ a_k < a_{k+1}, \ k \in \omega,$$

be the enumeration of A in strictly increasing order. Since A_T is an infinite T-antichain, it follows that $\sigma_e = \lim_k \sigma_{e,k} \in T \setminus A_T$, from which it follows that $x_e = \lim_k x_{e,k} \notin A$, for all $e \in \omega$. Therefore, A_T computes a function, namely $g(k) = a_k \in \omega$, $k \in \omega$, that is not equal to any h-c.e. function $f_e : \omega \to \omega$, $e \in \omega$.

As we previously mentioned, the proof of Theorem 4.11 requires the following lemma.

Lemma 4.14. For any given order function h, there is a uniform computable approximation $f(e, x, s) : \omega^3 \to \omega$ to every h-c.e. strictly increasing function, $f_e, e \in \omega$, such that for any given $e \in \omega$,

$$f_e(x) = \lim_{s \to \infty} f(e, x, s)$$
, for all $x \in \omega$,

and satisfies items (a)-(c) above.

Proof. Recall that $\{\varphi_e : e \in \omega\}$ denotes a computable enumeration of all partial computable functions. Moreover, via various elementary effective modifications we can assume, without any loss of generality, that:

- (i) for each $e \in \omega$ we have that $\varphi_e = \varphi_e(x,t) = \varphi_e(\langle x,t \rangle)$, for some fixed computable bijection (i.e. pairing function) $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$,
- (ii) $\varphi_{e,s}(x,t)\downarrow$ (i.e. $\varphi_e(x,t)\downarrow$ in *s*-many stages) only if $e, x, t \leq s$,
- (iii) for any given $e, x, s \in \omega$

$$\varphi_{e,s}(x,t) < \varphi_{e,s}(x+1,t)$$
 whenever $\varphi_{e,s}(x,t) \downarrow, \varphi_{e,s}(x+1,t) \downarrow,$

(iv) $\varphi_{e,s}(x,t) \downarrow$ whenever $\varphi_{e,s}(x,t+1) \downarrow^{18}$, and

(v) for each $s \in \omega$ there is at most one triple $\langle e, x, t \rangle \in \omega^3$, t > 0, such that

 $\varphi_{e,s-1}(x,t)$ \uparrow but $\varphi_{e,s}(x,t)\downarrow$,

and moreover any such triple satisfies $e, x \leq t \leq s$.

The rest of the proof describes a somewhat less trivial effective modification that essentially turns our uniform effective enumeration $\{\varphi_e(x,t) : e \in \omega\}$ into f.

For each $e, x, s \in \omega$, let $t_{e,x,s} \in \omega$ be maximal such that

$$\varphi_{e,s}(x,t) \downarrow \ge x, \ 0 \le t \le t_{e,x,s}, \ t \in \omega,$$

and let $n_{e,x,s} \in \omega$ count the number of $0 < t \leq t_{e,x,s}$ for which we have that

$$\varphi_{e,s}(x,t) \neq \varphi_{e,s}(x,t-1).$$

Although $t_{e,x,s}$ may not always exist, we always start counting $n_{e,x,0} = 0$ (even though $t_{e,x,0}$ will not exist at stage s = 0); moreover, if $t_{e,x,s}$ does not exist then we necessarily have that $n_{e,x,s} = 0$. We should also note that if $t_{e,x,s} \in \omega$ exists, then so does $t_{e,x,s+1} \ge t_{e,x,s}$. Now, we define

$$f(e, x, s) = \begin{cases} x, \text{ if } t_{e,x,s} \text{ does not exist,} \\ \varphi_{e,s}(x, t_{e,x,s}), \text{ if } t_{e,x,s} \text{ exists and } n_{e,x,s} < h(x), \\ f(e, x, s - 1), \text{ if } n_{e,x,s} \ge h(x). \end{cases}$$

By our construction of f and

- by (ii) above it follows that f satisfies (a);
- by (iii) above it follows that f satisfies (c); and
- by (iv) and (v) above it follows that for any given $s \in \omega$, there is at most one pair $e, x \in \omega, e, x \leq s$, such that $t_{e,x,s} < t_{e,x,s+1} = t_{e,x,s} + 1$, and so f satisfies (b).

Theorem 4.11 is interesting in the context of the following lemma and ω -models.

Lemma 4.15 (Folklore; [LMP19] Lemma 4.7). Let P be a predicate of the form $(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X,Y)]$, where Φ is an arbitrary predicate and Ψ is Π_2^0 . If some PA degree does not compute a solution of P, then every solution of P has hyperimmune degree.

Following the reasoning of Nies and Shafer in [NS20, Section 7], via [NS20, Theorem 7.6] let $P \subseteq \omega$ be of superlow PA degree (see [Nie09, Chapter 1; Theorem 1.8.37] for more details on superlow Turing degrees and the Superlow Basis Theorem) such that every $Q \leq_T P$, $Q \subseteq \omega$, is h-c.e. for $h(n) = k^n$, and some $k = k_Q \in \omega$. Now, let $h_0(n) = n^n$, $n \in \omega$; then h_0 is an order function that grows faster than every exponential function of the form $h_k(n) = k^n$, $k \in \omega$. Let $T = T_{h_0} = \bigcup_{s \in \omega} T_{s,h_0} \subseteq 2^{<\omega}$ be the corresponding infinite Σ_1^0 completely branching tree that we construct in Theorem 4.11 above. Theorem 4.11 implies that P does not compute an infinite T-antichain, and moreover Lemma 4.15 implies that every infinite antichain $A_T \subseteq T$ is of hyperimmune Turing degree. This argument leads to the following two corollaries.

Corollary 4.16. TAC implies HYP in the context of ω -models.

Corollary 4.17. TAC does not follow from WKL₀, even in the context of ω -models.

 $^{^{18}}$ We can assume item (iv) without any loss of generality since in the end we are only really interested in the *total* computable functions, i.e. those partial computable functions that converge on all inputs.

Proof. A well-known consequence of the Hyperimmune-Free Basis Theorem [JS72] says that WKL_0 does not imply HYP, even in the context of ω -models. However, the previous corollary says that TAC implies HYP in the context of ω -models, contradicting the premise that WKL_0 implies TAC.

5. Rings with infinitely many minimal prime ideals

The main goal of this section is to prove that Noether's Primary Decomposition Lemma PDL follows from $RT_2^2 + WKL_0$; along the way we will establish some contextually useful elementary algebraic properties of Noetherian rings in RCA_0 . More specifically, these properties will be useful in any analysis of PDL including the next section where we show how RPDL follows from TAC + WKL₀ over RCA₀.

5.1. Maximal Annihilators are Minimal Primes in the Noetherian context.

Lemma 5.1. Let R be a computable ring satisfying computable-ACC, and such that

$$R = \{x_i : i \in \omega\}$$

is an effective enumeration of the elements of R. Now, suppose that $i_0 \in \omega$ is least such that $x = x_{i_0} \in R$ satisfies $x_{i_0}^2 = 0_R$ and define $I \subset R$ recursively via

• $I_0 = \{x_{i_0}\} \subset R$, • $I_{s+1} = \begin{cases} I_s \cup \{x_{x_0+s+1}\}, \text{ if } x_{x_0+s+1}^2 = 0_R \& x_{i_0+s+1} \in Ann(I_s) = \cap_{y \in I_s} Ann(y), \\ I_s, \text{ otherwise.} \end{cases}$ • $I = \bigcup_{s \in \omega} I_s.$

• $I = \bigcup_{s \in \omega} I_s$.

Then it follows that $I \subset R$ is an ideal of R such that $I^2 = \langle 0_R \rangle_R$.

Proof. By our construction of $I \subset R$ it is easy to verify that $1 \notin R$ and that $I^2 = \langle 0_R \rangle_R$. It remains for us to show that $I \subset R$ is indeed an ideal of R.

Suppose that $a, b \in I$ and let $s_0 \in \omega$ be least such that $a, b \in I_s \subseteq I$. Then it follows that

- $a + b \in Ann(I_{s_0})$ and
- $ra \in Ann(I_{s_0})$, for all $r \in R$,

from which it follows that $a + b, ra \in I$ for all $r \in R$. Hence $I \subset R$ is an ideal of R.

Note that, under the assumption of the existence of $i_0 \in \omega$, the construction of the previous lemma is uniform in the computable ring R. This means that, assuming that $i_0 \in \omega$ always exists in the following construction, we can uniformly and computably iterate the construction of $I \subset R$ in the previous lemma to obtain a uniformly computable sequence of rings and corresponding ideals

$$I_n \subset R_n, \ n \in \omega,$$

$$R_{n+1} = R_n / I_n, \ n \in \omega,$$

$$R_0 = R,$$

yielding an infinite strictly ascending chain of ideals

$$J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n \subset \cdots \subset R, \ n \in \omega,$$

such that $J_n \subset R$ corresponds to $I_n \subset R_n$ under the canonical quotient homomorphism $\varphi_n : R \to R_n$. Therefore, it follows that either R contains a uniformly computable strictly ascending chain of ideals

$$J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n \subset \cdots \subset R,$$

which would violate our assumption that R satisfies computable-ACC, or else there exists $n \in \omega$ such that the R-quotient ring R_n does not contain any nilpotent elements. Moreover

by (Π_1^0-) induction it is not difficult to see that each $J_n \subset R_n$ satisfies $J_n^{2^n} = \langle 0_R \rangle_R$ -i.e. the product of any 2^n -many elements in J_n is equal to zero, implying that $J_n \subset R$ is uniformly nilpotent. We have thus proven the following proposition and corollary.

Proposition 5.2. Let R be a computable ring satisfying computable-ACC. Then R contains a computable uniformly nilpotent ideal $J \subset R$ such that R/J contains no nilpotent elements.

Lemma 5.3. (RCA₀) Let R be a Noetherian ring. Then there is a uniformly nilpotent ideal $J \subset R$ such that R/J contains no nilpotent elements. In other words, $J \subset R$ is a maximal (uniformly) nilpotent ideal.

This leads us to the following important proposition.

Proposition 5.4. Let R be a computable ring satisfying computable-ACC, and containing no nilpotent elements. Then, for each $0_R \neq x \in R$, there exists $r \in R$ such that $0 \neq rx \in R$ and $Ann(rx) \subset R$ is a minimal prime ideal containing neither r nor x.

Proof. Let $0_R \neq x \in R$ be given, and suppose that $\{r_i : i \in \omega\} = R$ is an effective subenumeration of R in which each element of R appears infinitely often. Now, construct the uniformly computable sequence of elements $\{x_i : i \in \omega\} \subset R$ such that

•
$$x = x_0 \in R$$
,
• $x_{i+1} = \begin{cases} r_i \cdot_R x_i, \text{ if } r_i \cdot_R x_i \neq 0 \in R, \\ x_i, \text{ otherwise.} \end{cases}$

By construction it follows that

- (1) $0_R \neq x_i \in R$ and
- (2) x_{i+1} is an *R*-multiple of $x = x_i$, and therefore (by Σ_1^0 -induction) x_i is an *R*-multiple of $x = x_0$,

for all $i \in \omega$.

Now, consider the corresponding sequence of annihilator ideals $\{I_n : n \in \omega\}$ defined by

$$I_n = Ann(x_n) \subset R, \ n \in \omega.$$

It follows that

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \subset R$$

is a uniformly computable increasing chain of ideals in R since $x_{i+1} \in R$ is an R-multiple of $x_i \in R$ for all $i \in \omega$, and via computable-ACC this ascending chain of ideals eventually stabilizes-i.e. there is an $N \in \omega$ such that

$$Ann(x_N) = I_N = I_n = Ann(x_n) \subset R, \ n \ge N.$$

We claim that $Ann(x_N) = I_N = P \subset R$ is a prime ideal not containing x.

To see that P is prime, let $a, b \in R$ be such that $ab \in P = I_N = Ann(x_N) \subset R$. Then we have that $abx_N = 0 \in R$. Assume that $a \notin Ann(x_N)$ so that $ax_N \neq 0 \in R$; in this case we will show that $b \in Ann(x_N)$. Note that since $abx_N = 0 \in R$ we have that $b \in Ann(ax_N) \subset R$. Let $i_a \in \omega$, $i_a > N$, be such that $a = r_{i_a} \in R$, then by our definition of $N \in \omega$ and our construction of $x_{i_a} \in R$, $i_a \in \omega$, it follows that

$$Ann(x_N) \subseteq Ann(ax_N) \subseteq Ann(x_{i_a}) = Ann(x_N) \subset R$$

and so we have that $Ann(ax_N) = Ann(x_N)$. Therefore $b \in Ann(ax_N) = Ann(x_N) = I_N$ and hence $P = I_N \subset R$ is a prime ideal.

Finally, note that if $x_N = rx \neq 0 \in R$, $r \in R$, the, since R contains no nilpotent elements, we have that $(rx)^2 \neq 0 \in R$ and hence $rx \notin Ann(rx) = P \subset R$ hence $r, x \notin P$. Furthermore, note that since $x_N \cdot_R P = 0$ then it follows that any prime ideal $Q \subset R$ must either contain COMPUTABILITY AND COMBINATORIAL ASPECTS OF MINIMAL PRIMES IN NOETHERIAN RINGS5

 $x_N = rx$, or else it must contain all of P. It follows that P is a minimal prime ideal of R (characterized by the fact that it does not contain x_N).

Corollary 5.5 (RCA₀). Let R be a Noetherian ring with no nilpotent elements. Then every minimal prime ideal $P \subset R$ is the annihilator of some $x \in R \setminus P$.

One can use Corollary 5.5 to characterize the nonzero elements of a Noetherian ring with no nilpotent elements, as follows.

Corollary 5.6 (RCA₀). Let R be a Noetherian ring with no nilpotent elements, and let $x \in R$. Then $x = 0_R$ if and only if x is contained in every minimal prime ideal of R.

Proof. It is clear that if $x = 0_R$ then x belongs to every prime ideal. On the other hand, if $x \neq 0_R$ then by Corollary 5.5 it follows that there exists $a \in R$ such that $0 \neq_R a \cdot x = y$ and P = Ann(y) is a minimal prime ideal not containing x.

The following relevant converse of Proposition 5.4 above also holds.

Proposition 5.7. Let R be a computable ring satisfying computable-ACC and with no nilpotent elements. Let $P \subset R$ be a minimal prime ideal, and $a \in R \setminus P$. Then there exists $x \in R \setminus P$ such that Ann(ax) = P.

Proof. Let

$$\{a = a_0, a_1, a_2, a_3, \cdots, a_N, \cdots\} = R \setminus P, \ N \in \omega$$

be an enumeration of $R \setminus P$, and for each $N \in \omega$ define

$$x_N = \prod_{i=0}^N a_i.$$

By construction it follows that

$$Ann(x_0) \subseteq Ann(x_1) \subseteq Ann(x_2) \subseteq \cdots \subseteq Ann(x_N) \subseteq \cdots, N \in \omega,$$

and since R satisfies computable-ACC there exists $N_0 \in \omega$ such that for all $N \geq N_0$ we have $Ann(x_N) = Ann(x_{N_0})$. To prove the current lemma it suffices to show that $P = Ann(x_{N_0})$.

It is easy to see that $Ann(x_{N_0}) \subseteq P$, since $x_N \notin P$ and P is prime. To see that $P \subseteq Ann(x_{N_0})$, let $c \in P$ and suppose for a contradiction that $cx_{N_0} \neq 0 \in R$. Then, by Proposition 5.4 above there exists $r_c \in R$ such that $r_c cx_{N_0} \neq 0 \in R$ and $Q = Ann(r_c cx_{N_0}) \subset R$ is a minimal prime ideal with $r_c, c, x_{N_0} \notin Q$. Note that $c \in P \setminus Q$ and thus $Q \neq P$, and since Q and P are minimal prime ideals of R, there exists $q \in Q \setminus P \subset R \setminus P$ such that

$$0_R = q \cdot r_c c x_{N_0} = (c r_c) \cdot q x_{N_0}.$$

Now, by definition of N_0 it follows that

$$0_R = (cr_c) \cdot x_{N_0} = r_c c x_{N_0}$$

a contradiction.

Corollary 5.8 (RCA₀). Let R be a Noetherian ring with no nilpotent elements. Let $P \subset R$ be a minimal prime ideal of R and let $a \in R \setminus P$. Then there exists $x \in R \setminus P$ such that P = Ann(ax).

Remark 5.9. (RCA₀) Let R be a ring, $P \subset R$ a minimal prime ideal, and $x \in R \setminus P$ such that P = Ann(x). Then it follows that $x \in Q$ for any other R-minimal prime ideal $Q \neq P$.

5.2. Prime Avoidance. We require one more well-known and important algebraic lemma before can give a proof of PDL via $WKL_0 + RT_2^2$.

Lemma 5.10 (Prime Avoidance Lemma, RCA_0). Let R be a ring, let $I \subseteq R$ be an ideal, and let $P_1, P_2, \ldots, P_N \subset R$, $N \in \omega$ be prime ideals such that $I \nsubseteq P_i$, $i = 1, 2, \ldots, N \in \omega$. Then there exists $0 \neq x \in R$ such that

$$x \in I \setminus \bigcup_{i=1}^{N} P_i \subseteq R$$

Proof. We employ Σ_1^0 -induction on $N \in \mathbb{N}$ and set parameters $I, P_1, P_2, \ldots, P_N \subset R$. The base case N = 1 is trivial given our assumptions. Now, for the induction step, for each $i = 1, 2, \ldots, N$ let

$$z_i \in I \setminus \bigcup_{\substack{j=1\\j \neq i}}^N P_j.$$

If we have that $z_i \notin P_i$ for some $1 \leq i \leq N$ then we are done, so we can assume that $z_i \in P_i$ for all $1 \leq i \leq N$. Now, set

$$z = z_1 z_2 \cdots z_{N-1} + z_N;$$

it follows that $z \in I \setminus \bigcup_{i=1}^{N} P_i$ as required.

5.3. **Proving PDL in Second-Order Arithmetic.** We are finally ready to prove PDL in Second-Order Arithmetic.

Theorem 5.11. Let R be a computable ring with infinitely many computable minimal prime ideals that satisfies computable-ACC. Then there is a computable 2-coloring $c : \omega \times \omega \rightarrow \{0, 1\}$ such that any infinite homogeneous set $H \subseteq \omega$ (see [DH09, Definition 1.3] for more details) for color $c_0 \in \{0, 1\}$ computes either:

- (a) an infinite strictly ascending chain of (annihilator) ideals if $c_0 = 0$, or else
- (b) an infinite sequence $x_0, x_1, \ldots \in R$ such that for each $i \in \omega$ we have that

$$x_{i+1} \notin \langle x_0, x_1, \dots, x_i \rangle_R,$$

if $c_0 = 1$.

Proof. Let R be a ring with infinitely many minimal prime ideals $P_0, P_1, P_2, \ldots \subset R$ and enumeration $R = \{r_n : n \in \omega\}$ such that for every $n_0 \in \omega$ there are infinitely many $n \in \omega$ such that $r_{n_0} = r_n$. By passing to the quotient ring R/J, where $J \subset R$ is as in Proposition 5.2 above, we may assume that R has no nilpotent elements in which case Proposition 5.7 above says that for each $n \in \omega$ there exists $p_n \in R \setminus P$ such that $P_n = Ann(p_n)$. Via our construction in the proof of Proposition 5.4 above, we can uniformly and computably construct a subenumeration of the R-minimal prime ideals $\{A_i : i \in \omega\}$ such that

- $a_{i,0} = r_i$, and
- for all $k > i, k \in \omega$, either

 $-r_k \cdot_R a_{i,k-1} \neq 0 \in R$, in which case we set $a_{i,k+1} = r_k \cdot_R a_{i,k-1}$, or else

 $-r_k \cdot_R a_{i,k-1} = 0 \in R$ and we have that $r_k \in A$.

As in the proof of Proposition 5.4 above, since R satisfies computable-ACC it follows that for each $i \in \omega$ there exists $n_i \in \omega$, $n_i \ge i$, such that

$$A = Ann(a_{i,n_i}) = Ann(a_{i,k})$$
 for any $k > n_i, k \in \omega$

is a minimal prime ideal of R. Moreover, for each $n \in \omega$ there exists $i \in \omega$ such that $r_i = p_n$ and since we have that $P_n = Ann(p_n)$ by our construction of A_i it follows that $A_i = P_n$.

Thus we can assume that we have a uniformly computable subenumeration of R-minimal prime ideals $\{P_n : n \in \omega\}$, which we can effectively refine to an infinite sequence of distinct minimal primes $\{Q_n : n \in \omega\} \subseteq \{P_n : n \in \omega\}$. Now, by Lemma 5.10, there exists an infinite computable sequence of elements x_0, x_1, x_2, \ldots such that for each $n \in \omega$ we have that

$$x_n \in Q_n \setminus \left(\bigcup_{i=0}^{n-1} Q_i\right).$$

For each $n_1 \leq n_2, n_1, n_2 \in \omega$, let

$$c(n_1, n_2) = \begin{cases} 0, \ x_{n_1} \notin Q_{n_2}, \\ 1, \ x_{n_1} \in Q_{n_2}. \end{cases}$$

and let H be an infinite homogeneous set for c with uniform color $c_0 \in \{0, 1\}$. Let $H = \{n_k : k \in \omega\}$ be a strictly increasing enumeration of H.

If $c_0 = 0$ then for any $i, j \in \omega$, i < j, we have that $x_{n_i} \notin Q_{n_j}$. Now, for each $N \in \omega$, if we set

$$a_N = \prod_{i=0}^N x_{n_i} \in R$$
 and $A_N = Ann(a_N) \subset R$,

then by Lemma 5.6 above and the fact that $x_{n_N} \in Q_N$ (by construction) but $x_{n_N} \notin A_{N+1}$ (which follows from the fact that $c_0 = 0$) we have that $q_{N+1} \in A_{N+1} \setminus A_N$, where $q_N \in R$ is such that $Q_N = Ann(q_N)^{19}$ and so

$$A_0 \subset A_1 \subset \cdots \subset A_N \subset \cdots \subset R$$

is a strictly increasing chain of ideals, uniformly computable in H.

On the other hand, if $c_0 = 1$ we have that $x_{n_1} \in Q_{n_2}$ for any given $n_1, n_2 \in \omega, n_1 < n_2$. In this case, for all $N \in \omega$ we have that

$$\langle x_{n_0}, x_{n_1}, \dots, x_{n_{N+1}} \rangle_R \subseteq \bigcap_{k=N+1}^{\infty} Q_{n_k} \text{ but } \langle x_{n_0}, x_{n_1}, \dots, x_{n_{N+1}} \rangle_R \nsubseteq \bigcap_{k=N}^{\infty} Q_{n_k}$$

since $x_{n_{N+1}} \notin Q_{n_N}$ by construction. It follows that

$$x_{n_{N+1}} \notin \langle x_{n_0}, x_{n_1}, \dots, x_{n_N} \rangle$$

for each $N \in \omega$.

Corollary 5.12. (RCA_0) WKL₀ + RT₂² *implies* PDL.

Proof. Let R be a commutative ring such that for each $N \in \mathbb{N}$ R contains N minimal prime ideals. In the previous theorem we described

- why we can assume, without any loss of generality, that R has no nilpotent elements, and also
- how we can use our hypotheses to effectively construct an infinite sequence of mutually distinct minimal prime ideals $\{Q_n : \in \mathbb{N}\}$ in R.

If $c: \omega \times \omega \to 2$ is the effective coloring that we described in the previous theorem, then via $\mathsf{RT}_2^2 c$ has an infinite homogeneous set $H \subseteq \mathbb{N}$ with uniform color $c_0 \in \{0, 1\}$. If $c_0 = 0$ then we described how to effectively produce an infinite strictly ascending chain of (annihilator) ideals in R via H. On the other hand, if c = 1 then we constructed an infinite sequence of elements $x_0, x_1, x_2, \ldots \in R$ such that for each $N \in \mathbb{N}$, x_{N+1} is not an R-linear combination

¹⁹Note that $q_N \in R$ exists via Lemma 5.7 above.

of x_0, x_1, \ldots, x_N . In this case, via WKL_0 , it follows that there exists a strictly ascending chain of ideals

$$I_0 \subset I_1 \subset \cdots \subset I_N \subset \cdots \subset R$$

such that for each $N \in \mathbb{N}$ we have that $x_{N+1} \in I_{N+1} \setminus I_N$ (see [Con19, Lemma 4.2, Corollary 4.3] and the discussion in between for more details and references).

6. UPPER BOUNDS FOR RPDL AND URPDL

We now leave the general Primary Decomposition Lemma (PDL) behind and focus on its restrictions RPDL and URPDL in relation to the Tree Antichain Theorem TAC, over $RCA_0 + B\Sigma_2$. Our overall goal in this context is to provide proofs of RPDL and URPDL in Second-Order Arithmetic that differ from our proof of PDL above (i.e. Corollary 5.12) by showing that:

- RPDL follows from $TAC + WKL_0$, and
- URPDL follows from TAC.

Definition 6.1. Let R be a computable ring with infinitely many minimal prime ideals satisfying computable-ACC. In the proof of Theorem 5.11 above we showed how to uniformly and effectively enumerate the minimal prime ideals of R. Fix such an enumeration $\mathcal{O} = \{P_k : k \in \omega\}$, and for each $0_R \neq x$ define

$$\mathcal{P}_x = \mathcal{P}_{x,\mathcal{O}} = \{k \in \omega : x \in P_k\}$$

to be the prime type of x (relative to \mathcal{O}).

Remark 6.2. Let R be a computable ring with infinitely many prime ideals that satisfies computable-ACC. Going forward we will always aim to construct an infinite strictly ascending chain of ideals in R. Therefore we can assume, by taking the quotient of R by the computable $(\Delta_1^0 - definable)$ ideal of nilpotent elements in Proposition 5.2 and Corollary 5.3, that R does not contain any nilpotent elements. Furthermore, in the proof of Theorem 5.11 above we showed how to construct a uniformly computable enumeration of all the minimal prime ideals of R,

$$\mathcal{O} = \mathcal{O}_R = \{ P_k : k \in \omega \}.$$

Going forward we will always implicitly assume that such a fixed enumeration $\mathcal{O} = \mathcal{O}_R$ is given along with any such R.

Remark 6.3. Corollary 5.6 says that if R in the previous definition has no nilpotent elements, then $x = 0_R$ if and only if the prime type of x is ω . Moreover, in light of Corollary 5.6 above, for any $x \in R$ we have that

$$Ann(x) = \{ y \in R : \mathcal{P}_x \cup \mathcal{P}_y = \omega \} = \bigcap_{k \notin \mathcal{P}_x} P_k \subset R.$$

Also, if $x, y \in R$ are such that $ax + by = 1_R$ for some $a, b \in R$. Then \mathcal{P}_x and \mathcal{P}_y must be disjoint sets of natural numbers. More generally, if $x_0, x_1, \ldots, x_N \in R$ are such that

$$\sum_{i=0}^{N} a_i x_i = 1_R$$

for some $a_0, a_1, \ldots, a_N \in R$, then we have that

$$\bigcap_{i=0}^{N} \mathcal{P}_{x_i} = \emptyset \subset \omega.$$

The following lemma will be very useful in constructing infinite strictly ascending chains of annihilator ideals in computable rings with infinitely many minimal primes.

Lemma 6.4. Let R be a computable ring with infinitely many minimal primes satisfying computable-ACC, and suppose that there is an incomputable sequence of elements $X = \{x_k : \in \omega\} \subseteq R$ such that for each $k \in \omega$,

$$\emptyset \neq \mathcal{P}_{x_{k+1}} \setminus \bigcup_{\ell=0}^{k} \mathcal{P}_{x_{\ell}} = \mathcal{P}_{x_{k+1}} \setminus \mathcal{P}_{X_{k}} \subseteq \omega,$$

where $X_k = \prod_{\ell=0}^k x_\ell$. Then we have that

 $Ann(X_0) \subset Ann(X_1) \subset \cdots \subset Ann(X_N) \subset \cdots \subset R$

is an X-uniformly computable infinite strictly ascending chain of R-annihilator ideals, and thus R does not satisfy X-computable-ACC.

Proof. Let $\mathcal{P}_{X_k} = \bigcup_{\ell=0}^k \mathcal{P}_{x_\ell} \subseteq \omega$, for each $k \in \omega$. Then by our hypothesis we have that

 $\mathcal{P}_{X_0} \subset \mathcal{P}_{X_1} \subset \mathcal{P}_{X_2} \subset \cdots \subset \mathcal{P}_{X_N} \subset \cdots \subset \omega,$

from which it follows that

$$\bigcap_{\ell \notin \mathcal{P}_{X_0}} P_{\ell} \subset \bigcap_{\ell \notin \mathcal{P}_{X_1}} P_{\ell} \subset \bigcap_{\ell \notin \mathcal{P}_{X_2}} P_{\ell} \subset \cdots \subset \bigcap_{\ell \notin \mathcal{P}_{X_N}} P_{\ell} \subset \cdots \subset R.$$

The conclusion now follows from Remark 6.3 above.

Theorem 6.5. Let R be a computable ring with infinitely many computable minimal prime ideals and satisfies computable-ACC. For each $n \in \omega$ define the Σ_1^0 set

$$\mathfrak{C}_n = \mathfrak{C}_{R,n} = \{ x \in R : |\mathcal{P}_x| \ge n \} \subset R$$

and suppose that there exists a partial computable function $f: R \to R \times R$ such that for all $x \in \mathfrak{C}_2$ we have that $f(x) \downarrow = \langle y, z \rangle$ for some $y, z \in R$ such that $\mathcal{P}_y, \mathcal{P}_z$ form a nontrivial²⁰ partition of \mathcal{P}_x . Then there exists a Σ_1^0 -definable completely branching tree $T_R \subseteq 2^{<\omega}$ such that any infinite T_R -antichain computes an infinite strictly ascending chain of R-ideals.

Proof. We enumerate an infinite Σ_1^0 completely branching tree $T = T_R = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ and corresponding uniformly computable functions $F_s : T_s \to R$, $F_{s+1} \supset F_s$, computable $F = \bigcup_{s \in \omega} F_s$, in stages $s \in \omega$. Moreover, for each $s \in \omega$ we will have that:

- (a) $\prod_{\lambda \in \Lambda_s} F(\lambda) = 0_R$, and
- (b) for any $\alpha \in T_s$ and any prefix-free set $A \subset T_s$ such that $[A] = [\alpha] \subseteq 2^{\omega}$, we have that $\{\mathcal{P}_{F(\beta)} : \beta \in A\}$ is a nontrivial²¹ partition of $\mathcal{P}_{F(\alpha)}$,

where Λ_s denotes the (finite) set of leaves of T_s . Now, if $A = \{\alpha_k : k \in \omega\} \subseteq T$ is an infinite T-antichain, then by (a) and (b) above it follows that the sequence $\{F(\alpha_k) : k \in \omega\} \subseteq R$ satisfies the hypotheses of Lemma 6.4, which yields an infinite strictly ascending chain of R-ideals. Therefore, to prove the current theorem it suffices to construct the aforementioned T.

At stage s = 0, define $T_0 = \{\emptyset\}$ and $F_0(\emptyset) = 0_R$. Note that, by our construction of T_0, F_0 and Σ_1^0 -induction, property (a) will follow from property (b) via Remark 6.3 above. Furthermore, via Σ_1^0 -induction, to verify property (b) it suffices to assume that $A = \{\alpha 0, \alpha 1\}$ is the set of successor nodes (children) of α .

At stage s + 1 > 0, $s \in \omega$, assume that we are given a finite completely branching tree $T_s \subset 2^{<\omega}$ with (s + 1)-many leaves. Via Σ_1^0 -induction we can assume that property (b)

$$^{20}\mathcal{P}_{y}, \mathcal{P}_{z} \neq \emptyset.$$

²¹Every $\mathcal{P}_{F(\beta)} \subset \omega$ is nonempty.

holds for T_s . Let Λ denote the leaves of T_s . By the Finite Pigeonhole Principle there is a $\lambda_0 \in \Lambda$ such that $F(\lambda_0) \in \mathfrak{C}_2$. Now, suppose that $f(F(\lambda)) = \langle y, z \rangle, y, z \in R$, and define

- $T_{s+1} = T_s \cup \{\lambda 0, \lambda 1\}$, and
- $F_{s+1}(\lambda 0) = y, F_{s+1}(\lambda 1) = z.$

The verification of property (b) for T_{s+1} follows from our hypotheses on f and Σ_1^0 -induction.

Corollary 6.6 (RCA₀). TAC *implies* URPDL.

Proof. Via Remark 6.3 above it follows that any ring with infinitely many uniformly coprime minimal prime ideals satisfies the hypotheses of Theorem 6.5, which is valid in RCA_0 . \Box

Our main goal for the remainder of the section is to show that RPDL follows from $\mathsf{WKL}_0 + \mathsf{TAC}$.

Lemma 6.7. Let R be a computable ring with infinitely many computable minimal prime ideals, each of which is PA-maximal. Assume that $x, y \in R$ are such that $\mathcal{P}_x \cap \mathcal{P}_y = \emptyset$. Then there exist $a, b \in R$ such that $ax + by = 1_R$.

Proof. The argument is similar to the one we outlined in Remark 2.11 above. Assume for a contradiction that no such $a, b \in R$ exist. Then, by [Sim09, Theorem IV.6.4], there is an infinite computable tree $T = T_{x,y} \subseteq 2^{<\omega}$ such that every infinite path through T codes a prime ideal $P \subset R$ containing both x and y. By our minimality and maximality assumptions on the primes of R it follows that

$$x, y \in P \subset R$$

for some minimal prime ideal P, contradicting our assumption that $\mathcal{P}_x \cap \mathcal{P}_y = \emptyset$.

Lemma 6.8. Let R be a computable ring with infinitely many computable minimal prime ideals, each of which is PA-maximal. Then for each $k \in \omega$ there exists $x_k \in R$ such that

$$\mathcal{P}_{x_k} = \{k\}$$

Proof. By Proposition 5.7 above, for each $k \in \omega$ there exists $p_k \in R \setminus P_k$ such that $P_k = Ann(p_k) \subset R$. Now, Remark 2.11 says that

$$x_k + ap_k = 1_R,$$

for some $x_k \in P_k$ and $a \in R$. Furthermore, by definition of p_k and Remark 6.3 above it follows that $\mathcal{P}_{p_k} = \omega \setminus \{k\}$, from which it follows that $\mathcal{P}_{x_k} = \{k\}$ as the lemma claims. \Box

Lemma 6.9. Let R be a computable ring with infinitely many computable prime ideals, each of which is PA-maximal. Then, either

- R does not satisfy computable-ACC, or else
- for each $0_R \neq x \in \bigcup_{k \in \omega} P_k$ we can uniformly and effectively find $0_R \neq y \in \bigcup_{k \in \omega} P_k$ such that $\mathcal{P}_x, \mathcal{P}_y$ form a nontrivial partition of ω .

Proof. Let $x = x_0$. Since $0_R \neq x_0$, by Remark 6.3 above and the previous two lemmas it follows that there exists (and we can effectively find) some $a_0 \in R$ such that $1_R = \langle x_0, a_0 \rangle_R$. Now, if $x_0 a_0 = 0_R$ we can take $y = a_0$; otherwise we have $0_R \neq x_1 = x_0 a_0$ and repeat the argument to (uniformly and effectively) construct $a_1 \in R$ such that $\langle x_1, a_1 \rangle_R = 1_R$ and it follows that $\mathcal{P}_{a_0} \cap \mathcal{P}_{a_1} = \emptyset$. If $0_R = x_1 a_1 = x_0 a_0 a_1$, set $y = a_0 a_1$; otherwise keep repeating the argument...

In the end either we end up with a valid $y \in R$, or else we end up generating a uniformly computable infinite sequence of elements $\{a_k : k \in \omega\}$ such that $\{\mathcal{P}_{a_k} : k \in \omega\}$ are mutually disjoint sets, and it follows that

$$Ann(b_0) \subset Ann(b_1) \subset Ann(b_2) \subset \cdots \subset Ann(b_N) \subset \cdots \subset R, \ N \in \omega,$$

 \square

is a uniformly computable infinite strictly ascending chain of R-ideals.

Theorem 6.10. Let R be a computable ring satisfying computable-ACC and containing infinitely many computable minimal prime ideals, each of which is PA-maximal. Then there is an infinite Σ_1^0 completely branching tree $T = T_R \subseteq 2^{<\omega}$ such that every infinite T-antichain computes an infinite strictly ascending chain of R-annihilator ideals.

Proof. By Theorem 6.5 above it suffices to construct a partial computable function $f: R \to R \times R$, such that for all $x \in \mathfrak{C}_2 = \{x \in R : |\mathcal{P}_x| \geq 2\}$, $f(x) = \langle y, z \rangle$, $y, z \in R$, are such that $\mathcal{P}_y, \mathcal{P}_z$ form a nontrivial partition of \mathcal{P}_x . If $x = 0_R$ then any $y, z \in R$ such that $yz = 0_R = x$ and $1_R \in \langle y, z \rangle_R$ will suffice. On the other hand, if $x \neq 0_R$, $x \in \mathfrak{C}_2$, then by Remark 6.3 and Lemmas 6.9, 6.7 above there exists $a \in R$ such that $xa = 0_R$ and $1_R \in \langle x, a \rangle_R$. Furthermore, by our hypothesis on x and Lemma 6.8 above it follows that there exists (and we can uniformly and effectively locate some) $y \in R$ such that

$$1_R \in \langle ax, y \rangle_R$$
 and $axy \neq 0_R$.

Furthermore, via Lemma 6.9 implies that there exists (and we can uniformly and effectively locate some) $z \in R$ such that

$$axyz = 0_R$$
 and $1_R \in \langle axy, z \rangle_R$.

It follows that $\mathcal{P}_y, \mathcal{P}_z$ form a nontrivial partition of \mathcal{P}_x . Finally, upon (uniformly and effectively) setting $f(x) = \langle y, z \rangle \in \mathbb{R} \times \mathbb{R}$ for all $x \in \mathfrak{C}_2$, Theorem 6.5 says that the desired infinite Σ_1^0 completely branching tree $T = T_R \subseteq 2^{<\omega}$ exists.

Corollary 6.11 (WKL₀). TAC *implies* RPDL.

7. A lower bound for URPDL over $RCA_0 + B\Sigma_2$

Theorem 7.1. Let $T = \bigcup_{s \in \omega} T_s \subseteq 2^{<\omega}$ be an infinite completely branching tree. Then there is a computable ring $R = R_T$ with infinitely many computable minimal prime ideals and such that every infinite strictly ascending chain of R-ideals computes an infinite T-antichain.

Proof. Let $R_0 = \mathbb{Q}[X_t : t \in \omega]$, and define a computable function $h : \omega \times \{0, 1\} \to T \subseteq 2^{<\omega}$ $h(t, 0) = \sigma_t 0 \in 2^{<\omega}$ and $h(t, 1) = \sigma_t 1 \in 2^{<\omega}$, $t \in \omega$, where $\sigma_t \in T_t \subset 2^{<\omega}$ is the unique T_s -leaf such that $\{\sigma_t 0, \sigma_t 1\} = T_{t+1} \setminus T_t$. Then we have a computable isomorphism

 $H: R_0 \to \mathbb{Q}[X_\sigma : \sigma \in T]$

generated by the computable reindexing of R_0 -indices $t \in \omega$ given by:

$$H(X_t) = \begin{cases} X_{h(t/2,0)} = X_{\sigma_t 0}, \ t \text{ even}, \\ X_{h((t-1)/2,1)} = X_{\sigma_t 1}, \ t \text{ odd}. \end{cases}$$

Let $R_{T,0} = \mathbb{Q}[X_{\sigma} : \sigma \in T] \cong R_0.$

Definition 7.2. Let $X \in R_0$ be a monomial. We have

$$H(X) = \prod_{\sigma \in T_t} X_{\sigma}^{\alpha_{\sigma}} \in R_{T,0}, \ \alpha_{\sigma}, t \in \omega,$$

where t is least such that $\{\sigma \in T : \alpha_{\sigma} > 0\} \subseteq T_t$. Let

$$\ell = \max\{|\sigma| \in \omega : \sigma \in T_t\},\$$

and for each $\tau \in 2^{<\omega}$ such that $|\tau| \ge \ell$ let

$$c(\tau) = c_X(\tau) = \sum_{\substack{\sigma \in T_t \\ \sigma \subseteq \tau}} \alpha_{\sigma} \in \omega$$

and note that $c(\tau) \in \omega$ is uniformly computable in the monomial $X \in R_0 \cong R_{T,0}$ and node $\tau \in 2^{<\omega}, |\tau| \ge \ell$. Now, for any given $k \ge \ell$ we define

$$C(k) = C_X(k) = \langle c(\tau_1), c(\tau_2), \cdots, c(\tau_{2^k}) \rangle \in \omega^{2^k},$$

where $\tau_1, \tau_2, \ldots, \tau_{2^k} \in 2^{=k}$ is the unique lexicographic ordering of $2^{=k} \subset 2^{<\omega}$, to be the multiplicity-covering code (MC-code) corresponding to $X \in R_0 \cong R_{T,0}$ of length $k \in \omega$.

We think of each monomial $X \in R_0 \cong R_{T,0}$ as corresponding to an open covering of 2^{ω} given by the indices of the indeterminates $\sigma \in T$ of $R_{T,0}$ that appear as factors in $X \in R_{T,0}$, and thus a given $\tau \in 2^{<\omega}$ can be covered by more than one index $\sigma \subseteq \tau$, or it can be covered by the same index "multiple times" if $\alpha_{\sigma} \geq 2$, $\sigma \subseteq \tau$.

Remark 7.3. For all $k > \ell$, $k, \ell \in \omega$, in Definition 7.2 above, we have that $C(k) \in \omega^{2^k}$ is determined by $C(\ell) \in \omega^{2^\ell}$.

Definition 7.4. Let $M_0, M_1 \in R_0 \cong R_{T,0}$ be \mathbb{Q} -monomials such that $\Sigma_0, \Sigma_1 \subset T \subset 2^{<\omega}$ are the finite sets of $R_{T,0}$ -indeterminate indices appearing in M_0, M_1 , respectively, and let

$$\ell_0 = \max\{|\sigma| : \sigma \in \Sigma_0\} \in \omega, \ \ell_1 = \max\{|\sigma| : \sigma \in \Sigma_1\} \in \omega.$$

Without any loss of generality assume that $\ell_0 \leq \ell_1$. In this case we say that M_0 and M_1 are *MC*-equivalent whenever $C_{M_0}(\ell_1) = C_{M_1}(\ell_1) \in \omega^{2^{\ell_1}}$.

It is not difficult to verify that MC-Equivalence is an equivalence relation.

Definition 7.5. For any given (\mathbb{Q}) -monomial $M \in R_0 \cong R_{T,0}$, let

 $\Sigma_M = \{ \sigma \in T : X_\sigma \text{ appears in } M \} \subset T \subseteq 2^{<\omega}, \ \ell_M = \max\{ |\sigma| : \sigma \in \Sigma \} \in \omega.$

For each $\tau = \tau_i \in 2^{=\ell_M}$, $1 \leq i \leq 2^{\ell_M}$, as in Definition 7.2 above, let

$$C_M = C_M(\ell_M) \in \omega^{2^{\epsilon_M}}, \quad C_M(i) = c(\tau_i) \in \omega.$$

Remark 7.6. For any two monomials $M_0, M_1 \in R_0 \cong R_{T,0}$ with product monomial $M = M_0 M_1 \in R_0 \cong R_{T,0}$ we have that $\ell_M = \max\{\ell_{M_0}, \ell_{M_1}\} \in \omega$ and

$$C_M = C_{M_0}(\ell_M) + C_{M_1}(\ell_M) \in \omega^{2^{\ell_M}}$$

where the sum above denotes coordinate-wise (i.e. vector) addition in $\omega^{2^{\ell_M}}$.

Definition 7.7. Recall that two finite binary strings are <u>comparable</u> whenever one is a prefix of the other.

Let $X, Y \subseteq 2^{<\omega}$. We say that Y covers X whenever every $\sigma \in X$ is comparable to some $\tau \in Y$.

Finally, note that if $T \subseteq 2^{\omega}$ is a completely branching tree and $X \subseteq 2^{<\omega}$, then X covers T if and only if X covers $2^{<\omega}$.

Let $\mathcal{M} \subset R_0 \cong R_{T,0}$ be the set of monomials of R_0 , and let

$$\mathcal{M}_{2^{<\omega}} = \{ M \in \mathcal{M} : (\forall 0 \le i \le 2^{\ell_M}) [C_M(i) > 0] \}.$$

In other words, $\mathcal{M}_{2^{<\omega}}$ denotes the set of $R_{T,0}$ -monomials $M \in \mathcal{M}$ such that the $(2^{<\omega})$ -indices appearing in M cover $2^{<\omega}$. Now, if

$$I_{0,0} = \langle \mathcal{M}_{2^{<\omega}} \rangle_{\mathbb{Q}} \subset R_0 \cong R_{T,0}$$

denotes the \mathbb{Q} -span of $\mathcal{M}_{2\leq\omega}$ in $R_0 \cong R_{T,0}$ then it easily follows that $I_{0,0}$ is a computable subset of $R_0 \cong R_{T,0}$. Moreover, by Remark 7.6 above, it follows that $I_{0,0}$ is actually a computable *ideal* of $R_0 \cong R_{T,0}$.

Now, for any given $\ell \in \omega$ and $C \in \omega^{2^{\ell}}$, let

$$\mathcal{M}_C = \mathcal{M}_{C,\ell} = \{ M \in \mathcal{M} : C_M = C \text{ and } \ell_M \leq \ell \} \subset \mathcal{M} \subset R_0 \cong R_{T,0}$$

be the set of R_0 -monomials with the fixed MC-code C, and define

$$\mathcal{H}_C = \left\{ \sum_{\substack{M \in F \subset \mathcal{M}_C, \\ F \text{ finite}}} q_M M : q_M \in \mathbb{Q}, \ \sum_{M \in F} q_M = 0 \right\} \subset R_0 \cong R_{T,0}$$

to be the set of "homogeneous" polynomials in R_0 all of whose monomial summands have MC-code C and whose rational coefficients sum to zero. It follows that $\mathcal{H}_C \subset R_0 \cong R_{T,0}$ is uniformly computable in $C \in \omega^{2^\ell}$. It is not difficult to check that for any given $C \in N^{2^\ell}$ we have that the set $\mathcal{H}_C \subset R_0 \cong R_{T,0}$ is closed under addition and scalar (i.e. \mathbb{Q} -)multiplication. Furthermore, via Remark 7.6 above it follows that the larger set

$$\mathcal{H} = \bigcup_{\substack{C \in \omega^{2^{\ell}}, \\ \ell \in \omega}} \mathcal{H}_C \subset R_0 \cong R_{T,0}$$

is also closed under multiplication by $\mathcal{M} \subset R_0 \cong R_{T,0}$, from which it follows that the \mathbb{Q} -span

$$I_{0,1} = \langle \mathcal{H} \rangle_{\mathbb{Q}} \subset R_0 \cong R_{T,0}$$

is an R_0 -ideal. Furthermore, we have that $I_{0,1}$ is a computable ideal since to decide whether a given polynomial $p \in R_0 \cong R_{T,0}$ is in $I_{1,0}$, one can effectively partition the Q-monomial summands $qM \in R_0 \cong R_{T,0}, q \in \mathbb{Q}, M \in \mathcal{M} \subset R_0 \cong R_{T,0}$, of p by their corresponding MC-equivalence classes $C_M \in \omega^{2^{\ell_M}}, l_M \in \omega$, and then check to see if the rational coefficients in each MC-equivalence class sum to zero.

We can now define

$$I_0 = I_{0,0} +_{R_0} I_{0,1} \subset R_0 \cong R_{T,0}.$$

It follows that I_0 is a computable ideal of $R_0 \cong R_{T,0}$. To decide whether or not a given polynomial $p \in R_0 \cong R_{T,0}$ is in I_0 , first remove all \mathbb{Q} -monomial summands of p that belong to $I_{0,0}$, thus obtaining a new polynomial $p' \in R_0 \cong R_{T,0}$. Then check whether $p' \in I_{1,0}$ as in the previous paragraph. Let

$$R_1 = R_0 / I_0$$

Although it will not serve a purpose for us and so we omit its proof, we invite the motivated reader to show that in fact we have

$$R_1 = R_{T,0} / \langle X_0 = X_{\emptyset}, X_{\sigma} - X_{\sigma 0} X_{\sigma 1} : \sigma, \sigma 0, \sigma 1 \in T \rangle_{R_{T,0}}.^{22}$$

Now, for any given $p \in R_0 \cong R_{T,0}$ we can write

$$p = \sum_{i=1}^{N} q_i M_i, \ q_i \in \mathbb{Q}, \ M_i \in \mathcal{M}, \ N \in \omega,$$

expressing p as a \mathbb{Q} -linear combination of monomials in $R_0 \cong R_{T,0}$. Now, let

$$\ell_p = \max\{\ell_{M_i} : 1 \le i \le N\} \in \omega;$$

²²Let $I'_0 = \langle X_{\emptyset}, X_{\sigma} - X_{\sigma 0} X_{\sigma 1} \rangle_{R_{T,0}} \subset R_{T,0} \cong R_0$ and note that every generator of I'_0 is contained in I_0 from which it follows that $I'_0 \subseteq I_0$. To prove that $I_0 \subseteq I'_0$ it is very helpful to first show that, given any MC-code $C \in \omega^{2^\ell}$, $\ell \in \omega$, there is a unique monomial $M_C \in R_0 \cong R_{T,0}$ of minimal degree such that $0 \leq \ell_{M_C} \leq \ell$, $\ell_{M_C} \in \omega$, $C_{M_C}(\ell) = C$, and $M - M_C \in I'_0$ for any monomial $M \in R_{T,0} \cong R_0$ that is MC-equivalent to M_C . Finally, if M_1, M_2, \ldots, M_N , $N \in \omega$, is a finite sequence of monomials as in the previous sentence with corresponding rational coefficients $q_1, q_2, \ldots, q_N \in \mathbb{Q}$ such that $\sum_{k=1}^N q_k = 0$ then we have that $q_k(M_k - M_C) \in I'_0$ for each $k = 1, 2, \ldots, N$ and it follows that $\sum_{k=1}^N q_k(M_k - M_C) = \sum_{k=1}^N q_k M_k \in I'_0$.

it follows that ℓ_p is the length of a longest node appearing as an index in $p \in R_{T,0}$. Now, we construct a computable set $\mathcal{U} \subset R_0$ with $p \in \mathcal{U}$ if and only if for all $1 \leq k \leq 2^{\ell}$, $k \in \omega$, there exists $1 \leq i_k \leq N$, $i_k \in \omega$, such that

$$C_{M_{i_i}}(k) = 0 \in \omega_i$$

where $C_{M_{i_k}} \in \omega^{2^{\ell}}$ is the multiplicity covering code of M_i at level $\ell \in \omega, \ell \geq \ell_{M_i}$, and moreover

$$\sum_{\substack{0 \le i \le N, \\ M_i \text{ MC-equivalent to } M_{ii.}}} q_i \neq 0.$$

It follows that for every $p \in \mathcal{U} \subset R_0 \cong R_{T,0}$ we have that $\overline{p} \neq 0_{R_1} \in R_1 = R_0/I_0$ and also that any given $\ell' \geq \ell$ witnesses that $p \in \mathcal{U}$ just as $\ell \in \omega$ did above. It is clear that \mathcal{U} is a computable subset of R_0 since we have already seen that $I_{1,0} \subset R_0$ is computable and computing the MC code of a given monomial/polynomial can be done uniformly.

We claim that $\mathcal{U} \subset R_0$ is multiplicative. To see why, let $p_1, p_2 \in \mathcal{U} \subset R_0 \cong R_{T,0}$,

$$p_{1} = \sum_{i=1}^{N_{1}} q_{i,1}M_{i,1}, \ q_{i,1} \in \mathbb{Q}, \ M_{i,1} \in \mathcal{M}, \ N_{1} \in \omega,$$
$$p_{2} = \sum_{i=1}^{N_{2}} q_{i,2}M_{i,2}, \ q_{i,2} \in \mathbb{Q}, \ M_{i,2} \in \mathcal{M}, \ N_{2} \in \omega,$$
$$\ell_{1} = \max\{\ell_{M_{i,1}} : 1 \le i \le N_{1}\} \in \omega,$$
$$\ell_{2} = \max\{\ell_{M_{i_{2}}} : 1 \le i \le N_{2}\} \in \omega,$$

and

$$\ell = \max\{\ell_1, \ell_2\} \in \omega.$$

Now, for any given $1 \leq k \leq 2^{\ell}$, if $C_1, C_2 \in \omega^{2^{\ell}}$ are the ω -lexicographically least MC codes corresponding to monomials of p_1, p_2 , respectively, such that $C_1(k) = C_2(k) = 0$, then by Remark 7.6 above it follows that the MC code $C_1 + C_2 \in \omega^{2^{\ell}}$, $(C_1 + C_2)(k) = 0 \in \omega$ corresponds to some monomial appearing in the polynomial product $p_1p_2 \in R_0$, and is the ω -lexicographically least such MC code. Moreover, it is not difficult to check that if $0 \neq Q_1, Q_2 \in \mathbb{Q}$ are the sums of the coefficients of monomials with MC code C_1, C_2 in p_1 and p_2 , respectively, then $0 \neq Q_1Q_2 \in \mathbb{Q}$ is the sum of the \mathbb{Q} -coefficients corresponding to those monomial summands of p_1p_2 with MC code $C_1 + C_2$. It follows that $\mathcal{U} \subset R_0 \cong R_{T,0}$ is a multiplicative set not containing any element of $I_0 = I_{0,0} + I_{1,0} \subset R_0 \cong R_{T,0}$, and therefore we can localize the ring $R_1 = R_0/I_0 = \overline{R_0}$ at this computable subset $\overline{\mathcal{U}} \subset R_1$, $0_{R_1} \notin \overline{\mathcal{U}}$. Let $R = R_T$ denote the resulting computable ring; R will be our main ring of interest for the rest of the proof.

Before we proceed we should note two key facts about R, both of which follow easily from our previous remarks.

- For all $\sigma \in T$, if $\sigma 0, \sigma 1 \in T$ then $\overline{X_{\sigma}} = \overline{X_{\sigma 0}X_{\sigma 1}} \in R$, $X_{\sigma}, X_{\sigma 0}, X_{\sigma 1} \in R_0 \cong R_{T,0}$ (*R* inherits this property from $R_1 = R_0/I_0$). Therefore, any prime ideal $P \subset R$ containing $\overline{X_0} \in R$ must also contain either $\overline{X_{\sigma 0}} \in R$ or else $\overline{X_{\sigma 1}} \in R$. However, as we shall see in the next item, any proper ideal such as $P \subset R$ cannot contain both of these indeterminates.
- If $\sigma, \tau \in 2^{<\omega}$ are incomparable, then any R-ideal containing both $\overline{X_{\sigma}}, \overline{X_{\tau}} \in R$ must also contain the identity element 1_R . To see why note that by our construction of $\overline{U} \subset R$ (more specifically, by our localization in going from R_1 to R) it follows that $\overline{X_{\sigma}} + \overline{X_{\tau}}$ is a unit in R.

The following lemma is a useful generalization of the second item above.

Lemma 7.8. Every element $0_R \neq r \in R \setminus \overline{\mathcal{U}}$ can be expressed in the form

$$r = \overline{uM} \in R$$

for some unit $u \in \mathcal{U} \subset R_0 \cong R_{T,0}$ and some monomial $M \in R_{T,0} \setminus I_0$. Proof. Let $0_R \neq r \in R, r = \overline{p}, p \in R_0 \setminus \mathcal{U}$,

$$p = \sum_{i=1}^{N} q_i M_i, \ q_i \in \mathbb{Q}, \ M_i \in \mathcal{M}, \ i \in \omega.$$

For each $i = 1, 2, ..., N \in \omega$ let $\Sigma_i \subset T \subseteq 2^{<\omega}$ be the finite set *T*-indices of indeterminates appearing in the monomial $M_i \in \mathcal{M}$, and define

$$\ell_i = \max\{|\sigma| : \sigma \in \Sigma_i\} \in \omega,\$$

and also

$$\ell = \max\{\ell_i : 0 \le i \le N\} \in \omega.$$

Now, by repeatedly replacing indeterminates of the form $X_{\sigma}, \sigma \in T, |\sigma| < \ell$, appearing in a given monomial summand of $p, M_i \in \mathcal{M} \subset R_{T,0} \cong R_0, 1 \le i \le N$, with the R-equivalent product $X_{\sigma 0}X_{\sigma 1} \in \mathcal{M} \subset R_{T,0} \cong R_0, \sigma 0, \sigma 1 \in T$, we can assume without any loss of generality that each indeterminate $X_{\tau}, \tau \in T \subseteq 2^{<\omega}$ appearing in $p \in R_{T,0} \cong R_0$ satisfies $|\tau| = \ell \in \omega$. Then, if we let $M_0 \in \mathcal{M} \subset R_{T,0} \setminus I_0$ be the unique greatest common divisor monomial of M_1, M_2, \ldots, M_N in \mathcal{M} , and factorize

$$p = M_0 \cdot_{R_{T,0}} \sum_{i=1}^N q_i M'_i, \ M'_i = \frac{M_i}{M_0}, \ i \in \omega,$$

then by our construction of $\mathcal{U} \subset R_0 \cong R_{T,0}$ and the fact that M_0 is the greatest common divisor of the monomial summands of $p \in R_0 \cong R_{T,0}$, it follows that

$$\sum_{i=1} q_i M'_i \in \mathcal{U}_i$$

and so we can set $M = M_0 \in \mathcal{M} \subset R_0 \cong R_{T,0}$ and $u = \sum_{i=1}^N q_i M'_i \in \mathcal{U}$ thus proving the current lemma.

The following can be easily verified by our previous remarks about R.

- For each $s \in \omega$, and finite $T_s \subset 2^{<\omega}$, T_s has exactly (s+1)-many leaves $\lambda_{s,1}, \lambda_{s,2}, \ldots, \lambda_{s,s+1}$. Let $\Lambda_s = \{\lambda_{s,k} : 1 \le k \le s+1\}$ denote the set of T_s -leaves. It follows that Λ_s covers T_s , and since T is completely branching we have that Λ_s also covers $2^{<\omega}$.
- For each $s \in \omega$:
 - $-\prod_{k=1}^{s+1} \overline{X_{\lambda_{s,k}}} = 0_R$ and - for any $0 \le k < \ell \le s+1$ we have that $X_k + X_\ell \in \mathcal{U} \subset R_0 \cong R_{T,0}$ and so
 - $\overline{X_k + X_\ell}$ is an *R*-unit.
- Let $P \subset R$ be prime. Then, by the previous point, for each $s \in \omega$, P contains exactly one T_s -leaf $\lambda = \lambda_{s,P} = \lambda_{s,k_P} \in T_s$, $1 \leq k_P \leq s+1$. Moreover, we have that

$$\overline{X_{\lambda}} \in A_{s,k} = Ann(\prod_{\substack{k=1\\k \neq k_P}}^{s+1} \overline{X_{\lambda_{s,k}}}) \subseteq P.$$

- Conversely, for each $s \in \omega$ and $1 \leq k \leq s+1$, there is at least one minimal prime ideal $P = P_{s,k} \subset R$ such that $\lambda_{s,k} \in A_{s,k} \subseteq P$.
- Let $P \subset R$ be prime. Either:

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- there is a T-leaf $\lambda \in T$, $\lambda 0, \lambda 1 \notin T$, such that $P = \langle \overline{X_{\lambda}} \rangle_{R}$ ²³ or else
- there is a unique $f \in [T] \subseteq 2^{\omega}$ such that for all $\sigma \subset f, \sigma \in 2^{<\omega}, \overline{X_{\sigma}} \in P^{24}$.
- For any two distinct R-prime ideals $P \neq Q$, there is a unique maximal $\sigma \in T$ such that $\overline{X_{\sigma}} \in P \cap Q$. Moreover, since $P \neq Q$ we must have that $\sigma 0, \sigma 1 \in T$, and by our construction of R we have that

$$P \cap Q = \langle \overline{X_{\sigma}} \rangle_R$$

It follows that any $x \in P \cap Q$ is an *R*-multiple of $\overline{X_{\sigma}} = \overline{X_{\sigma 0}} \cdot_R \overline{X_{\sigma 1}}$ and $\overline{X_{\sigma 0}} +_R \overline{X_{\sigma 1}}$ is an *R*-unit. In other words, the minimal prime ideals of *R* are uniformly coprime.

- For each $s \in \omega$, R contains (s + 1)-many distinct minimal prime ideals. In our proof of Theorem 5.11 above we showed how to use this fact to construct a uniformly computable enumeration of infinitely many distinct minimal R-prime ideals.
- *R* contains a uniformly computable enumeration of infinitely many minimal prime ideals, each pair of which is uniformly coprime.
- RCA_0 proves that R satisfies the hypotheses of URPDL, and URPDL says that any such ring is not Noetherian.

Let

$$J_0 \subset J_1 \subset \cdots \subset J_N \subset \cdots \subset R, \ N \in \omega,$$

be an infinite strictly ascending chain of R-ideals. Moreover, Lemma 7.8 says that for each $i \in \omega$ there exists $\overline{x_i} \in J_{i+1} \setminus J_i \subset R$, for some R_0 -monomial $x_i \in \mathcal{M}$. For each $i \in \omega$ let $\Sigma_i \subset T \subseteq 2^{<\omega}$ be the set of $2^{<\omega}$ -indices that appear in x_i and define

$$\ell_i = \max\{|\sigma| : \sigma \in \Sigma_i\} \in \omega.$$

By the Finite Pigeonhole Principle we must have that $\limsup_i \ell_i = \infty$, and by passing to an infinite computable subset of $i \in \omega$ we can assume without any loss of generality that $\ell_i < \ell_{i+1}$ for all $i \in \omega$. Furthermore, since $\overline{x_i} \in J_{i+1} \setminus J_i \subset R$, we must have that the monomial $x_i \in \mathcal{M}$ properly divides the monomial $x_{i+1} \in \mathcal{M}$ via a non-*R*-unit quotient. In terms of MC-codes this means that

$$C_i(k) \ge C_{i+1}(k)$$
 and $C_i(k_0) > C_{i+1}(k_0)$,

for all $i \in \omega$, $1 \le k \le 2^{\ell_{i+1}}$, and at least one $1 \le k_0 \le 2^{\ell_{i+1}}$. Here $C_i = C_{x_i}, C_{i+1} = C_{x_{i+1}} \in \omega^{2^{\ell_{i+1}}}$ denote the MC-codes of the R_0 -monomials x_i, x_{i+1} , respectively.

For each $i \in \omega$ write

$$x_i = \prod_{j=1}^{N_i} X_{\sigma_{i,j}}^{\alpha_{i,j}} \in \mathcal{M} \subset R_0 \cong R_{T,0}, \ \sigma_{i,j} \in T, \ \alpha_{i,j} \in \omega, \ \alpha_{i,j} > 0,$$

and for each $i \in \omega$, let $t_i \in \omega$ be least such that

$$\sigma_{i,j} \in T_{t_i} \subset T$$
, for all $j = 1, 2, \ldots, N_i$.

From our hypothesis that $\ell_i < \ell_{i+1}$ in the previous paragraph it follows that $t_i < t_{i+1}$. Furthermore, by our construction of $I_{0,1} \subset I_0 = I_{0,0} + I_{0,1} \subset R_0 \cong R_{T,0}$ and $R_1 = R_0/I_0$, it follows that if $\Lambda_t \subset T_t$ denotes the leaves of $T_t \subset T$, then for any given $i \in \omega$ and $t \ge t_i$ there is an R_0 -monomial $x_{i,t}$ that is MC-equivalent (and thus R-equivalent) to x_i , and such that the T-indices of every indeterminate appearing in $x_{i,t}$ comes from Λ_t . Also, for each $t \ge t_i$,

²³In this case P is called *isolated*.

²⁴In this case P is called *embedded*.

 $i \in \omega$, $x_{i,t}$ can be obtained uniformly and effectively from $x_i = x_{i,t_i}$. Now, for all $i \in \omega$ and $t \ge t_i$ write

$$x_{i,t} = \prod_{j=1}^{N_{i,t}} X_{\sigma_{i,t,j}}^{\alpha_{i,t,j}} \in \mathcal{M} \subset R_0 \cong R_{T,0}, \ \sigma_{i,t,j} \in \Lambda_t, \ \alpha_{i,t,j} \in \omega \ \alpha_{i,t,j} > 0.$$

In the previous paragraph we explained that we have $C_i > C_{i+1}$ for all $i \in \omega$, and so it follows that for each $i \in \omega$ there is some $T_{t_{i+1}}$ -leaf $\tau_i \in \Lambda_{t_{i+1}}$ such that the $R_{T,0}$ -indeterminate X_{τ_i} appears in the $R_{T,0}$ -monomial $x_{i,t_{i+1}}$ with strictly larger exponent than it has in $x_{i+1,t_{i+1}}$. Let $\alpha(i) \in \omega$ denote the exponent of X_{τ_i} in $x_{i+1,t_{i+1}}$, $i \in \omega$. Since x_{i+1} divides $x_i, x_{i+1}, x_i \in \mathcal{M}$, for all $i \in \omega$, it follows that

$$0 \le \alpha(i) \le M,$$

where $M \in \omega$ is the maximal exponent appearing in x_0^{25} Therefore, by the Infinite Pigeonhole Principle (i.e. $B\Sigma_2$) there exists some $0 \le m \le M$, $m \in \omega$, and infinitely many $i \in \omega$ such that

$$\alpha(i) = m$$

Finally, by our construction of $\alpha(i) \in \omega$ it follows that the computable set

$$A = \{\tau_i : \alpha(i) = m\} \subset T$$

is an infinite T-antichain, as required by TAC.

Corollary 7.9 (RCA₀ + $B\Sigma_2$). TAC is equivalent to URPDL.

Corollary 7.10 ($RCA_0 + B\Sigma_2$). RPDL *implies* TAC.

We leave the following question open.

Question 7.11. What is the reverse mathematical strength of RPDL over either RCA₀ or RCA₀ + B Σ_2 ? In particular, is RPDL equivalent to either TAC + WKL₀ (our upper bound) or TAC (our lower bound) over RCA₀(+B Σ_2)?

7.0.1. Restricting the Induction Scheme. Define

WTAC: For any infinite Σ_1^0 completely branching tree $T \subset 2^{<\mathbb{N}}$ with splitting enumeration $T = \bigcup_{s \in \mathbb{N}} T_s, T_s \subset T_{s+1}$, there exists a nonincreasing²⁶ function $f: T \to \mathbb{N}$ such that $(\forall N \in \mathbb{N})(\exists k > N, \sigma_1, \sigma_2, \ldots, \sigma_k \in T)$ such that for all $i = 1, 2, \ldots, k$ $f(\sigma_i) > \min\{f(\sigma_i 0), f(\sigma_i 1)\}.$

It is not difficult to show that, over RCA_0 , TAC is equivalent to WTAC with the added restriction that $f(\emptyset) = 1$ (from which it follows that $f(\sigma) \in \{0, 1\}$ for all $\sigma \in T$), and so TAC implies WTAC. Moreover, using the techniques that we developed in the proof of Theorem 7.1 above, one can also show that URPDL implies WTAC over RCA_0 . To get a brief sense of why this is the case, suppose that we are given an infinite Σ_1^0 completely branching tree $T \subset 2^{<\mathbb{N}}$, if $R = R_0/I_0[\mathcal{U}^{-1}]$ is the corresponding principle ideal ring²⁷ that we constructed in the proof of Theorem 7.1 above, and

$$I_1 \subset I_2 \subset \cdots \in I_k \subset \cdots \subset R, \ k \in \mathbb{N},$$

is an infinite strictly ascending chain of R-ideals with corresponding principle monomial generators

$$x_k = \prod_{\sigma \in S_k} X_{\sigma}^{\alpha_{\sigma}} \in R, \ S_k \subset T, \ \alpha_{\sigma} \in \mathbb{N},$$

 $^{^{25}}M$ is well-defined up to both MC– and R-equivalence.

 $^{^{27}}R$ is not a domain here.

where $S \subset T$ is prefix-free and finite. In this case we create a correspondence between x_k and the values

$$f(\sigma) = \alpha_{\sigma} \in \mathbb{N}, \ \sigma \in S_k$$

via the exponents α_{σ} of the indeterminates of X_k , namely X_{σ} , $\sigma \in S_k \subset T$. We leave it to the reader who has read and understood our proof of Theorem 7.1 above to fill in the rest details.

The following theorem summarizes our results over RCA_0 , without the additional assumption of $\mathsf{B}\Sigma_2$.

Theorem 7.12 (RCA_0).

$$TAC \rightarrow URPDL \rightarrow WTAC.$$

Question 7.13. What are the first-order parts of PDL, RPDL, URPDL, TAC, and WTAC?

8. An algebraic characterization of $WKL_0 + TAC$ over $B\Sigma_2$

In the previous subsection we established that URPDL is equivalent to TAC over $RCA_0 + B\Sigma_2$. Although we do not know the exact strength of RPDL over $RCA_0 + B\Sigma_2$, we can specify a related algebraic principle, given in [Eis95, Theorem 2.14], and verify that it is equivalent to $WKL_0 + TAC$ over $RCA_0 + B\Sigma_2$.

NMMA : If R is a Noetherian ring such that every prime ideal of R is maximal, then R is Artinian.

NMMA is the conjunction of the following two weaker principles.

- FNMMA : If R is a Noetherian ring with finitely many prime ideals, each of which is both minimal and maximal, then R is Artinian.
- INMMA : If R is a Noetherian ring with infinitely many prime ideals²⁸, each of which is both minimal and maximal, then R is Artinian.

It is clear that INMMA follows from RPDL, since RPDL implies that (the antecedent of) INMMA is vacuous. Therefore, via Corollary 6.11 above, $WKL_0 + TAC$ implies INMMA. Furthermore, if T is a given infinite Σ_1^0 completely branching tree, and $R = R_T$ is the corresponding computable ring that we constructed in Theorem 7.1 above, then

$$R \supset \langle X_0 \rangle_R \supset \langle X_0^2 \rangle_R \supset \langle X_0^3 \rangle_R \supset \cdots \supset \langle X_0^N \rangle_R \supset \cdots$$

is a uniformly computable infinite strictly descending chain of ideals in R and thus R does not satisfy computable-DCC (i.e. R is not Artinian). Moreover, Theorem 7.1 shows that every infinite strictly ascending chain of ideals in R computes an infinite T-antichain (over $RCA_0 + B\Sigma_2$). Therefore, Theorem 7.1 above essentially shows that INMMA implies TAC over $RCA_0 + B\Sigma_2$.

Lemma 8.1 ($\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$). INMMA implies TAC, and is implied by $\mathsf{WKL}_0 + \mathsf{TAC}$.

Now, to show that NMMA = FNMMA + INMMA is equivalent to $WKL_0 + TAC$ over $RCA_0 + B\Sigma_2$, it suffices to show that FNMMA is equivalent to WKL_0 over RCA_0 . The proof that WKL_0 implies FNMMA is very similar to the proof that WKL_0 implies ART given in [Con19, Theorem 8.3]; in particular the following theorem is a computable structure theorem for rings R containing finitely many minimal prime ideals that are also PA-maximal, similar to the Computable Structure Theorem for (Local) Artinian Rings [Con19, Theorem 7.1, Corollary 7.3].

Theorem 8.2 (Computable Structure Theorem for Rings with finitely many minimal prime ideals that are also PA-maximal). Let R be a computable ring with (only) finitely many prime ideals that are both minimal and PA-maximal, and satisfies computable-ACC. Then,

if M_1, M_2, \ldots, M_N , $N \in \omega$, enumerate (all of) the distinct minimal prime ideals of R, N is a uniform bound on the length of any strictly ascending (or descending) chain of R-ideals.

Proof. The proof is very similar to various arguments found in [Con10, Con19]; we will outline the main points of the proof (valid in RCA_0).

- Since each M_i , $1 \leq i \leq N$, is PA-maximal it follows that for all $m, n \in \omega$ and $1 \leq i \neq j \leq N$ there exists $m_{i,m} \in M_i^m$ and $m_{j,n} \in M_j^n$ such that $m_{i,m} + m_{j,n} = 1_R$. Otherwise, for some $1 \leq k \leq N$ we could construct an infinite computable tree $T \subseteq 2^{<\omega}$ such that every infinite path $f \in [T] \subseteq 2^{\omega}$ codes an R-ideal I_f such that $M_k \subset I_f \subset R$, contradicting our assumption that M_k is PA-maximal.
- Additionally, our proof of Lemma 5.3 above says that J is uniformly nilpotent, i.e. there exists $N_0 \in \omega$ such that

$$J^n = \prod_{i=1}^N M_i^n = \langle 0_R \rangle.$$

Now, via Bounded Π_1^0 -Comprehension let

$$\langle n_0, n_1, \cdots, n_N \rangle \in (n)^N$$

be $(n)^N$ -minimal such that

$$\prod_{i=1}^{N} M_i^{n_i} = \langle 0_R \rangle$$

• Now, it follows from the previous two points that for each $1 \leq i \neq j \leq N$, $M_i^{n_i} + M_j^{n_j} = R$, and in this case the Chinese Remainder Theorem says that there is a computable isomorphism

$$\varphi: R \to \prod_{i=1}^{N} R/M_i^{n_i}$$
 with kernel $J = \bigcap_{i=1}^{N} M_i^{n_i} = \prod_{i=1}^{N} M_i^{n_i} = \langle 0_R \rangle.$

• For each $1 \leq k \leq N$, let $x_{k,1}, x_{k,2}, \ldots, x_{k,n_k-1} \in R/M_k^{n_k}$ be such that $\prod_{i=1}^{n_k-1} x_{k,i} \neq 0_{R/M_k^{n_k}}$. The annihilator ideals

$$Ann(x_{k,1}) \subset Ann(x_{k,1}x_{k,2}) \subset \ldots \subset Ann(\prod_{i=1}^{n_k-1} x_{k,i}) \subset R/M_k^{n_k}$$

can be used to construct a finite chain of R-ideals

$$\langle 0_R \rangle = I_0 \subset I_1 \subset \cdots \subset I_{\sum_i n_i}$$

such that each successive quotient I_{i+1}/I_i , $0 \le i < \sum_i n_i$ is an R/M_j -vector space, for some $1 \le j \le N$. This is the Structure Theorem for Rings with finitely many minimal primes that are also PA-maximal.

• Finally, if R contains an infinite uniformly computable strictly descending chain of ideals, then the finite chain described in the previous point (and the corresponding vector spaces mentioned there) can be used to construct an infinite computable tree $T \subseteq 2^{<\omega}$ all of whose infinite paths code infinite strictly ascending chains of R-ideals.

Corollary 8.3 (RCA₀). WKL₀ *implies* FNMMA.

We now turn our attention to showing that FNMMA implies WKL₀ over RCA₀. To do so, given any two nonempty disjoint computably enumerable sets $A, B \subset \omega$, we will eventually construct a ring $R = R_{A,B}$ such that:

- *R* does not satisfy computable-DCC,
- R contains exactly two minimal prime ideals that are also maximal, and
- every infinite strictly ascending chain of ideals in R codes an (A, B)-separator $D \subset \omega$ such that $A \subseteq D$ and $D \cap B = \emptyset$.

Lemma 8.4. Let $A, B \subset \omega$ be nonempty computably enumerable disjoint sets. There is a computable integral domain $R_0 = R_{0,A,B}$ containing an infinite uniformly computable strictly descending chain of ideals

$$R_0 \supset I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_N \supset \cdots, N \in \omega,$$

and such that every infinite strictly ascending chain of R_0 -ideals codes an (A, B)-separator $D \subset \omega$.

Proof. The proof is very similar to that of [DLM07, Theorem 3.2], with which we assume the reader is familiar. The following items sketch the proof of the current theorem. Let A, B be given infinite disjoint computably enumerable sets.

- Let $R_0 = \mathbb{Q}[X_k : k \in \omega]$, and let F denote the field of fractions of R_0 . We represent every nonzero $p \in R_0$ as a finite sum of unique monomials with nonzero \mathbb{Q} -coefficients, and we say that each such monomial *appears in p*. We will construct R as a computably enumerable subring of F by effectively enumerating the generators of R.
- For each $0 \neq p \in R$, let $N_p \in \omega$ denote the largest index of any indeterminate appearing in p, and let $\text{GCD}(p) \in R_0$ denote the greatest common divisor of the monomials appearing in p. Note that GCD(p) is itself a monomial. We enumerate the generators of R as follows:

$$R_0 \cup \left\{ \frac{X_k}{p} : p \in R_0, \text{ GCD}(p) = 1, \ k \in A, \ k > N_p \right\} \cup \\ \cup \left\{ \frac{1 - X_k}{p} : p \in R_0, \text{ GCD}(p) = 1, \ k \in B, \ k > N_p \right\} \subset F.$$

- Our construction is very similar to that of [DLM07, Theorem 3.2]; the only real difference being that we have required our denominators to have GCD equal to one. Now, whereas every nontrivial ideal of [DLM07, Theorem 3.2] computes an (A, B)-separator, our restriction on the GCD of denominators leads to the conclusion that every R-ideal that is not generated by a single monomial computes an (A, B)-separator, and every R-ideal that is generated by a single monomial is computable. This essentially the heart of (the difference between) our arugment and the proof of [DLM07, Theorem 3.2], which we now outline:
 - Let $I \subset R$ be an ideal. Via the Π_1^0 -Well-Ordering Principle, let $X \in R_0$ be the unique monomial of maximal degree such that X divides the numerator of every $x \in I$. Then it follows that

$$I = \langle X \rangle_R \cdot_R J,$$

for some R-ideal J, and by our construction of X and the GCD requirement on the denominators of R, we can apply the argument of [DLM07, Theorem 3.2] to J and show that if $J \neq \langle 0_R \rangle$ and J does not compute an (A, B)-separator, then J = R and thus $I = \langle X \rangle_R$.

- Furthermore, our GCD criterion on R-denominators imply that for any R_0 -monomial X, X divides the numerator of any $x \in \langle X \rangle_R$. It follows that $\langle X \rangle_R$ is computable, for any R_0 -monomial X.

• Note that R does not satisfy computable-DCC, since

$$R \supset \langle X_0 \rangle_R \supset \langle X_0^2 \rangle_R \supset \cdots \supset \langle X_0^N \rangle_R \supset \cdots, \ N \in \omega$$

is an infinite uniformly computable strictly descending chain of R-ideals.

• Finally, we claim that every infinite strictly ascending chain of ideals

 $\langle 0 \rangle_R = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_N \subset \cdots \subset R, \ N \in \omega,$

contains an ideal $I_M \subset R$, $M \in \omega$, that computes an (A, B)-separator. Otherwise, by our previous remarks we would have that

$$I_0 = \langle X_N^{\alpha} \rangle_R$$
, and, more generally, $I_k = \langle X_N^{\alpha-k} \rangle_R$, $N, \alpha, k \in \omega$;

contradicting the (Σ_1^0-) Well-Ordering Principle $(\mathsf{I}\Sigma_1)$.

Corollary 8.5 (RCA_0). FNMMA *implies* WKL₀.

Lemma 8.6 (RCA_0). FNMMA is equivalent to WKL_0 .

Proof. Corollaries 8.3 and 8.5 above.

Theorem 8.7 ($\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$). NMMA = FNMMA + INMMA is equivalent to $\mathsf{WKL}_0 + \mathsf{TAC}$.

Proof. Lemmas 8.1 and 8.5 above.

References

- [AK00] C. J. Ash and J. Knight. Computable Structures and the Hyperarithmetical Hierarchy, volume 144 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, Amsterdam, 2000.
- [BPS17] L. Bienvenu, L. Patey, and P. Shafer. On the logical strengths of partial solutions to mathematical problems. *Transactions of the London Mathematical Society*, 4(1):30–71, 2017.
- [CDSS12] P. Cholak, D. D. Dzhafarov, R. A. Shore, and N. Schweber. Computably enumerable partial orders. Computability, 1(2):1243–1251, 2012.
- [CJS01] P. A. Cholak, C. G. Jockusch, and T. A. Slaman. On the strength of Ramsey's theorem for pairs. J. Symbolic Logic, 66(1):1–55, 2001.
- [Con10] C.J. Conidis. Chain conditions in computable rings. *Transactions of the AMS*, 362(12):6523–6550, 2010.
- [Con19] C.J. Conidis. The computability, definability, and proof-theory of Artinian rings. Advances in Mathematics, 341(1):1–39, 2019.
- [DF99] D.S. Dummit and R.M. Foote. Abstract Algebra. John Wiley & Sons, 1999.
- [DH09] D. D. Dzhafarov and J. L. Hirst. The Polarized Ramsey's Theorem. Archive for Mathematical Logic, 48(2):141–157, 2009.
- [DH10] R.G. Downey and D.R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer-Verlag, 2010.
- [DHLS03] R. G. Downey, D. R. Hirschfeldt, S. Lempp, and R. Solomon. Computability-theoretic and prooftheoretic aspects of partial and linear orderings. *Israel Journal of Mathematics*, 138:271–289, 2003.
- [DLM07] R.G. Downey, S. Lempp, and J.R. Mileti. Ideals in computable rings. Journal of Algebra, 314:872– 887, 2007.
- [Eis95] D. Eisenbud. Commutative algebra with a view toward algebraic geometry. Springer-Verlag, 1995.
- [Flo12] S. Flood. Reverse mathematics and a Ramsey-type König's lemma. J. Symbolic Logic, 77(4):1272– 1280, 2012.
- [GM17] N. Greenberg and A. Melnikov. Proper divisibility in computable rings. Journal of Algebra, 474(3):180–212, 2017.
- [GMS13] N. Greenberg, A. Montalbán, and T. A. Slaman. Relative to any non-hyperarithmetic set. Journal of Mathematical Logic, 13, 2013. Article ID 1250007.
- [HS07] D.R. Hirschfeldt and R.A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. Journal of Symbolic Logic, 71:171–206, 2007.

- [HSS09] D.R. Hirschfeldt, R.A. Shore, and T.A. Slaman. The atomic model theorem and omitting partial types. *Transactions of the American Mathematical Society*, 361:5805–5837, 2009.
- [JKL⁺09] C. J. Jockusch, B. Kastermans, S. Lempp, M. Lerman, and R. Solomon. Stability and posets. Journal of Symbolic Logic, 74(2):693–711, 2009.
- [JS72] C. G. Jockusch and R. I. Soare. Π_1^0 classes and degrees of theories. Trans. Amer. Math. Soc., 173(1):33–56, 1972.
- [Lan93] S. Lang. Algebra. Springer-Verlag, 1993.
- [LMP19] L. Liu, B. Monin, and L. Patey. A computable analysis of variable words theorems. Proc. Amer. Math. Soc., 147(2):823–834, 2019.
- [LST13] M. Lerman, R. Solomon, and H. Towsner. Separating principles below Ramsey's theorem for pairs. Journal of Mathematical Logic, 13, 2013.
- [Mat04] H. Matsumura. Commutative Ring Theory. Cambridge University Press, 2004.
- [Mon] A. Montalban. Computable Structure Theory: Beyond the Arithmetic. Cambridge University Press. To appear.
- [Mon21] A. Montalban. Computable Structure Theory: Within the Arithmetic. Cambridge University Press, 2021.
- [Nie09] A. O. Nies. Computability and Randomness. Oxford University Press, 2009.
- [Noe21] E. Noether. Idealtheorie in ringbereichen. Mathematische Annalen, 83(1):24–66, 1921.
- [NS20] E. Noether and W. Schmeidler. Moduln in nichtkommutativen bereichen, insbesondere aus differential- und differenzenausdrücken. *Mathematische Zeitschrift*, 8(1):1–35, 1920.
- [NS20] A. O. Nies and P. Schafer. Randomness notions and reverse mathematics. Journal of Symbolic Logic, 85(1):271–299, 2020.
- [Pat16] L. Patey. Les mathématiques à rebours de théorèmes de type Ramsey. PhD thesis, Université Paris Diderot, February 2016 2016.
- [Sim09] S.G. Simpson. Subsystems of Second Order Arithmetic, second edition. Cambridge University Press, 2009.
- [Soa16] R.I. Soare. *Turing Computability*. Springer-Verlag, 2016.
- [Wan13] W. Wang. Omitting cohesive sets. Nanjing Daxue Xuebao Shuxue Bannian Kan, 30:40–47, 2013.

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