COMPUTABILITY-THEORETIC ASPECTS OF AN ANTICHAIN THEOREM FOR INFINITE EXTENDIBLE TREES OF NON-TRIVIAL RANK

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Abstract. We introduce an antichain principle for infinite extendible trees of nontrivial rank and show that, in the context of Reverse Mathematics, its strength is distinct from every other principle currently cataloged in the "Reverse Mathematical Zoo."

1. Introduction

Two major branches of applied Computability Theory include Computable Structure Theory and Reverse Mathematics. While Computable Structure Theory typically aims to examine and classify the computability content of mathematical structures and their substructures, Reverse Mathematics examines and classifies the logical content of theorems associated with such structures. One interesting class of structures that have received much attention since the turn of the century are partial orders; see any of [GMS13, DHLS03, JKL$^+$, CDSS12, HS07] for more details. For a background in Computability Theory, see [Soa16]; for a background in Computable Structure Theory, consult [AK00]; [Sim09] contains background information in Reverse Mathematics.

This article is primarily a contribution to Reverse Mathematics, and as such examines and attempts to classify the strength of a particular mathematical theorem over the weak base theory known as the Recursive Comprehension Axiom scheme, or RCA_0.[1] More specifically, we introduce and analyze a theorem regarding the existence of antichains in a certain class of partial orders that are a subclass of trees. To achieve our goal we will examine the computability-theoretic aspects of antichains in this subclass of partial orders.

For our purposes we are primarily interested in the class of partial orders T consisting of downward closed subsets of binary strings (ordered by inclusion) with infintely many branchings. It is easy to show (see Remark 2.4 below for more details) that the infinitely many branchings result in the existence of infinite antichains $A_T \subset T$ within T. Moreover, the author has recently discovered, but has not yet published, results showing that such orderings T and their associated antichains A_T play a significant role in the Reverse Mathematics of Commutative Noetherian Rings. Although our primary goal here is to introduce what we believe to be an interesting class of partial orderings which yield interesting results from the point of view of Reverse Mathematics, a secondary aim of ours is to lay the groundwork for future results concerning the Reverse Mathematics of Commutative Noetherian Rings.

Our main goal in this article is to catalog some of the basic computability-theoretic aspects of antichains of the form A_T above, one consequence of which is that the existence of such

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[1]More precise definitions and references are given in the next section.

antichains in every such partial order is (computationally) distinct from every other combinatorial principle that has thus far been examined and cataloged in the "Reverse Mathematical Zoo." More details follow in the next section.

2. Background

2.1. **Trees in Baire space and Cantor space.** Let $\omega = \{0, 1, 2, \ldots\}$ denote the standard natural numbers, while $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the possibly nonstandard natural numbers. Any of the following definitions that mention $\mathbb{N}$ has an implicit corresponding version for ω. Let $N^{<\mathbb{N}}$, $N \in \mathbb{N}$, denote the set of finite $N-$ary sequences ordered by extension, and let $\mathbb{N}^{<\mathbb{N}}$, denote the set of finite sequences of natural numbers ordered by extension. Any defintion mentioning finite binary sequences, i.e. $2^{<\mathbb{N}}$, has an implicit corresponding version for both $N^{<N}$ and $\mathbb{N}^{<\mathbb{N}}$ as well. We will explicitly write our finite sequences in either $2^{<\mathbb{N}}$ or $\mathbb{N}^{<\mathbb{N}}$ using angled brackets, like so

$$\langle a_0, a_1, a_2, \ldots, a_n \rangle \in 2^{<\mathbb{N}}, \ n \in \mathbb{N}, \ a_i \in \omega, \ 0 \leq i \leq n.$$

For all $\sigma \in 2^{<\mathbb{N}}$ and $k \in \{0, 1\}$, $\sigma k \in 2^{<\mathbb{N}}$ denotes the 1-bit concatenation (extension) of σ with (by) k. Let $\mathbb{N}^{\leq N}_{<}$ denote finite sequences of *strictly increasing* natural numbers; it is easy to see that $\mathbb{N}^{<\mathbb{N}}$ is computably isomorphic to $\mathbb{N}^{\leq \mathbb{N}}_{<}$ and we will implicitly use this obvious fact in Theorem 4.1 below. For any natural number $l \in \mathbb{N}$, let $2^{=l} \subset 2^{<\mathbb{N}}$ denote those finite binary sequences of natural numbers of length l. Let $\emptyset$ denote the root of $2^{<\mathbb{N}}$, and for all $\sigma \in 2^{<\mathbb{N}}$, let $|\sigma| \in \mathbb{N}$ denote the length of σ. For any $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, we write $\tau \subseteq \sigma$ to denote the fact that τ is a prefix of σ; we write $\tau \subset \sigma$ to denote the fact that τ is a *proper* prefix of σ. Note that $\subseteq$ yields a partial ordering on $2^{<\mathbb{N}}$. We say that $T \subseteq 2^{<\mathbb{N}}$ is a *tree* if for all $\sigma \in T$ and $\tau \subseteq \sigma$ we have that $\tau \in T$. Let $2^{\mathbb{N}}$ denote the set of *infinite* sequences of natural numbers. We write $\sigma \subseteq f$, $\sigma \in 2^{<\mathbb{N}}$, $f \in 2^{\mathbb{N}}$, to mean that σ is a finite initial segment of f. For any given tree $T \subseteq 2^{<\mathbb{N}}$, let $[T] \subseteq 2^{\mathbb{N}}$ denote the set of infinite binary sequences $f \in 2^{\mathbb{N}}$, such that for each $n \in \mathbb{N}$ we have that the finite initial segment of f of length n, denoted $f{\restriction}n \in 2^{<\mathbb{N}}$, is in T, i.e. we have

$$f{\restriction}n = \langle f(0), f(1), f(2), \cdots, f(n) \rangle \in T \subset 2^{<\mathbb{N}},$$

where $f(k) \in 2 = \{0, 1\}$ denotes the k^{th} bit of f. We say that a given $\sigma \in T$ is *extendible* whenever there exists $f \in [T] \subseteq 2^{\mathbb{N}}$ such that $\sigma \subset f$. We say that the tree $T \subset \omega^{<\omega}$ is extendible whenever T every element of T is extendible. For any given $\sigma \in 2^{<\mathbb{N}}$, let

$$[\sigma] = \{f \in 2^{\mathbb{N}} : f \supset \sigma\}$$

and for any subset $A \subset T$ let

$$[A] = \bigcup_{\sigma \in A} [\sigma] \subseteq 2^{\mathbb{N}}.$$

If $T \subset 2^{<\mathbb{N}}$ is a tree, $\lambda \in T$, but $\lambda 0, \lambda 1 \notin T$, (i.e. if $\lambda \in T$ has no $T-$extensions) then we say that λ is a *leaf* (of T).

2.2. **Computability Theory.** Our computability-theoretic notation is standard and follows that of [Soa16]. A computable nondecreasing unbounded function $h : \mathbb{N} \to \mathbb{N}$ is called an *order function*. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *computably approximable* or *limit computable* whenever there exists a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $f(x) = \lim_s g(x, s)$ exists for all $x \in \mathbb{N}$. Moreover, we say that the computable approximation g *obeys the order function* h, if, for every $x \in \mathbb{N}$, we have that

$$|\{s : g(x, s) \neq g(x, s+1)\}| \leq h(x),$$

where $|A| \in \mathbb{N}$ denotes the size of $A \subset \mathbb{N}$. Furthermore, we say that $X \subseteq \mathbb{N}$ is $h-$c.e. whenever X is limit computable via some $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that obeys h. Let $\{\varphi_e : e \in \mathbb{N}\}$

be a fixed uniformly computable enumeration of the partial computable functions, $\varphi_e : \mathbb{N} \to \mathbb{N}$, and let $\{\Phi_e : e \in \mathbb{N}\}$ denote a fixed uniformly computable enumeration of the oracle computable functionals, i.e. $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$. Recall that $\varphi(x) \downarrow$ denotes that the partial computable function φ eventually halts on input $x \in \mathbb{N}$, and that $\varphi_s(s) \downarrow$ ($\varphi_{e,s}(x) \downarrow$) says that the ($e^{th}$) partial computable function halts on input $x \in \mathbb{N}$ in at most $s \in \mathbb{N}$ steps. Similar defintions apply for $\Phi_e^\alpha(x) \downarrow$ and $\Phi_{e,s}^\alpha(x) \downarrow$, $e, x \in \mathbb{N}$, $\alpha \in 2^{<\mathbb{N}}$. A partial computable function $\varphi : \mathbb{N} \to \mathbb{N}$, is said to be *total* whenever $\varphi(x) \downarrow$ for all $x \in \mathbb{N}$. We say that $X \subset \mathbb{N}$ is *hyperimmune* whenever there is some function $f : \mathbb{N} \to \mathbb{N}$, $f \leq_T X$, i.e. X computes f, such that for every $e \in \mathbb{N}$ either there exist infinitely many natural numbers $x \in \mathbb{N}$, such that $f(x) > \varphi_e(x)$ whenever $\varphi_e : \mathbb{N} \to \mathbb{N}$ is total. Finally, we say that $X \subset \mathbb{N}$ is *DNR* (*diagonally nonrecursive*) whenever there is some $g \leq_T X$, $g : \mathbb{N} \to \mathbb{N}$, such that for all $e \in \mathbb{N}$, $g(e) \neq \Phi_e(e)$ whenever $\Phi_e(e) \downarrow$. Moreover, if $\emptyset' \subset \mathbb{N}$ denotes Turing's Halting Set and $g(e) \neq \Phi_e^{\emptyset'}(e)$, whenever $\Phi_e^{\emptyset'}(e) \downarrow$, $e \in \mathbb{N}$, then we say that X is $2-$*DNR*. If, in addition we have that $g(e) < h(e)$, for some nondecreasing function h and every $e \in \mathbb{N}$, then we say that X is $2 - h-$*DNR*.

2.2.1. *The Finite Injury Priority Method.* Finally, most of our proofs employ the Finite Injury Priority Method. Roughly speaking, this proof technique shows how one can satisfy a countable sequence of "requirements" (that usually culminates in the proof of a larger theorem) R_e, $e \in \mathbb{N}$, where R_e is said to have priority e and lower natural numbers correspond to higher priority requirements. Furthermore, for each $e \in \mathbb{N}$, the user of this method constructs a "proof module" M_e whose aim is to satisfy each R_e in isolation, with the added (key) condition that higher priority modules can only "disrupt" or "reset" lower priority modules and moreover this can only happen finitely many times over the course of the entire construction in which all modules act. From the point of view of Reverse Mathematics, this method usually requires a certain level of induction known as $\mathsf{B}\Sigma_2$ to be defined in the following subsection. More information on the Finite Injury Priority Method can be found in [Soa16, Chapter 7].

2.3. **Reverse Mathematics.** Reverse Mathematics is the subfield of Computability Theory and Proof Theory that aims to classify mathematical theorems, in the context of Second-Order Arithmetic and countable structures, according to their effective content. More specifically, in Reverse Mathematics one works over a weak base theory known as the Recursive Comprehension Axiom RCA_0 that, in the context of $\omega-$models and full induction[2], says:

- $\emptyset$ exists,
- whenever $X \subseteq \mathbb{N}$ exists and $Y \leq_T X$, then Y also exists, and
- whenever $X, Y \subseteq \mathbb{N}$ exist, then
$$X \oplus Y = \{2n : n \in X\} \cup \{2n + 1 : n \in Y\} \subseteq \mathbb{N}$$
exists.

The theorems typically analyzed in this context assert the existence of certain sets inside given structures. Roughly speaking, to show that theorem T_1 implies another theorem T_2 over RCA_0 it suffices to show that, given any computable structure corresponding to T_2, one can use finitely many iterations of solution sets for T_1 to compute a solution set for the given T_2-structure. For more details consult [Sim09].

We now state some theorems of Second-Order Arithmetic that will be relevant throughout the rest of our article. References (in order of appearance) for each of these theorems in the context of Reverse Mathematics are [Sim09, HS07, HSS09, NS]. More information on

[2]More information on induction schemes in the context of Second-Order Arithmetic follows.

Martin-Löf Randomness can be found in [DH10, Chapter 6]. A set $X \subseteq \mathbb{N}$ is *MLR* (Martin-Löf Random) if it passes every Martin-Löf test. Furthermore, X is *2-MLR* if it passes every Martin-Löf test relative to Turing's Halting Set $\emptyset'$.

$\mathsf{WKL_0}$: (Weak König's Lemma) Every infinite tree $T \subseteq 2^{<\mathbb{N}}$ contains an infinite path/chain.

CAC: (Chain-Antichain Theorem) Every infinite partial order contains either an infinite chain, or an infinite antichain

HYP : For every set $X \subseteq \mathbb{N}$ there is a set of pairs $Y \subseteq \mathbb{N} \times \mathbb{N}$ such that Y is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is hyperimmune relative to X.

$\mathsf{2-MLR}$: For every set $X \subseteq \mathbb{N}$ there is a set $Y \subset \mathbb{N}$ such that Y is $2-$MLR relative to X (i.e. Y is MLR relative to the Turing jump of X, denoted $X' \subset \mathbb{N}$).

An introduction to diagonally nonrecursive functions in the context of Reverse Mathematics can be found in [NS, Section 7]. Fix a nondecreasing function $h : \mathbb{N} \to \mathbb{N}$.

$\mathsf{h-2-DNR}$: For every set $X \subseteq \mathbb{N}$ there is a set $Y \subseteq \mathbb{N} \times \mathbb{N}$ that is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is $h-2-$DNR relative to X, i.e. f_Y is $h-$DNR relative to the Turing jump of X, denoted $X' \subset \mathbb{N}$.

$\mathsf{2-DNR}$: For every set $X \subseteq \mathbb{N}$ there is a set $Y \subseteq \mathbb{N} \times \mathbb{N}$ that is the graph of a function $f_Y : \mathbb{N} \to \mathbb{N}$ that is $2-$DNR relative to X, i.e. f_Y is DNR relative to X'.

More information on DNR Turing degrees and their relationship to Martin-Löf randomness can be found in [HS07, Nie09]. A well-known but unpublished result of J. Miller shows that $\mathsf{2-DNR}$ is equivalent to the Rainbow Ramsey Theorem for Pairs $\mathsf{RRT^2_2}$ over $\mathsf{RCA_0}$; see [NS, Theorem 7.4] and the following paragraph for more details.

2.3.1. *First-Order Reverse Mathematics.* We assume that the reader is familiar with the arithmetical hierarchy; for more information on this topic we invite the reader to consult either [Soa16, Chapter 4] or [AK00, Chapter 2]. Now, $\mathsf{RCA_0}$ includes a restricted induction scheme that only applies to Σ^0_1 formulas where a computable predicate is preceded only by existential quantifiers. Aside from asserting the existence of certain sets, theorems of Second-Order Arithmetic may also have First-Order (i.e. arithmetical, or number-theoretic) consequences and thus may require additional induction schemes (beyond Σ^0_1-induction) in their proofs. For example, it is well-known that CAC cannot be proved from Σ^0_1-induction alone. Rather, the first-order part of CAC includes a bounding principle for Σ^0_2-formulas that implies, but is not equivalent to Σ^0_1-induction, and will play an essential role in some of our proofs below.

$\mathsf{B\Sigma_2}$: Let $\psi(x)$ be a Σ^0_2-formula. Then, for any given $n \in \mathbb{N}$, if there exist $x_1, x_2, \ldots, x_n \in \mathbb{N}$ such that $\psi(x_i)$ holds for $1 \leq i \leq n$, then there exists $N \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in \mathbb{N}$ such that $\psi(y_i)$ holds for $1 \leq i \leq n$ and $\max\{y_i : 1 \leq i \leq n\} < N$.

An $\omega-$model is a model of Second-Order Arithmetic whose first-order part is the standard natural numbers $\omega = \{0, 1, 2, \ldots\}$ and therefore satisfies induction for all formulas. It is useful to keep in mind that, in the context of Reverse Mathematics, to show that a theorem T_1 *does not* imply another theorem T_2 it suffices to produce an $\omega-$model of $\mathsf{RCA_0}$ in which T_1 holds but T_2 does not.

2.3.2. *The Tree Antichain Theorem.* We now introduce the main theorem that we will examine from the point of view of Reverse Mathematics, namely the Tree Antichain Theorem, abbreviated TAC.

Definition 2.1 ($\mathsf{RCA_0}$). *We say that a tree $T \subseteq 2^{<\mathbb{N}}$ is completely branching if for all $\sigma \in T$ such that $\sigma k \in T$ for some $k \in \{0,1\}$ we have that $\sigma(1-k) \in T$ as well. In other words, every node $\sigma \in T$ is either a leaf, or else $\{\sigma 0, \sigma 1\} \subset T$.*

Additionally, for any given infinite completely branching tree $T \subseteq 2^{<\mathbb{N}}$, we say that $\{T_s : s \in \mathbb{N}\}$ is an enumeration of T whenever:

- $T_s \subseteq T$, *for all* $s \in \mathbb{N}$;
- $T_0 = \emptyset$;
- *for each* $s > 0$, $s \in \mathbb{N}$, *there exists a unique* $T_{s-1}-$*leaf* λ *such that* $T_s = \{\lambda 0, \lambda 1\} \cup T_{s-1}$; *and*
- $T = \cup_{s \in \mathbb{N}} T_s$.

It follows that T is Σ^0_1-definable (i.e. computably enumerable) if and only if there exists a uniformly computable enumeration of T.

Definition 2.2 ($\mathsf{RCA_0}$). *Let TAC be the theorem that says "every infinite Σ^0_1-definable completely branching tree $T \subseteq 2^{<\mathbb{N}}$, with corresponding enumeration $T = \cup_{s \in \mathbb{N}} T_s$, contains an infinite ($2^{<\mathbb{N}}-$)antichain."*

Remark 2.3 (An alternate characterization of TAC in Second-Order Arithmetic over $\mathsf{RCA_0}$). *The following equivalent version of TAC does not explicitly mention the computability-theoretic notion of Σ^0_1-definability in the guise of enumerations.*

$\mathsf{TAC_1}$: *Let $T \subseteq 2^{<\mathbb{N}}$ be an extendible tree containing infinitely many splittings, i.e. infinitely many $\sigma \in T$ such that $\sigma 0, \sigma 1 \in T$. Then T contains an infinite ($2^{<\mathbb{N}}-$)antichain.*

We leave it to the reader to verify that the TAC and $\mathsf{TAC_1}$ are equivalent over $\mathsf{RCA_0}$. From now on we will only work with TAC of Definition 2.2 above since we find it slightly more convenient.[3]

Remark 2.4. TAC *is not a standard theorem of mathematics that one would expect to find featured in a textbook, and so requires a (short and easy) proof. To see why TAC holds in Second-Order Arithmetic, note that $T \subseteq 2^{<\mathbb{N}}$ is an infinite partial order and therefore (by the Chain-Antichain Theorem) either contains an infinite chain or else it contains an infinite antichain. If T contains an infinite antichain then we are done, otherwise $T \subseteq 2^{<\mathbb{N}}$ contains an infinite chain/path $f \in [T] \subseteq 2^{\mathbb{N}}$. Now, for all $k \in \mathbb{N}$ let $\sigma_k = f\restriction k \in T \subseteq 2^{<\mathbb{N}}$ denote the unique initial segment of f of length k. Since T is completely branching, for each $k \in \mathbb{N}$ we have that both $\sigma_k 0, \sigma_k 1 \in T$. For each $k \in \mathbb{N}$ let $\tau_k = \sigma_k j \in T$ for the unique $j \in \{0,1\}$ such that $\tau_k \neq \sigma_{k+1} \subset f \in 2^{\mathbb{N}}$, i.e. $\tau_k \not\subseteq f$. It follows that $\{\tau_k : k \in \mathbb{N}\} \subseteq T \subseteq 2^{<\mathbb{N}}$ is an infinite antichain, as required.*

Note that we have actually given an effective proof of TAC via the Chain-Antichain property CAC for infinite partial orders.

2.4. **TAC is not equivalent to any "known" subsystem of Second-Order Arithmetic.** The overall main objective of this article is to catalog some of the computability-theoretic aspects of TAC and show that TAC is not equivalent to any other subsystem of Second-Order Arithmetic that has thus far been studied and is included in the "Reverse Mathematical Zoo"[4] that has been developed and promulgated by Dzhafarov and others (it suffices to do so in the context of $\omega-$models). Now, to see why this is the case, via the diagram given in `https://production.wordpress.uconn.edu/mathrmzoo/`

[3]At this point we should point out that the trees $T \subseteq 2^{<\mathbb{N}}$ mentioned in $\mathsf{TAC_1}$ have non-trivial (Cantor-Bendixson) rank, as defined in [Soa16, Definition 8.7.5]. The title of our article references this fact.

[4]See `rmzoo.math.uconn.edu` for more details, and `rmzoo.math.uconn.edu/diagrams` for visualizations of the Reverse Mathematical Zoo.

wp-content/uploads/sites/841/2014/09/diagram_oip.pdf, which we will refer to as simply diagram_oip.pdf, the reader can verify that, in the context of $\omega-$models, every subsystem mentioned there that is also a consequence of CAC either implies AMT (The Atomic Model Theorem; see [HSS09] for more details) or is implied by AMT (over RCA). Furthermore, together [Pat, Theorem 9.1.2] and Theorem 3.4 below say that TAC does not imply AMT, while [HSS09, Corollary 3.5] says that AMT does not imply TAC. Therefore, TAC cannot be equivalent to any subsystem of Second-Order Arithmetic mentioned in diagram_oip.pdf.

2.5. **TAC's relationship to the theory of Commutative Noetherian Rings.** As we previously mentioned in our introduction, TAC plays a role in the theory of Commutative Noetherian Rings. We now outline some of our currently unpublished results in this area. The following theorem of Noetherian Algebra originally led us to TAC.

NFP: Every Notherian ring has only finitely many minimal prime ideals.

Thus far we can show that (over $\mathsf{RCA_0}$)

$$\mathsf{CAC} + \mathsf{WKL_0} \rightarrow \mathsf{NFP} \rightarrow \mathsf{TAC}$$

and are currently working on showing that NFP implies $\mathsf{WKL_0}$.

3. Upper Bounds on TAC

The main goal of this section is to establish the weakness of TAC in two respects. First, we will point out (via Remark 2.4 above) that TAC obviously follows from the Chain-Antichain Theorem for infinite partial orders. Then, we will show that TAC also follows from $2-\mathsf{MLR}$, i.e. the existence of (relatively) 2-random sets.

Theorem 3.1. TAC *follows from* CAC *over* $\mathsf{RCA_0}$.

Proof. Remark 2.4 in the previous section. □

The following theorem will be useful in showing that $2-\mathsf{MLR}$ implies TAC below; it is essentially due to Kučera and we refer the reader to [NS, Section 7] and [BPS17, Theorem 2.8] for more details.

Theorem 3.2. *Let $h(x) = 2^x$, $x \in \mathbb{N}$. Then every 2-MLR set $X \subseteq \mathbb{N}$ computes an $h-2-$DNR function f.*

Corollary 3.3 ($\mathsf{RCA_0}$). $2-\mathsf{MLR}$ *implies* $h-2-\mathsf{DNR}$ *for $h(x) = 2^x$, $x \in \mathbb{N}$.*

Theorem 3.4 ($\mathsf{RCA_0}$). $h-2-\mathsf{DNR}$ *implies* TAC *for $h(x) = 2^x$, $x \in \mathbb{N}$.*

Proof. Let $T =\subseteq 2^{<\mathbb{N}}$, be an infinite Σ^0_1 completely branching tree, with corresponding enumeration $T = \bigcup_{s\in\mathbb{N}} T_s$, and let $h(x) = 2^x$, $x \in \mathbb{N}$. Now, let $\mathcal{A} \subset \mathbb{N}^{<\mathbb{N}}$ be the computable finitely branching tree such that at level $k \in \mathbb{N}$, all nodes σ of length k have exactly 2^{k+1} successor nodes, i.e. $\sigma\ell_k \in \mathcal{A}$, $0 \leq \ell_k < 2^{k+1}$, $\ell \in \mathbb{N}$. We will construct a partial computable oracle reduction $\Phi : \mathcal{A} \rightarrow T$ such that for every $h-2-$DNR function $f \in [\mathcal{A}] \subset \mathbb{N}^{\mathbb{N}}$ we have that

$$A = \{\Phi^g(k)\downarrow= a_k : k \in \mathbb{N}\} \subset T$$

is an infinite antichain, for some $g \leq_T f$, as required by TAC.

To construct Φ, inductively assume that for any given $\sigma \in \mathcal{A}$,

$$A_\sigma = \{\Phi^\sigma(i)\downarrow= a_i : 0 < i < |\sigma|,\ i \in \mathbb{N}\} \subset T$$

is a finite antichain of size $|\sigma| - 1$, and then wait for a computable finite approximation $T_s \subset T$, $s \in \mathbb{N}$, that contains an antichain $B_\sigma \subset T_s$, $|B_\sigma| = 2^{|\sigma|+1}$,

$$B_\sigma = \{b_i : 0 \leq i < 2^{|\sigma|+1}\} \subset T,$$

disjoint from A_σ and such that $A_\sigma \cup B_\sigma \subset T_s$ is a finite antichain of T. Now set

$$\Phi^{\sigma \ell_k}(k+1) \downarrow = b_{\ell_k} \in T,\ 0 \leq \ell_k < 2^k,$$

in this case. This inductive process defines a partial computable oracle reduction $\Phi : \mathcal{A} \to T$. Moreover, it is clear by construction that if Φ^f is defined on $\mathbb{N}^+$ then

$$A = \{\Phi^f(k) : k \in \mathbb{N}^+\} \subset T$$

is an infinite antichain. Next we will show that if f is $h-2-$DNR then f computes some $g \in [\mathcal{A}]$ such that Φ^g is defined on all of $\mathbb{N}^+$.

Before we continue it may be of help to the reader to note that the following three statements are easily shown to be equivalent for any given finite antichain $A \subset T$:

(1) $\emptyset \neq [T] \setminus [A] \subseteq 2^{\mathbb{N}}$, i.e. there is an infinite path through T for which no $\sigma \in A$ is an initial segment,
(2) the set of nodes of T not extending A is infinite, in particular these nodes form an infinite Σ^0_1 completely branching tree, and
(3) A is extendible to an infinite antichain of T.

We invite the reader to verify this easy equivalence that is really at the heart of the current proof. We will mainly mention (1) for the rest of the proof.

Now, note that if $A_0 \subset T$ is a finite antichain such that

$$\emptyset \neq [T] \setminus [A_0],$$

and $B_0 \subset T$ is another finite antichain such that the union $A_0 \cup B_0 \subset T$ is also a finite antichain, then (since the elements of B are mutually incomparable) it follows that there can *only be at most one* $\sigma \in B_0 \subset T$ for which we have

$$\emptyset = [T] \setminus [A_0 \cup \{\sigma\}],$$

and moreover this σ is Σ^0_2-definable, uniformly in A_0 and B_0. Thus, by our construction of Φ above, it follows that if $\sigma \in \mathcal{A}$, $|\sigma| = k \in \mathbb{N}$, is extendible to some $g_0 \in [\mathcal{A}]$, $g_0 \supset \sigma$, such that Φ^{g_0} is defined on $\mathbb{N}^+$, then there is at most one $0 \leq \ell_k < 2^{k+1}$, $\ell_k \in \mathbb{N}$, for which there is no $g_1 \in [\mathcal{A}]$, $g_1 \supset \sigma\ell_k$, such that Φ^{g_1} is total on $\mathbb{N}^+$. Furthermore, by definition it follows that any given $h-2-$DNR function f can compute some $g \in [\mathcal{A}]$ that avoids all such "dead ends" $0 \leq \ell_k < 2^{k+1}$, uniformly in $k \in \mathbb{N}$, and therefore produce a $g \leq_T f$ such that Φ^g is total on $\mathbb{N}^+$. □

Corollary 3.5 (RCA_0)**.** $2-\mathsf{MLR}$ *implies* TAC.

Proof. See Corollary 3.3 and Theorem 3.4 above. □

4. Lower Bounds on TAC

The main goal of this section is to establish the strength of TAC in two respects. First, we show that TAC implies HYP, over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$.

Theorem 4.1 ($\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$)**.** TAC *implies* HYP.

Proof. We will construct an infinite Σ^0_1 completely branching tree $T = \cup_{s\in\mathbb{N}} T_s \subset 2^{\mathbb{N}}$ such that

- $T_0 = \emptyset$,
- $T_{s+1} = T_s \cup \{\lambda 0, \lambda 1\}$, for some leaf $\lambda \in T_s$, and

- every infinite antichain A of T is of hyperimmune Turing degree via one of two reductions $\Phi^A, \Psi^A : \mathbb{N} \to \mathbb{N}$ that we will also contruct along with T.

For any $\sigma \in T \subseteq 2^{<\mathbb{N}}$ let $s \in \mathbb{N}$ be minimal such that $\sigma \in T_s \setminus T_{s-1}$ and define

$$s_\sigma = \begin{cases} 2s, \text{ if } \sigma(|\sigma|-1) = 0, \\ 2s+1, \text{ if } \sigma(|\sigma|-1) = 1. \end{cases}$$

Then, for any given subset $\Sigma \subseteq T$, let $\Xi(\Sigma) = \{s_\sigma : \sigma \in \Sigma\} \subset \mathbb{N}$. It follows that the correspondence Ξ of the previous sentence yields a natural uniformly computable order-preserving (i.e. continuous) map between the finite subsets of T and $\mathbb{N}^{<\mathbb{N}}$, i.e. $\Xi : 2^{<T} \to \mathbb{N}^{<\mathbb{N}}_{<} \cong \mathbb{N}^{<\mathbb{N}}$, and also between infinite subsets of T and $\mathbb{N}^{\mathbb{N}}$, i.e. $\Xi : 2^T \to \mathbb{N}^{\mathbb{N}}$, such that for every finite $A \subset \mathbb{N}$ and corresponding $\sigma_A \in 2^{<T}$, we have that

$$|\Xi(A)| = |\sigma_A| = |A| \in \mathbb{N}.$$

By $\mathsf{B}\Sigma_2$, for each $x \in \mathbb{N}$, let

$$M_x = \max_{s\in\mathbb{N}}\{(\exists x' \leq x)[\phi_{x',s}(x)\downarrow \text{ and } \phi_{x',s}(x)\uparrow]\} \in \mathbb{N}.$$

Our argument is a finite injury priority argument with requirements

R_e : If the $\varphi_e : \mathbb{N} \to \mathbb{N}$ is total then there exists $x_e \in \mathbb{N}$ such that for each $T-$antichain $\alpha_e \in \Xi^{-1}[(M_{x_e})^{=x_e}] \subset 2^{<T}$ there exists $x_{\alpha_e} \in \mathbb{N}$ such that

$$\varphi_e(x_{\alpha_e}) < \Phi^{\alpha_e}(x_{\alpha_e})\downarrow.$$

Fix an infinite antichain $A = \{\sigma_k : k \in \mathbb{N}\} \subset T$ such that $s_{\sigma_{k_0}} < s_{\sigma_{k_1}}$ whenever $k_0 < k_1$, $k_0, k_1 \in \mathbb{N}$, and

$$\Xi(A) = \langle s_{\sigma_0}, s_{\sigma_1}, \cdots \rangle \in \mathbb{N}^{\mathbb{N}}.$$

Now, assume that the requirement R_e, $e \in \mathbb{N}$, is satisfied via $x_e \in \mathbb{N}$ and let

$$\alpha_e = A\restriction x_e = \{\sigma_k : k \leq x_e\} \subset T, \quad i.e.\ \alpha_e \in 2^{<T}.$$

In this case there are two possibilities. The first possibility says that we could have

$$\sigma_{x_e} \in T_{M_{x_e}} \subset T \subset 2^{<\mathbb{N}}$$

which is equivalent to saying that

$$\sigma_i \in T_{M_{x_e}} \subset 2^{<\mathbb{N}},\ i = 0, 1, \ldots, x_e \in \mathbb{N},$$

or that

$$s_{\sigma_i} \leq M_{x_e}, i = 0, 1, 2, \ldots, x_e \in \mathbb{N},$$

or that

$$\langle s_{\sigma_0}, s_{\sigma_1}, \ldots, s_{\sigma_{x_e}} \rangle \in (M_{x_e})^{=x_e},$$

or (finally) that

$$\alpha_e \in \Xi^{-1}[(M_{x_e})^{=x_e}] \subset 2^{<T}.$$

In this case the fact that R_e is satisfied guarantees us that our reduction $\Phi^{\alpha_e}(x_e)\downarrow > \varphi_e(x_e)$. Otherwise, we would have that $s_{\sigma_{x_e}} > M_{x_e}$. Now, if we set

$$\Psi^\alpha(|\alpha|-1) = s_{\alpha(|\alpha|-1)} \in \mathbb{N},\ \alpha \in 2^{<T},\ \alpha(k) \in T,\ k < |\alpha|,$$

then by our construction above it follows that

$$\Psi^{\alpha_e}(x_e) = s_{\sigma_{x_e}} > M_{x_e} \geq \varphi_{x'}(x_e),\ 0 \leq x' \leq x_e,$$

whenever $\varphi_{x'}(x_e)\downarrow$. We have now shown that if requirement R_e is satisfied via $x_e \in \mathbb{N}$ and φ_e is total, then, for any given antichain $A \subset T$ we either have that $\Phi^A(x_e) > \varphi_e(x_e)$, or else we have that $\Psi^A(x_e) > \varphi_{x'}(x_e)$, for all $0 \leq x' \leq x_e$. It follows that either

(1) there exist infinitely many $e \in \mathbb{N}$ and corresponding $x_e \in \mathbb{N}$ for which $\Psi^A(x_e) \downarrow> \varphi_{x'}(x_e)$, for all $0 \leq x' \leq x_e$, or else
(2) for almost all natural numbers e, we have that $\Phi_e^A(x_e) \downarrow> \varphi_e(x_e)$.

In the first case (1) it is easy to see that $\Psi^A : \mathbb{N} \to \mathbb{N}$ is hyperimmune via the (uniform set of) witnesses $\{x_e : e \in \mathbb{N}\} \subseteq \mathbb{N}$. In the second case (2), given a total computable function $\varphi = \varphi_{e_0}$, via the Padding Lemma [Soa16, Lemma 1.5.2] it follows that there exist infinitely many $e \in \mathbb{N}$ such that $\varphi_{e_0} = \varphi_e$ and $\Phi^A(x_e) > \varphi_e(x_e)$, implying that $\Phi^A : \mathbb{N} \to \mathbb{N}$ is hyperimmune.

We defined $\Psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ above, i.e.

$$\Psi^\alpha(|\alpha| - 1) = s_{\alpha(|\alpha|-1)} \in \mathbb{N}.$$

Therefore, all that is left to do now is to produce/eumerate an infinite Σ_1^0 completely branching tree $T = \cup_{s\in\mathbb{N}} T_s$, and corresponding reductions $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$, such that every requirement R_e, $e \in \mathbb{N}$, is satisfied. Our verification of this fact via the finite injury priority method will closely follow [DH10, Theorem 8.21.1] and/or [KM17, Theorem 6.1], both of which are modeled on Martin's proof that the hyperimmune Turing degrees have measure one.

For each $e \in \mathbb{N}$, let $M_{e,s} \in \mathbb{N}$ denote a uniform computable approximation to $M_e \in \mathbb{N}$ such that

$$\lim_s M_{e,s} = M_e$$

for each $e \in \mathbb{N}$. Now, fix $e \in \mathbb{N}$ and assume, via $\mathsf{B}\Sigma_2$[5], that s is the last stage at which any requirement of the form $R_{e'}$, $0 \leq e' < e$, acts and resets R_e. By our construction (that follows), in this case we will have already produced corresponding finite sequences

$$\emptyset = \sigma_{-1} \subset \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{e-1} \in T,$$

and

$$0 = s_{-1} < s_0 < s_1 < \cdots s_{e-1} \in \mathbb{N}$$

such that for each $-1 \leq k < e$ we will have that σ_k is a uniform prefix for every element of $T \setminus T_{s_k}$.

To satisfy requirement R_e, $e \in \mathbb{N}$, at stage $s \in \mathbb{N}$, first we choose a large $x_e = x_{e,s} \in \mathbb{N}$ such that $x_e > |T_s| + 2$. By our construction it will follow that x_e is redefined only if R_e is reset by a higher priority requirement $R_{e'}$, $0 \leq e' < e$, at some later stage of the construction. Hence, in the current context (see our application of $\mathsf{B}\Sigma_2$ above for more details) x_e is never again reset. Next, choose the leftmost T_s−leaf $\lambda \supseteq \sigma_{e-1}$ and set $T_{s+1} = T_s \cup \{\lambda 0, \lambda 1\}$. Now, since $x_e > |T_s| + 2$ it follows that for every $\Sigma \in \Xi^{-1}[\mathbb{N}^{=\ell_e}] \subset 2^{<T}$, if $\sigma = \Sigma(|\Sigma| - 1) \in T$ then $\sigma \supset \sigma_{e-1}$. Let

$$\tau_1, \tau_2, \ldots, \tau_{N_e} \in \Xi^{-1}[(M_{e,s})^{=\ell_e}] \subset 2^{<T},\ N_e \in \mathbb{N},\ N_e > 1,$$

be an enumeration of the finite antichains of $\Xi^{-1}[(M_{e,s})^{=\ell_e}]$, and

$$\rho_k = \tau_k(|\tau_k| - 1) \in T,\ 1 \leq k \leq N_e.$$

Let $k = 1$. To satisfy R_e we take $\rho_k \in T_s$ and choose a large number $x_{e,k} = x_{e,\rho_k} \in \mathbb{N}$ such that

$$x_{e,k} > \max_x \{x : (\exists \sigma \in T_s)[\Phi_s^\sigma(x) \downarrow]\}.$$

Then we set

$$\sigma_e = \sigma_{e,k} = \rho_k \supset \sigma_{e-1},$$

[5]By our remarks in the following paragraphs it will eventually follow that each requirement R_e is reset at most $1 + 2 + 3 + \cdots + e = e(e + 1)/2$−many times, thus justifying our use of $\mathsf{B}\Sigma_2$ in the current context of the finite injury priority method.

thus promising (for now) every new node that we add to T from now on will extend $\rho_k \in T$. Under this hypothesis (i.e. if this hypothesis persists indefinitely then) it follows that the finite antichain $\tau_k \in 2^{<T}$ above is not the initial segment of any infinite antichain of T. Furthermore, we say that $x_{e,k} = x_{e,\rho_k}$ is currently reserved for requirement e. We may change our current choice of ρ, however, when we see that $\varphi_{e,s'}(x) \downarrow$ at some later stage $s' > s$, $s' \in \mathbb{N}$. In this case we react by:

- setting $\Phi^{\rho_k}_{s'}(x_{e,k}) = \varphi_{e,s'}(x_{e,k}) + 1$,
- resetting all requirements $R_{e'}$, $e' > e$, $e' \in \mathbb{N}$, and
- if $k + 1 \leq N_e$ then:
  - resetting the current $\rho = \rho_k$ to its new value $\rho = \rho_{k+1}$ (with corresponding τ_k), and
  - choosing some new $x_{e,k+1} = x_{e,\rho_{k+1}} \in \mathbb{N}$ as we did in the first sentence of the current paragraph with s' replacing s.

This iterative effective procedure works until we exhaust all $k = 1, 2, \ldots, N_e \in \mathbb{N}$ and corresponding $\rho_k = \tau_k(|\tau_k| - 1) \supset \sigma_{e-1}$. In this case, by our construction above it follows that if $A \subset T$ is an infinite antichain with initial segment $\tau_k \subset A$, $1 \leq k \leq N_e$, and if $\varphi_e : \mathbb{N} \to \mathbb{N}$ is total, then we have that

$$\Phi^A(x_{e,k}) = \Phi^{\tau_k}(x_{e,k}) = \varphi_e(x_{e,k}) + 1 > \varphi_e(x_{e,k}).$$

Finally, if we never exhaust our enumeration $k = 1, 2, \ldots, N_e$, it follows that φ_e is not total. In this case R_e is trivially satisfied.

Meanwhile, at the current stage $s \in \mathbb{N}$ of the construction we ensure that $\Phi^\rho(x) \downarrow$ for all $\rho \in 2^{<T_s}$ and $x \in \mathbb{N}$, $x \leq s$, unless x is currently reserved for requirement e and $\rho \supseteq \rho_k$, $1 \leq k \leq N_{e,s}$.

Finally, if we have that $M_{e',s} < M_{e',s+1}$, for some $e' \leq e$, $e' \in \mathbb{N}$, then we reset all requirements $R_{e''}$, $e'' \geq e'$, $e'' \in \mathbb{N}$, and proceed to stage $s + 1$ with corresponding finite sequences

$$\{\sigma_k = \sigma_{k,s+1} : -1 \leq k \leq e - 1\} \subset T_s = T_{s+1} \subset 2^{<\mathbb{N}}$$

and

$$\{s_k = s_{k,s+1} : -1 \leq k \leq e - 1\} \subset \mathbb{N}.$$

This completes the construction of our infinite Σ^0_1 completely branching tree $T = \cup_{s\in\mathbb{N}} T_s \subset 2^{<\mathbb{N}}$, and functionals $\Phi, \Psi : 2^{<T} \times \mathbb{N} \to \mathbb{N}$. Note that, by our construction above and $\mathsf{B}\Sigma_2$, it follows that each requirement R_e, $e \in \mathbb{N}$, can only be reset finitely many times by the higher priority requirements $R_{e'}$, $e' < e$, and this suffices to prove the current theorem. □

We now turn our attention to showing that TAC does not follow from $\mathsf{WKL_0}$, even in the context of $\omega-$models.

Theorem 4.2. *For any order function $h : \omega \to \omega$,* TAC *implies the existence of a set $A \subset \omega$ that is not $h-$c.e.*

Proof. Let $h : \omega \to \omega^+$ be a computable order function. To do this, in the coming paragraphs we will employ a finite injury argument to construct an infinite Σ^0_1 completely branching tree $T \subset 2^{<\omega}$ with corresponding approximation $T = \cup_{s\in\mathbb{N}} T_s$ such that every potential Δ^0_2 set that obeys the order function h is not an infinite $T-$antichain. Let $f(e, x, s) : \omega^3 \to 2^{<\omega}$ be a uniformly computable approximation to every $h - \Delta^0_2-$set $X_e \subseteq 2^{<\omega}$ such that for every $e, x \in \omega$,

$$f_e(x) = \lim_s f(e, x, s) \in T$$

obeys h (therefore exists) and yields a ($2^{<\omega}-$)nondecreasing function of x. Then for every $e \in \omega$, $e > 0$, we will satisfy the requirement

R_e : There exists $x_e \in \omega$ such that if $f_e(x_e) = \lim_s f_e(x_e) = \sigma_e \in T$ then $\sigma \supset \sigma_e$ for almost all $\sigma \in T$. It follows that σ_e cannot be included in any infinite antichain of T and so the image of f_e is not an infinite $T-$antichain.

We now construct $T = \cup_{s\in\omega} T_s \subset 2^{<\omega}$ in stages $s \in \omega$. At stage $s = 0$ we enumerate the root $\emptyset \in 2^{<\omega}$ into T_0 and $\Sigma_0 = \{\} \subset 2^{<\omega}$, and reset all requirements R_e, $e \in \omega$. Now, at stage $s > 0$ assume that we are given a (possibly empty) finite sequence of $T-$nodes

$$\Sigma_s = \{\sigma_j = \sigma_{j,s} : 0 \leq j \leq k\}, \quad \emptyset = \sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \in T_s \subset T,\ k \in \omega,$$

and corresponding

$$x_0 = x_{0,s} < x_1 = x_{1,s} < x_2 = x_{2,s} < \cdots < x_k = x_{k,s}$$

such that for each $0 \leq e \leq k$ we have that either:

- $f_e(x_e) \notin T_s$, or
- $f_e(x_e) \not\supseteq \sigma_{e-1}$, or else
- $f_e(x_e) = \sigma_e \in T_s$.

Now, suppose that there exists $0 \leq e \leq k$ for which $f_e(x_e, s-1) \neq f_e(x_e, s)$. In this case we

- reset all requirements $R_{e'}$, $e' > e$, $e' \in \omega$, including all currently defined $\sigma_{e'}$ and $x_{e'}$, for all $e' > e$,
- set $\sigma_{e',s} = \sigma_{e',s+1}$ for all $0 \leq e' < e$, and
- set $\sigma_{e,s+1} = \begin{cases} f_e(x_e), \text{ if } f_e(x_e) \in T_s \text{ and } f_e(x_e) \supset \sigma_{e-1}, \\ \sigma_{e-1,s}0 \in T_s, \text{ otherwise.} \end{cases}$, and finally
- proceed to the next stage of the construction with $T_s = T_{s+1}$,

  $$\Sigma_{s+1} = \{\sigma_{e',s+1} : 0 \leq e' \leq e\} \subset T_{s+1},$$

  and corresponding

  $$x_0 = x_{0,s+1} < x_1 = x_{1,s+1} < x_2 = x_{2,s+1} \cdots < x_e = x_{e,s+1} \in \omega.$$

On the other hand, if such a $0 \leq e \leq k$ (as described above) does not exist at stage s, then we set out to satisfy R_{k+1} by:

- setting $T_{s+1} = T_s \cup \{\lambda 0, \lambda 1\}$, where $\lambda \in T_s$ is the leftmost leaf of T_s extending $\sigma_k \in T_s$, with the following caveat:
  - If either $\lambda 0, \lambda 1 \in \{\sigma_e : 0 \leq e \leq k\} \subseteq T_{s+1}$ then let $0 \leq e \leq k$, $e \in \omega$, be minimal such that $\sigma_e \in \{\lambda 0, \lambda 1\}$ and reset all requirements $R_{e'}$, $e' > e$. Then proceed to the next stage $s+1$ with

    $$\Sigma_{s+1} = \{\sigma_{j,s} = \sigma_{j,s+1} : 1 \leq j \leq e\} \subset T_{s+1}$$

    and corresponding

    $$\{x_{j,s+1} = x_{j,s} : 0 \leq j \leq e\} \subset \omega.$$

  And finally,
- setting $x_{k+1} = x_{k+1,s} = |T_s| + 1 \in \omega$ and

  $$\sigma_{k+1} = \begin{cases} f_{k+1}(x_{k+1}), \text{ if } f_{k+1}(x_{k+1}, s) \in T_{s+1}, \\ \sigma_k 0 \in T_{s+1}, \text{ otherwise.} \end{cases}$$

  By our choice of x_{k+1} it follows that if the image of $f_{e_0}(\cdot, t_0) : \omega \to 2^{<\omega}$ on domain $x = 0, 1, 2 \ldots, x_{k+1} \in \omega$ yields a T_s-antichain for some fixed $e_0, t_0 \in \omega$, then $f_{e_0}(x_{k+1}, t') \not\supseteq \sigma_k$ at all subsequent stages $t' \geq t_0$ and so it follows that we will always have

  $$\sigma_{j,s'} = \sigma_j \subset \sigma_{j+1} = \sigma_{j+1,s'}$$

whenever $f_j(\cdot, t) : \{0, 1, 2, \ldots, x_j\} \to T_t,\ j, t \in \omega$ yields a T_t-antichain.

For each $x \in \omega$, let

$$H(x) = \prod_{i=0}^{x} 2 \cdot h(x) = 2^{x+1} \prod_{i=0}^{x} h(x),$$

where $h : \omega \to \omega^+$ is our fixed order function above. To finish off the proof, we invite the reader to confirm the success of our given finite injury priority argument above by verifying that:

- Each requirement R_e can only be reset by higher priority requirements $R_{e'}$, $e' < e$, and more specifically R_e is reset by $R_{e'}$, $e' < e$, whenever our computable approximation $f_{e'}$ changes or T_s grows to contain $f_{e'}$ and (all together) this may happen at most $H(e-1)-$many times. It follows that for each $e \in \omega$ there is a stage $s_e \in \omega$ such that R_e is no longer reset after stage s_e.
- Assuming that R_e is no longer reset at or after stage $s_e \in \omega$ and

$$f_e(x) = \lim_s f_e(x, s) : \omega \to \omega.$$

To verify that the range of f_e is not an infinite $T-$antichain note that by our construction above there exists $x_e = x_{e,s_e}$ such that for all $s \geq s_e$ and corresponding

$$x = x_{e,s} = x_{e,s_e}$$

we have that either:
  - $f_e(x) \notin T$, or else
  - $f_e(x) = \sigma_e \in T$ and almost all nodes of T extend σ_e.

□

Corollary 4.3. TAC *does not follow from* WKL_0 *over* RCA_0. *More specifically, there is an* $\omega-$*model of* $\mathsf{WKL}_0 + \neg\mathsf{TAC}$.

Proof. [NS, Proposition 7.6] asserts the existence of an $\omega-$model of WKL_0, $\mathfrak{M} = (\omega, \mathcal{S})$, such that for every $S \in \mathcal{S}$ there exists $k \in \omega$ such that S is k^n-c.e. Therefore, every $S \in \mathcal{S}$ is $h-$c.e. for any computable order function that dominates every exponential function. However, it follows from Theorem 4.2 above that $\mathfrak{M}$ cannot be a model of TAC. □

5. An Open Question

This article leaves many open questions concerning the precise strength of TAC, but we think that the following one is the most important for now.

Question 5.1. *Does* $2-\mathsf{DNR}$ *imply* TAC *over* RCA_0*? In the context of* $\omega-$*models?*

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