GALVIN'S "RACING PAWNS" GAME, INTERNAL HYPERARITHMETIC COMPREHENSION, AND THE LAW OF EXCLUDED MIDDLE

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Abstract. We show that the fact that the first player ("white") wins every instance of Galvin's "racing pawns" game (for countable trees) is equivalent to arithmetic transfinite recursion. Along the way we analyse the satisfaction relation for infinitary formulas, of "internal" hyperarithmetic comprehension, and of the law of excluded middle for such formulas.

1. Introduction

The proof of closed determinacy (Gale-Stewart [GS53]) is often summed up as "do not lose", or slightly more formally, picking a move which does not result in a situation from which the opponent has a winning strategy. This proof masks a transfinite recursive process. For a more revealing argument, let $\mathcal{U}$ be an open subset of Baire space ω^ω, and consider the game $G_{\mathcal{U}}$, in which the players (say I and II) alternate picking natural numbers to eventually produce an element of Baire space; player I wins the play if the sequence produced is an element of $\mathcal{U}$. For countable ordinals α, we define subsets U_α and V_α of $\omega^{<\omega}$ (the collection of all finite sequences of natural numbers). The set V_α will be a collection of positions from which player II does not have a winning strategy; the set U_α will be a collection of positions from which player I has a winning strategy. We let U_0 be the collection of strings σ for which $[\sigma]$, the collection of all infinite extensions of σ, is contained in $\mathcal{U}$. Given U_α, we let V_α be the collection of strings σ, all of whose immediate extensions $\sigma\hat{}n$ lie in U_α. And given $V_{<\alpha} = \bigcup_{\beta<\alpha} V_\beta$, we let U_α be the collection of strings σ which have some immediate extension $\sigma\hat{}n$ in $V_{<\alpha}$. Then player I has a winning strategy for the game $G_{\mathcal{U}}$ if the empty sequence $\langle\rangle$ is "ranked", that is, if it is an element of $U_{<\omega_1}$, and player II has a winning strategy if $\langle\rangle$ is not ranked. The strategy for player I, given a position $\sigma \in U_\alpha$, is to choose an extension in $V_{<\alpha}$; the strategy for player II, given a position $\sigma \notin V_{<\omega_1}$, is to choose an extension outside $U_{<\omega_1}$.

Proof-theoretically, the argument above uses the minimal subsystem of second-order arithmetic in which a good theory of ordinals is available, namely arithmetical transfinite recursion ($\mathtt{ATR}_0$); an overspill argument in $\mathtt{ATR}_0$ shows the existence of the sets U_α and V_α. A reversal is often given (as in [Sim09]) by Σ^1_1-separation. However, it is possible to give a direct argument: from clopen determinacy, in fact, one can deduce the existence of the transfinite iteration $X^{(\alpha)}$ of the Turing jump of a set X. While probably well-known, we have not found such an argument in the literature. In this paper we develop the theory of infinitary logic within second-order arithmetic, and among other results, use it to give a proof of $\mathtt{ATR}_0$ from clopen determinacy.

We also investigate the strength of determinacy for a particular class of games. Galvin's "racing pawns" game F_T is played on a well-founded tree T. Two players, W and B, take turns moving one of two pawns, marked "white" and "black", starting at the root of the tree, and at each step moving to a child (an immediate successor). The player W has the first move. The winner is the player whose pawn reaches a leaf first. However, each player can move *either* pawn; the complexity of the game follows from the fact that an optimal move may be either pushing one's own pawn toward a leaf, or the opponent's pawn away from leaves.

A tricky proof (Galvin; see [Gra85]) shows that for any well-founded tree T, the player W has a winning strategy for the game F_T. Grantham [Gra85] carried out a detailed ordinal analysis of the games F_T and described a winning strategy for W along the lines described above for an open game $G_{\mathfrak{U}}$; Grantham's analysis for F_T is of course much more complicated. We show that Galvin's result is equivalent to $\mathtt{ATR}_0$. In other words, while seemingly much more restricted than general clopen or open games, the determinacy of the racing pawns games is equivalent to the determinacy of all open games.

In one direction, we analyze Galvin's original proof and show using the existence of ω-jumps ($\mathtt{ACA}_0^+$) that the player B has no winning strategy for the game F_T; we then code Galvin's game into a clopen game $G_{\mathfrak{U}}$ and invoke clopen determinacy to show that Galvin's theorem is provable in $\mathtt{ATR}_0$. We remark that it is still open whether the bound $\mathtt{ACA}_0^+$ can be improved. In the other direction, we use infinitary logic. We in fact show that two natural statements regarding infinitary logic – the law of excluded middle, and a comprehension principle which we name internal hyperarithmetic comprehension – are equivalent to $\mathtt{ATR}_0$, and follow from Galvin's theorem.

For more background on arithmetical transfinite recursion, and on reverse mathematics in general, see [Sim09]. For a detailed account of the interplay of determinacy and second-order arithmetic, see [MSar].

1.1. **Formalising Galvin's theorem.** From now, we work in the system $\mathtt{RCA}_0$ of recursive comprehension, consisting of Σ^0_1-induction and Δ^0_1-comprehension, namely the system which corresponds to computable mathematics.

A *tree* is a partial ordering with a least element, for which every principal initial segment is linearly ordered and finite; usually the structure on a tree is augmented by the immediate predecessor relation. Every tree is effectively isomorphic to a subset of $\omega^{<\omega}$ which is closed under taking initial segments, with the ordering given by string extension. So from now we assume that all trees are downward closed subsets of $\omega^{<\omega}$. A tree is *well-founded* if it has no infinite paths, that is, if for no $f \in \omega^\omega$ do we have $f\restriction_n \in T$ for all n.

For a well-founded tree T, we code the racing pawns game on T by a clopen game $G_{\mathcal{W}(T)}$. The details of such coding are not that important, as long as they are effective. The most direct coding is as follows. Let $\sigma \in \omega^{\leqslant\omega}$. Identifying the player W with the player I, and the player B with the player II, recursively for $n < |\sigma|$, the instruction $\sigma(n) = 2m$ is interpreted as telling the player moving at step n (I if n is even, II if n is odd) to move the white pawn from its current location τ to the location $\tau\hat{\ }m$; the instruction $\sigma(n) = 2m + 1$ is interpreted as telling the player to move the black pawn from τ to $\tau\hat{\ }m$.

Not all moves are legal: if a string σ tells a player to move a pawn from τ to $\tau\hat{}m$, but $\tau\hat{}m \notin T$, then the move is illegal in the game F_T. In the game $G_{\mathcal{W}(T)}$, this would be interpreted as the player forfeiting the game. The success set $\mathcal{W}(T)$ is thus defined to be the collection of (infinite) strings which determine plays at which B forfeits first, or the white pawn reaches a leaf of T. We *define* the racing pawns game F_T to be the game $G_{\mathcal{W}(T)}$.

It is immediate that $\mathcal{W}(T)$ is open. In fact, because T is well-founded, it is clopen: every infinite sequence has a finite initial segment which either determines an illegal play or which directs one of the pawns to a leaf of T. Indeed, a clopen code for $\mathcal{W}(T)$ (an open code for $\mathcal{W}(T)$ and an open code for its complement) can be effectively obtained from T. So $\mathtt{RCA}_0$ implies that for any well-founded tree T, $\mathcal{W}(T)$ exists.

Definition 1.1. Galvin's theorem, which we denote by $\mathtt{WW}$, is the statement that for any well-founded tree T, the player W has a winning strategy for the racing pawns game F_T.

In Section 2.1 we show:

Theorem 1.2 ($\mathtt{ACA}_0^+$)**.** *For any well-founded tree T, the player* B *does not have a winning strategy for the game F_T.*

Since $\mathtt{ATR}_0$ implies clopen determinacy, since $\mathtt{RCA}_0$ is sufficient to show that it is impossible for both players to have a winning strategy for a given game, and since $\mathtt{ATR}_0$ implies $\mathtt{ACA}_0^+$, we get that $\mathtt{ATR}_0$ implies Galvin's theorem $\mathtt{WW}$. As we mentioned above, we do not know if Theorem 1.2 can be improved; perhaps it is possible to prove in $\mathtt{RCA}_0$ that B does not have a winning strategy for any game F_T.

We do get a reversal, but before we indicate how, we explain why we would like to pass to a narrower class of games. The point is that set-theoretically, Galvin's game F_T depends only on the isomorphism type of the tree T. The coding necessary for formalising the game in second-order arithmetic introduces extra information – the numbers in the sequences which are the elements of T. This information can be thought of as *labels* on the nodes of the tree T. This introduces unwarranted strength to the game. For example, in a general (labelled) tree $T \subseteq \omega^{<\omega}$, the collection of leaves of T is $\Pi^0_1(T)$, and not necessarily computable in T, and so even telling when a play has already resulted in a win is not effective.

Consider the tree described in Figure 1. Let $\langle \varnothing'_s \rangle$ be an effective enumeration of the halting set $\varnothing'$. The tree T consists of the empty sequence $\langle\rangle$, of the sequence $\langle 0 \rangle$, the sequences $\langle 0n \rangle$ for all $n < \omega$, and the sequences $\langle 0ns \rangle$ for all n and s such that $n \in \varnothing'_s$. Clearly T is computable, and so $\mathtt{RCA}_0$ implies its existence.

Claim 1.3. *Any winning strategy for W for the game F_T computes $\varnothing'$.*

Proof. Fix $e \notin \varnothing'$, and let σ be a winning strategy for W for the game F_T.

We first observe that W's first move according to σ must be moving the white pawn to $\langle 0 \rangle$. This is because the only other possible move is moving the black pawn to $\langle 0 \rangle$. But then B can win the game by moving the black pawn to $\langle 0e \rangle$.

Now playing against σ, wanting to enquire whether a given number n is in $\varnothing'$ or not, we let B move the white pawn to $\langle 0n \rangle$. W's next move (following σ) tells us whether n is in $\varnothing'$ or not. If W moves the white pawn to a node $\langle 0ns \rangle$, where

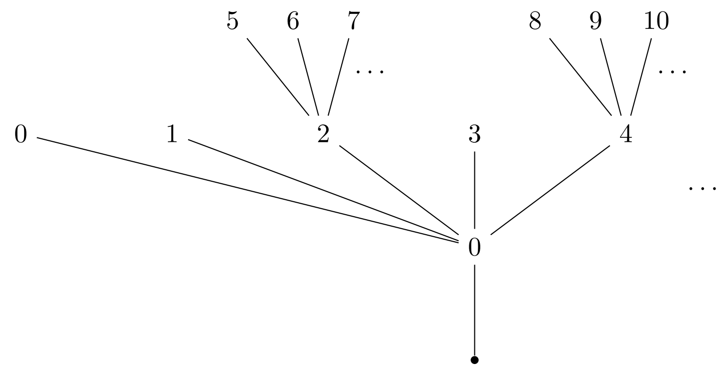

FIGURE 1. Computing $\varnothing'$ with a labelled tree. The number 2 enters $\varnothing'$ at stage 5, and the number 4 enters $\varnothing'$ at stage 8. The numbers 0,1 and 3 are not elements of $\varnothing'$.

$n \in \varnothing'_s$, then certainly $n \in \varnothing'$. Otherwise, we claim that $n \notin \varnothing'$. That is, we claim that $\langle 0n \rangle$ is a leaf of T, and so that W has already won the play.

Suppose for contradiction that $\langle 0n \rangle$ is not a leaf of T. There are two possibilities: if W next moves the white pawn, to a node $\langle 0ns \rangle$ where $n \notin \varnothing'_s$, then as $\langle 0ns \rangle \notin T$, this would be a loss for white, which contradicts the assumption that σ is a winning strategy for W. Otherwise, W moves the black pawn to $\langle 0 \rangle$; then B can respond by moving the black pawn to the leaf $\langle 0e \rangle$ and winning – again a contradiction. $\square$

Formalising in $\mathtt{RCA}_0$, we see that $\mathtt{WW}$ implies $\mathtt{ACA}_0$. This is somewhat unsatisfying, however, because it takes advantage of the difficulty in determining leaves. To address this, we introduce a restricted notion of trees.

Definition 1.4. A tree $T \subseteq \omega^{<\omega}$ is *unlabeled* if for every $\rho \in T$, the set $\{d \mid \rho\hat{\ }d \in T\}$ is an initial segment of ω (not necessarily proper, and possibly empty).

If T is unlabeled, then T can compute whether a string $\rho \in T$ is a leaf simply by checking if $\rho\hat{\ }0 \in T$. We shall refer to trees as *labeled trees* when we wish to emphasize that we are not restricting our attention to unlabeled trees.

Definition 1.5. The statement $\mathtt{WWU}$ is the restriction of Galvin's theorem to unlabelled trees, namely the statement that for any well-founded unlabeled tree T, the player W has a winning strategy for the racing pawns game F_T.

Certainly $\mathtt{WW}$ implies $\mathtt{WWU}$. Among other results, our main theorem (1.20) will state that $\mathtt{WW}$ and $\mathtt{WWU}$ are both equivalent to $\mathtt{ATR}_0$ (over $\mathtt{RCA}_0$).

1.2. **Infinitary propositional logic.** Classically, for a language $\mathcal{L}$, the infinitary logic $\mathcal{L}_{\omega_1,\omega}$ is obtained from the atomic $\mathcal{L}$-formulas by closing under quantification, negation, and countable conjunctions and disjunctions. There are ordinals hidden in this definition – formally, we need to define by recursion on $\alpha < \omega_1$ the collection of formulas of rank α. Usually, implied within the definition, is the fact that the ranks of formulas are comparable, so that we indeed get an increasing collection of formulas. Comparability of well-orderings is equivalent to $\mathtt{ATR}_0$. But if we drop the assumption of comparability, the *definition* of infinitary formulas can be carried out in $\mathtt{RCA}_0$.

In this paper, an *ordinal* is simply a well-ordering of a subset of ω. The standard equivalent definitions of well-orderings (using infinite descending chains or least elements) are equivalent in $\mathtt{RCA}_0$ (Hirst [Hir05]). In our notation, we imagine though that ordinals follow the von-Neumann pattern. If $\alpha = (\alpha, <_\alpha)$ is an ordinal and $\beta \in \alpha$, then we also write $\beta < \alpha$, and we identify β with the initial segment $\{\gamma \in \alpha : \gamma <_\alpha \beta\}$. If β is the $<_\alpha$-greatest element of α then we also write $\alpha = \beta + 1$. The empty ordering is denoted by 0.

Rather than working with an arbitrary signature $\mathcal{L}$ and with quantifiers, we restrict ourselves to the most elementary infinitary logic: sentential propositional logic. The two atomic sentences are `True` and `False`. For connectives we use disjunction, conjunction and negation. Informally, given an ordinal α, a propositional sentence of rank α is the result of applying a connective to a set of propositional sentences of smaller rank. Formally, the object defined will consist of the sentence together with all of its subsentences. To be concrete:

Definition 1.6. Let α be an ordinal. A *propositional sentence of rank* α is a sequence of functions $\langle\psi_\beta\rangle_{\beta\leqslant\alpha}$ such that for all $\beta \leqslant \alpha$,

- $\psi_\beta(0) \in \{\neg, \wedge, \vee\}$;
- for all $n \geqslant 1$, $\psi_\beta(n) \in \beta \cup \{-1, -2\}$;
- if $\psi_\beta(0) = \neg$, then for all $n \geqslant 2$, $\psi_\beta(n) = -2$.

The definition is to be interpreted as follows. The sequence $\langle\psi_\beta\rangle_{\beta\leqslant\alpha}$ is a sequence of sub-sentences of the sentence ψ_α. We expand it by letting $\psi_{-1} = \mathtt{True}$ and $\psi_{-2} = \mathtt{False}$. Each sentence ψ_β is the result of applying the connective $\psi_\beta(0)$ to the sequence of sentences $\left\langle\psi_{\psi_\beta(n)}\right\rangle_{n\geqslant 1}$, except that if $\psi_\beta(0) = \neg$ then we really mean $\psi_\beta = \neg\psi_{\psi_\beta(1)}$, so the information given by $\psi_\beta(n)$ for $n \geqslant 2$ is irrelevant – in this case we require that $\psi_{\psi_\beta(n)} = \mathtt{False}$ for $n \geqslant 2$ (we are negative people).

However, in the sequel we will not worry about the precise formalisation; we will informally write sentences of the form $\psi = \bigvee_n \psi_n$, $\psi = \bigwedge_n \psi_n$ and $\psi = \neg\varphi$.

Having defined the syntax, we need to consider the semantics – the interpretation of an infinitary propositional sentence in a model of $\mathtt{RCA}_0$. As expected, the standard definition of semantics can only be carried out in $\mathtt{ATR}_0$. Weaker systems lack the comprehension power to show that the standard satisfaction relation exists.

We choose to understand sentences by the games that they define. The idea is best illustrated by an example. Consider the sentence

$$\psi = \bigvee_{n<\omega} \bigwedge_{m<\omega} \bigvee_{k<\omega} \psi_{n,m,k},$$

where each $\psi_{n,m,k}$ is either `True` or `False`. Satisfaction of ψ corresponds to a game: say player I wants to show that ψ is true. She needs to pick a number n. The opponent, player II, then responds by picking m. If player I can always respond with some k such that $\psi_{n,m,k}$ is true, then ψ is true. In other words, ψ is true if and only if player I has a winning strategy for the game $G_{\mathcal{U}}$, where $\mathcal{U}$ is the clopen set consisting of all sequences beginning with $\langle n, m, k\rangle$ such that $\psi_{n,m,k} = \mathtt{True}$.

For symmetry, we associate with both players (I and II) games which they win if a given sentence is true. For our ease, we make the following definition. Let $\mathcal{U} \subseteq \omega^\omega$.

- Player I *can force into* $\mathcal{U}$ if she has a winning strategy for the game $G_{\mathcal{U}}$.
- Player II *can force into* $\mathcal{U}$ if he has a winning strategy for the game $G_{\omega^\omega\setminus\mathcal{U}}$.

In other words, a player $i \in \{\mathrm{I}, \mathrm{II}\}$ can force into $\mathcal{U}$ if that player has a strategy that against any play of the opponent will result in an element of $\mathcal{U}$. To motivate parts of the following definition, we note:

Observation 1.7 ($\mathtt{RCA}_0$). Let $\mathcal{U} \subseteq \omega^\omega$. A player $i \in \{\mathrm{I}, \mathrm{II}\}$ can force into $\mathcal{U}$ if and only if their opponent can force into $\bigcup_n n\hat{}\mathcal{U}$.

Here $n\hat{}\mathcal{U} = \{n\hat{}f : f \in \mathcal{U}\}$. For brevity, let $\mathcal{U}^* = \bigcup_n n\hat{}\mathcal{U}$.

For an infinitary propositional sentence ψ we define two clopen sets $\mathcal{S}_\mathrm{I}(\psi)$ and $\mathcal{S}_\mathrm{II}(\psi)$ by recursion:

- $\mathcal{S}_\mathrm{I}(\mathtt{True}) = \mathcal{S}_\mathrm{II}(\mathtt{True}) = \omega^\omega$ and $\mathcal{S}_\mathrm{I}(\mathtt{False}) = \mathcal{S}_\mathrm{II}(\mathtt{False}) = \varnothing$.
- $\mathcal{S}_\mathrm{I}(\bigvee \psi_n) = \bigcup_n n\hat{}\mathcal{S}_\mathrm{II}(\psi_n)$ and $\mathcal{S}_\mathrm{II}(\bigvee \psi_n) = \mathcal{S}_\mathrm{I}(\bigvee \psi_n)^*$.
- $\mathcal{S}_\mathrm{II}(\bigwedge \psi_n) = \bigcup_n n\hat{}\mathcal{S}_\mathrm{I}(\psi_n)$ and $\mathcal{S}_\mathrm{I}(\bigwedge \psi_n) = \mathcal{S}_\mathrm{II}(\bigwedge \psi_n)^*$.
- $\mathcal{S}_\mathrm{I}(\neg\psi) = \omega^\omega \backslash \mathcal{S}_\mathrm{II}(\psi)$, and $\mathcal{S}_\mathrm{II}(\neg\psi) = \omega^\omega \backslash \mathcal{S}_\mathrm{I}(\psi)$.

The definition of the clopen sets $\mathcal{S}_\mathrm{I}(\psi)$ and $\mathcal{S}_\mathrm{II}(\psi)$ is performed by *effective transfinite recursion* on the rank of ψ. If $\overline{\psi} = \langle \psi_\beta \rangle_{\beta \leqslant \alpha}$ for an ordinal α, then effectively in $\overline{\psi}$ (and α), we construct a function from $\alpha + 1$ to ω mapping $\beta \leqslant \alpha$ to a pair of $\overline{\psi}$-computable indices for clopen codes for $\mathcal{S}_\mathrm{I}(\psi_\beta)$ and $\mathcal{S}_\mathrm{II}(\psi_\beta)$. This can be carried out in $\mathtt{RCA}_0$, and so $\mathtt{RCA}_0$ implies that the sequence $\langle (\mathcal{S}_\mathrm{I}(\psi_\beta), \mathcal{S}_\mathrm{II}(\psi_\beta)) \rangle_{\beta \leqslant \alpha}$ exists. Uniqueness of this sequence, and hence of $\mathcal{S}_\mathrm{I}(\psi)$ and $\mathcal{S}_\mathrm{II}(\psi)$ for all ψ is also provable in $\mathtt{RCA}_0$. The point is that if both $\langle Y_\beta \rangle_{\beta \leqslant \alpha}$ and $\langle Z_\beta \rangle_{\beta \leqslant \alpha}$ satisfy the recursive definition of this sequence, then from a point $\beta \leqslant \alpha$ such that $Y_\beta \neq Z_\beta$ we can effectively find some $\gamma < \beta$ such that $Y_\gamma \neq Z_\gamma$.

Observation 1.8 ($\mathtt{RCA}_0$). There is an arithmetic formula $\varphi(\sigma, \psi)$ which states that σ is a winning strategy for $G_{\mathcal{S}_\mathrm{I}(\psi)}$. Indeed, since $\mathcal{S}_\mathrm{I}(\psi)$ is closed, it suffices to state that for every $\tau \in \omega^{<\omega}$, at the end of the partial game of σ played against τ, σ has not already lost.

By De Morgan's law, all sentences are equivalent to sentences omitting the negation connective (but building from both $\mathtt{True}$ and $\mathtt{False}$). For an infinitary propositional sentence ψ, we define two infinitary propositional sentences $P(\psi)$ and $N(\psi)$ by recursion:

- $P(\mathtt{True}) = N(\mathtt{False}) = \mathtt{True}$ and $P(\mathtt{False}) = N(\mathtt{True}) = \mathtt{False}$.
- $P(\neg\psi) = N(\psi)$ and $N(\neg\psi) = P(\psi)$.
- $P(\bigvee \psi_n) = \bigvee P(\psi_n)$ and $N(\bigvee \psi_n) = \bigwedge N(\psi_n)$.
- $P(\bigwedge \psi_n) = \bigwedge P(\psi_n)$ and $N(\bigwedge \psi_n) = \bigvee N(\psi_n)$.

This definition is again performed by effective transfinite recursion on the rank of ψ. If $\overline{\psi} = \langle \psi_\beta \rangle_{\beta \leqslant \alpha}$ for an ordinal α, then effectively in $\overline{\psi}$ (and α), we construct the sequence $\langle \theta_\beta \rangle_{\beta \leqslant \alpha}$, with $\theta_\beta = (P(\psi_\beta), N(\psi_\beta))$. Again, $\mathtt{RCA}_0$ proves that $\langle \theta_\beta \rangle_{\beta \leqslant \alpha}$ exists, and is unique.

Intuitively, $P(\psi)$ is a sentence equivalent to ψ which is obtained by pushing all negations to the base level, and $N(\psi)$ is a similar sentence, equivalent to $\neg\psi$. These equivalences are made formal by the following lemma.

Lemma 1.9 ($\mathtt{RCA}_0$)**.** *For any infinitary propositional sentence ψ, $\mathcal{S}_\mathrm{I}(\psi) = \mathcal{S}_\mathrm{I}(P(\psi)) = \omega^\omega \backslash \mathcal{S}_\mathrm{II}(N(\psi))$ and $\mathcal{S}_\mathrm{II}(\psi) = \mathcal{S}_\mathrm{II}(P(\psi)) = \omega^\omega \backslash \mathcal{S}_\mathrm{I}(N(\psi))$.*

Proof. First we argue that this holds by transfinite induction on the complexity of ψ, then we explain why it holds in $\mathtt{RCA}_0$ as well. The point is that $\mathtt{RCA}_0$ proves transfinite Π^0_1-induction.

For $\psi \in \{\texttt{True}, \texttt{False}\}$, this is immediate.

For $\neg\psi$,

$$\begin{aligned} \mathcal{S}_{\mathrm{I}}(\neg\psi) &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(\psi) \\ &= \mathcal{S}_{\mathrm{I}}(N(\psi)) \\ &= \mathcal{S}_{\mathrm{I}}(P(\neg\psi)). \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{S}_{\mathrm{I}}(\neg\psi) &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(\psi) \\ &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(P(\psi)) \\ &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(N(\neg\psi)). \end{aligned}$$

The arguments for $\mathcal{S}_{\mathrm{II}}(\neg\psi)$ are symmetric.

For $\bigvee \psi_n$,

$$\begin{aligned} \mathcal{S}_{\mathrm{I}}(\bigvee \psi_n) &= \bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(\psi_n) \\ &= \bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(P(\psi_n)) \\ &= \mathcal{S}_{\mathrm{I}}(\bigvee P(\psi_n)) \\ &= \mathcal{S}_{\mathrm{I}}(P(\bigvee \psi_n)). \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{S}_{\mathrm{I}}(\bigvee \psi_n) &= \bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(\psi_n) \\ &= \bigcup n\hat{\ }(\omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(N(\psi_n))) \\ &= \omega^\omega \backslash \bigcup n\hat{\ }\mathcal{S}_{\mathrm{I}}(N(\psi_n)) \\ &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(\bigwedge N(\psi_n)) \\ &= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(N(\bigvee \psi_n)). \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{S}_{\mathrm{II}}(\bigvee \psi_n) &= \mathcal{S}_{\mathrm{I}}(\bigvee \psi_n)^* \\ &= \left(\bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(\psi_n)\right)^* \\ &= \left(\bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(P(\psi_n))\right)^* \\ &= \mathcal{S}_{\mathrm{I}}(\bigvee P(\psi_n))^* \\ &= \mathcal{S}_{\mathrm{II}}(\bigvee P(\psi_n)) \\ &= \mathcal{S}_{\mathrm{II}}(P(\bigvee \psi_n)) \end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{S}_{\mathrm{II}}(\bigvee \psi_n) &= \mathcal{S}_{\mathrm{I}}(\bigvee \psi_n)^* \\
&= \Big(\bigcup n\hat{\ }\mathcal{S}_{\mathrm{II}}(\psi_n)\Big)^* \\
&= \Big(\bigcup n\hat{\ }(\omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(N(\psi_n)))\Big)^* \\
&= \Big(\omega^\omega \backslash \bigcup n\hat{\ }\mathcal{S}_{\mathrm{I}}(N(\psi_n))\Big)^* \\
&= \Big(\omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(\bigwedge N(\psi_n))\Big)^* \\
&= \omega^\omega \backslash \mathcal{S}_{\mathrm{II}}(\bigwedge N(\psi_n))^* \\
&= \omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(\bigwedge N(\psi_n)) \\
&= \omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(N(\bigvee \psi_n))
\end{aligned}$$

The arguments for $\bigwedge \psi_n$ proceed as the arguments for $\bigvee \psi_n$.

Thus we have argued that if $\overline{\psi} = \langle \psi_\beta \rangle_{\beta \leqslant \alpha}$ for an ordinal α, and β is such that the lemma does not hold for ψ_β, then there is some $\gamma < \beta$ such that the lemma does not hold for ψ_γ. Moreover, since equality of clopen sets is Π^0_1, we can *effectively* (in $\overline{\psi}$ and α) find such a γ.

Thus, if the lemma fails for ψ, $\mathtt{RCA}_0$ can construct an infinite decreasing sequence of subsentences at which it fails, contradicting α being well-founded. □

Lemma 1.10 ($\mathtt{RCA}_0$). *For any infinitary propositional sentence ψ, player* I *can force into $\mathcal{S}_{\mathrm{I}}(\psi)$ if and only if player* II *can force into $\mathcal{S}_{\mathrm{II}}(\psi)$.*

Proof. The proof is by cases, depending on the structure of ψ. We emphasise that the proof is *not* a transfinite induction on the complexity of ψ.

For $\psi = \mathtt{True}$ and $\psi = \mathtt{False}$, the lemma is clear. For ψ a conjunction or a disjunction, the lemma follows from Observation 1.7. By Lemma 1.9, we may assume that ψ contains no negations. □

Definition 1.11. An infinitary propositional sentence is *strategically true* if player $i \in \{\mathrm{I}, \mathrm{II}\}$ can force into $\mathcal{S}_i(\psi)$.

Basic behaviour of semantics is provable in $\mathtt{RCA}_0$:

Proposition 1.12 ($\mathtt{RCA}_0$).

(1) $\mathtt{True}$ *is strategically true and* $\mathtt{False}$ *is not strategically true.*
(2) $\bigvee \psi_n$ *is strategically true if and only if for some n, ψ_n is strategically true.*
(3) $\bigwedge \psi_n$ *is strategically true if and only if for all n, ψ_n is strategically true, and there is a list $\langle \sigma_n \rangle$ of strategies for some player i, with σ_n witnessing that i can force into $\mathcal{S}_i(\psi_n)$.*
(4) *For no sentence ψ are both ψ and $\neg\psi$ strategically true.*
(5) *A sentence ψ is strategically true if and only if $\neg\neg\psi$ is strategically true.*

However, the familiar behaviour of propositional logic depends on the law of excluded middle. We say that a sentence ψ is *strategically false* if $\neg\psi$ is strategically true.

Definition 1.13. The *law of excluded middle*, denoted by $\mathtt{LEM}$, states that every infinitary propositional sentence is either strategically true or strategically false.

It is immediate that clopen determinacy implies LEM: for a sentence ψ, consider the game $G_{\mathcal{S}_{\mathrm{I}}(\psi)}$. By clopen determinacy, either player I or player II has a winning strategy for this game. If player I does, then ψ is strategically true. If player II has the winning strategy, then player II can force into $\omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(\psi) = \mathcal{S}_{\mathrm{II}}(\neg\psi)$, and thus $\neg\psi$ is strategically true.

Definition 1.14. The axiom *conjunction introduction*, denoted by CI, states that for every infinitary propositional sentence $\bigwedge_n \psi_n$, if every ψ_n is strategically true then $\bigwedge_n \psi_n$ is strategically true.

Claim 1.15 ($\mathtt{RCA}_0$). *The law of excluded middle implies conjunction introduction.*

Proof. Unfurling the definition, we see that

$$\mathcal{S}_I\left(\neg \bigwedge \psi_n\right) = \bigcup_n n\hat{}(\omega^\omega \backslash \mathcal{S}_I(\psi_n)).$$

If $\bigwedge \psi_n$ is not strategically true, then by the law of excluded middle, player I can force into $\bigcup_n n\hat{}\omega^\omega \backslash \mathcal{S}_I(\psi_n)$. Let n be the first move according to a strategy σ witnessing this fact. Then the rest of the strategy σ shows that player II can force into $\omega^\omega \backslash \mathcal{S}_I(\psi_n)$ (i.e. player II has a winning strategy for the game $G_{\mathcal{S}_I(\psi_n)}$). In other words, player I cannot force into $\mathcal{S}_I(\psi_n)$, and so ψ_n is not strategically true. □

Indeed, this result is not surprising. In light of Observation 1.8 and Proposition 1.12(3), conjunction introduction follows from Σ^1_1-AC, the principle of Σ^1_1-choice. In Section 3 we prove the following:

Theorem 1.16 ($\mathtt{RCA}_0$). *The law of excluded middle implies* $\mathtt{ATR}_0$.

Since $\mathtt{ATR}_0$ implies Σ^1_1-AC, it follows that LEM proves CI.

We relate the racing pawns game to LEM in Section 2.2:

Theorem 1.17 ($\mathtt{RCA}_0$). WWU *implies* LEM.

We also investigate a comprehension principle related to infinitary logic. A sequence $\langle\psi_n\rangle_{n<\omega}$ of infinitary propositional sentences can be thought of as an *infinitary propositional formula* $\psi(x)$, with $\psi(n)$ stating that ψ_n is strategically true. The sets defined by infinitary propositional formulas are the same as the subsets of ω defined by computable infinitary formulas of first-order arithmetic, and so we expect (and can prove in $\mathtt{ATR}_0$) that they coincide with the relatively hyperarithmetic sets.

Definition 1.18. The principle of *internal hyperarithmetic comprehension*, denoted by IHC, is the statement that for any infinitary propositional formula $\psi(x)$, the set $\{n < \omega : \psi(n)\}$ defined by ψ exists.

The principle IHC is not equivalent to Δ^1_1-comprehension, which is weaker than $\mathtt{ATR}_0$, and in fact the separation can be observed by ω-models, the most prominent example being the model of all hyperarithmetic sets. The reason is that IHC requires comprehension for formulas internal to the model, which may include ill-founded formulas. For example, in the model of hyperarithmetic sets, Harrison's linear ordering is a well-ordering, and it supports an infinitary formula (namely the iteration of the Turing jump) which if it defined a set, that set would compute all hyperarithmetic sets.

In Section 3 we prove:

Theorem 1.19 ($\mathtt{RCA}_0$)**.** LEM *implies* IHC*, and* IHC *implies* $\mathtt{ATR}_0$*.*

And so since clopen determinacy immediately implies LEM, IHC is equivalent to $\mathtt{ATR}_0$. Indeed, the proof that IHC implies $\mathtt{ATR}_0$ does not pass through clopen determinacy, and so as promised above, Theorem 1.19 gives a direct argument showing that clopen determinacy implies $\mathtt{ATR}_0$.

We sum our results in the following theorem:

Theorem 1.20. *The following are equivalent over* $\mathtt{RCA}_0$*:*

1. *Galvin's theorem for labeled trees (*WW*).*
2. *Galvin's theorem for unlabeled trees (*WWU*).*
3. *The law of excluded middle (*LEM*).*
4. *Internal hyperarithmetic comprehension (*IHC*).*
5. *Arithmetical transfinite recursion (*$\mathtt{ATR}_0$*).*

We leave the following question open:

Question 1.21. *Can Theorem 1.2 be proved in a weaker system than* $\mathtt{ACA}_0^+$*? Does it hold in* $\mathtt{RCA}_0$*?*

2. Racing Pawns

Here we show that $\mathtt{ATR}_0$ implies Galvin's theorem, and that Galvin's theorem implies the law of excluded middle.

As discussed above, to show that $\mathtt{ATR}_0$ implies WW, it is sufficient to prove Theorem 1.2: a proof from $\mathtt{ACA}_0^+$ that player B does not have a winning strategy for the game F_T, where T is a well-founded tree.

2.1. **Player B does not have a winning strategy.**

Proof of Theorem 1.2. We first present Galvin's argument. Let T be a well-founded tree. Let σ be a strategy for player B in the game F_T. Galvin's idea is to play infinitely many games in parallel. At the root of the tree T we place infinitely many pawns, p_n, one for each $n < \omega$. The game G_n considers the pawn p_n as the white pawn, and the pawn p_{n+1} as the black pawn. At each game, the player B follows the instructions of the strategy σ. The multi-game is played in several rounds, until one of the pawns reaches a leaf of T, that is, until one of the games G_n ends.

At the beginning of each round, we move the first pawn p_0 to an arbitrary child of its current location, being a move of the player W in the game G_0. As promised, this prompts a move by the player B in the game G_0, as determined by the strategy σ. The instruction is to move either the white pawn (p_0) or the black pawn (p_1). In the first case, we end the round and start the next round. In the second case, we think of p_1's move as a move by the player W in the game G_1, and as promised, we now let player B move according to σ in the game G_1, moving either the white pawn (p_1) or the black pawn (p_2). In the first case, we consider p_1's move as a move by player W in the game G_0 (moving the black pawn), and go on to player B's response in the game G_0, following σ. In the second case, we consider p_2's move as a move by player W in the game G_2 (moving the white pawn), and follow σ's response in the game G_2. We repeat... in general, at some step of the round, a pawn has been moved by player W in a game G_n. Assuming that a leaf has not been reached, we follow σ's instruction for player B's response, moving either the white pawn (p_n) or the black pawn (p_{n+1}). In the latter case, we consider p_{n+1}'s move as a play by W in the game G_{n+1} and move to the next step, playing G_{n+1}. In the second case, if $n > 0$, we consider p_n's move as a play by W in the game G_{n-1} and move to the next step, playing G_{n-1}. If $n = 0$, we end the round.

It is possible that a round goes on for infinitely many steps. But in this case, every pawn p_n makes only finitely many moves during the round: an infinite sequence of moves which does not pass through a leaf of T witnesses that T is ill-founded. This means that the position of each pawn p_n is well-defined at the end of the round, and so we can proceed to a new round. Further, each game witnesses an even number of moves during the round, and so, by induction, each game begins the next round waiting for a move by player W.

Similarly, we see that it is impossible to play infinitely many rounds of the multi-game. At the beginning of each round we move the first pawn p_0; we cannot do so infinitely many times without reaching a leaf along the way. This shows that the last round must end with some pawn, say p_n, reaching a leaf. But this means that the player W won the play of the game G_n, as p_n is the white pawn of the game G_n, while player B followed the strategy σ. This shows that σ is not a winning strategy for player B for the game F_T.

In the context of second-order arithmetic, let X be a set which computes both T and σ. Inductively, we see that the sequence of positions of the pawns at the beginning of the k^{th} round of the multi-game is computable from $X^{(k)}$ (uniformly in k). This is because given the starting position, carrying out the round is computable from X. In particular, this means that if some pawn moves infinitely many times during the k^{th} round, then $X^{(k)}$ computes an infinite path through T. If there is no such path, then each pawn moves finitely many times, and $X^{(k+1)} = (X^{(k)})'$ can follow the k^{th} round and tell when each pawn has stopped moving for the rest of the round, thus finding the pawn's position at the beginning of the next round.

If there are infinitely many rounds, then $X^{(\omega)}$ can follows p_0's path and so find an infinite path in T. In other words, working in a model of second-order arithmetic, if $X^{(\omega)}$ exists (within the structure) and T has no infinite paths in the structure, then the multi-game, which also exists in the structure, is only played for finitely many rounds, and so must end with a pawn reaching a leaf of T and yielding the counter-example witnessing that σ is not a winning strategy for B. Thus Galvin's argument can be carried out in $\mathtt{ACA}_0^+$, the system which ensures the existence of $X^{(\omega)}$. □

2.2. When player W has a winning strategy.

Theorem 2.1 ($\mathtt{RCA}_0$). $\mathtt{WWU}$ *implies the law of excluded middle.*

Proof. Given an infinitary propositional sentence ψ, we will construct a tree T such that W's winning strategy for the game F_T gives a winning strategy for $G_{\mathcal{S}_1(\psi)}$, either for player I or player II. Recall that an open set is given by a set $U \subseteq \omega^{<\omega}$, with the interpretation that the corresponding open set is

$$\mathcal{U} = \{f \in \omega^\omega \mid f\restriction_n \in U \text{ for some } n \in \omega\}.$$

We may assume that U is an anti-chain.

Since $\mathcal{S}_{\mathrm{I}}(\psi)$ is clopen, it is represented by some anti-chain U_1, and $\omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(\psi)$ is represented by some anti-chain U_2. As a first attempt, consider the tree T whose leaves are the strings in U_1, and the strings $\sigma\hat{\ }0$ for $\sigma \in U_2$. That is,

$$T = (U_1 \cup U_2)^{\subseteq} \cup (U_2\hat{\ }0)\,,$$

where $U^{\subseteq}$ indicates the closure of U under the taking of initial segments, and $U_2\hat{\ }0 = \{\sigma\hat{\ }0 \mid \sigma \in U_2\}$. Note that since $U_1 \cup U_2$ is an anti-chain covering all of ω^ω, the set $(U_1 \cup U_2)^{\subseteq}$ is computable from $U_1 \cup U_2$: a string σ is an initial segment of an element of $U_1 \cup U_2$ if and only if no proper initial segment of σ is an element of $U_1 \cup U_2$. Thus the tree T exists by recursive comprehension. This tree is well-founded because U_1 and U_2 represent complementary open sets.

Suppose for the moment that the racing pawns game on this tree were to play out as two sequential sprints: first W and B take turns moving the white pawn until it reaches an element of $U_1 \cup U_2$; then, if the game is not yet won, B and W again alternate moving the black pawn until it reaches an element of $U_1 \cup U_2$, this time with B taking the first move. At the end of such a play, if the white pawn reaches an element of U_2 and the black pawn reaches an element of U_1, B has won. So a winning strategy for W that adheres to this restricted play-style will either guarantee that the white pawn reaches an element of U_1 or, failing that, guarantee that the black pawn reaches an element of U_2. Thus it gives a winning strategy for the game $G_{\mathcal{S}_{\mathrm{I}}(\psi)}$, either as player I or as player II. So the theorem would be proven.

Of course, although we can assume that B plays in the manner described above (by restricting our attention to those which do), there is no reason a priori to assume that W's winning strategy will do so. W might move the black pawn before the first sprint is over, or it might move the white pawn before the second sprint is over. There is also the possibility that the white pawn reaches an element of U_2 on B's turn; then the rules of the game do not allow B to take the first move of the second sprint. Indeed, an obvious winning move for W in this case is to move the white pawn to the adjacent leaf.

We address the final concern first, because it is a simple change: we add the sets of strings $U_2\hat{\ }00 \cup U_2\hat{\ }000$ to the tree. Now if the white pawn reaches a $\sigma \in U_2$ on B's turn, W can take its turn moving the white pawn to $\sigma\hat{\ }0$, and the black pawn sprint can begin next. In fact, $U_2\hat{\ }00$ alone would suffice for this, but we will later need that if the white pawn did not reach an element of U_1, then it did not end its sprint on either a leaf or the parent of a leaf.

Now we describe how we ensure that W plays as desired. Consider the tree in Figure 2. Suppose the black pawn is at the root of this tree, and suppose the white pawn is somewhere on the tree which is not a leaf nor the parent of a leaf; so W has not yet won the game, and W cannot win in a single move. Suppose also that W is playing a winning strategy, and it is B's turn to play. Then if B moves the black pawn to some $\langle n \rangle$, W must respond by moving the black pawn again. For if W instead moves the white pawn, then since by assumption W has still not yet won the game, B can win by moving the black pawn to $\langle n0 \rangle$.

In general, by attaching a leaf to every odd height vertex, we can ensure that W always takes its turn moving the black pawn whenever the black pawn reaches an odd height. This also ensures that W never moves the black pawn from an even height vertex to an odd height vertex; if it were ever to do so, B could immediately win by moving the black pawn to the appropriate leaf.

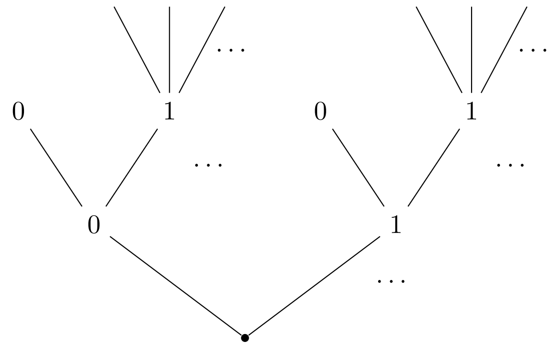

FIGURE 2. By having a leaf originating from every odd height vertex, W is required to move the black pawn whenever B moves it.

Returning to our earlier tree, consider those σ in the tree of odd length which are not in $U_1 \cup \bigcup_{i=0}^{3} U_2\hat{\ }0^i$. If every such σ is the parent of a leaf, then we know that a winning strategy for W will play as desired: W will not move the black pawn from the root before the game ends, as that would result in a loss; during the white sprint, W will move the white pawn from even heights to odd heights, and B will move it from odd heights to even heights; during the black sprint, if the game is not yet over, W will not move the white pawn before the black sprint ends; and during the black sprint, if the game is not yet over, W will move the black pawn from odd heights to even heights, and B will move it from even heights to odd heights. These can all be shown using bounded induction.

Of course, there is no reason to assume that every odd height vertex is the parent of a leaf, so we add such leaves when constructing our tree T. To aid with that, consider the function $f : \omega^{<\omega} \to \omega^{<\omega}$, with $|f(\sigma)| = |\sigma|$ and for all $n < |\sigma|$,

$$f(\sigma)(n) = \begin{cases} \sigma(n) & \text{if } n \text{ is even,} \\ \sigma(n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

(We must slide over all the even levels to make room for the extra leaf at that level.) This function is computable, so exists by recursive comprehension; and furthermore, for any set of strings A, $f[A]$ exists. We define our tree T to be

$$\begin{aligned} T := & \left\{ f(\sigma) \mid \sigma \in (U_1 \cup U_2)^{\subseteq} \right\} \\ & \cup \left\{ f(\sigma)\hat{\ }0 \mid |\sigma| \text{ is odd, and } \sigma \in (U_1 \cup U_2)^{\subseteq} \setminus (U_1 \cup U_2) \right\} \\ & \cup U_2\hat{\ }0 \cup U_2\hat{\ }00 \cup U_2\hat{\ }000. \end{aligned}$$

Recursive comprehension suffices to show that T exists.

T is well-founded, because if g were an infinite path through T, then $g(n) > 0$ for all odd n, and so $f^{-1}(g)$ would be an element of ω^ω not covered by $U_1 \cup U_2$; $f^{-1}(g)$ would exist by recursive comprehension.

T is also unlabeled. Consider any τ and any $n < m$ with $\tau\hat{\ }m \in T$. Then $\tau \in T$. If $|\tau|$ is odd and $n = 0$, then $\tau\hat{\ }0 \in T$ by construction. If $n \neq 0$ or $|\sigma|$ is even, then since $m > 0$, $f^{-1}(\tau\hat{\ }m) \in (U_1 \cup U_2)^{\subseteq}$. Since $U_1 \cup U_2$ is an anti-chain, $f^{-1}(\tau) \notin U_1 \cup U_2$. Since $U_1 \cup U_2$ cover ω^ω, some extension (not necessarily proper) of $f^{-1}(\tau\hat{\ }n)$ must be in $U_1 \cup U_2$, so $\tau\hat{\ }n \in T$ by construction.

As argued earlier, for a W winning strategy, if some B play results in the white sprint reaching an element of U_2, then W's strategy from that point on computes a strategy for player II to force into $\omega^\omega \backslash \mathcal{S}_{\mathrm{I}}(\psi) = \mathcal{S}_{\mathrm{II}}(\neg\psi)$ (using f and f^{-1} to perform the computation). If no B play results in the white sprint reaching an element of U_2, every B play must result in it reaching an element of U_1. So W's strategy computes a strategy for player I to force into $\mathcal{S}_{\mathrm{I}}(\psi)$. □

3. Infinitary Sentences

Here we show that LEM (the law of excluded middle) implies IHC (internal hyperarithmetic comprehension) and that IHC implies $\mathtt{ATR}_0$. Since clopen determinacy immediately implies LEM, this will be sufficient to prove Theorem 1.19.

Theorem 3.1 ($\mathtt{RCA}_0$)**.** LEM *implies* IHC*.*

Proof. Given an infinitary propositional formula $\psi(x)$, consider the infinitary propositional sentence $\theta = \bigwedge_n (\psi(n) \vee \neg\psi(n))$. By Proposition 1.12(2), LEM implies that $\psi(n) \vee \neg\psi(n)$ is strategically true. Then by Claim 1.15, LEM implies that θ is strategically true.

So there is a strategy by which player II can force into $\mathcal{S}_{\mathrm{II}}(\theta)$. Unpacking the definitions,

$$\mathcal{S}_{\mathrm{II}}(\theta) = \bigcup_n [n\hat{\ }0\hat{\ }\mathcal{S}_{\mathrm{II}}(\psi(n)) \cup n\hat{\ }1\hat{\ }\mathcal{S}_{\mathrm{II}}(\neg\psi(n))].$$

So if player I begins by playing n, and then player II, following this strategy, responds by playing $m \in \{0, 1\}$, then the remainder of this strategy forces into $\mathcal{S}_{\mathrm{II}}(\psi(n))$ (if $m = 0$) or $\mathcal{S}_{\mathrm{II}}(\neg\psi(n))$ (if $m = 1$). So the set of n such that player II's strategy responds to n by playing 0 is precisely $\{n \mid \psi(n)\}$. This set exists by recursive comprehension. □

3.1. **Some consequences of IHC.** Before we can prove IHC implies $\mathtt{ATR}_0$, we need several preliminary results. First, we will need $\mathtt{ACA}_0$, arithmetic comprehension.

Lemma 3.2. *[*$\mathtt{RCA}_0$*]* IHC *implies* $\mathtt{ACA}_0$*.*

Proof. Fix a set X. The set $Y = \{(e, s) \mid \Phi^X_{e,s}(e)\downarrow\}$ is recursive in X, and so exists by recursive comprehension. Let $\psi_{e,s} = \mathtt{True}$ if $(e, s) \in Y$, and $\psi_{e,s} = \mathtt{False}$ if $(e, s) \notin Y$. Let $\psi(e) = \bigvee_s \psi_{e,s}$. Note that the sentence $\psi(e)$ is recursive from Y uniformly in e, and so the sequence $\langle\psi(e)\rangle$ exists by recursive comprehension. By Proposition 1.12(2), $\psi(e)$ is strategically satisfied precisely if $e \in X'$. So by IHC, $Z = \{e \mid \psi(e)\}$ exists, and Z is precisely X'. □

We also need CI, the axiom of conjunction introduction. To show that IHC implies CI, we show that IHC allows us to effectively determine a satisfying strategy for any strategically satisfied sentence ψ. We first describe what this strategy is.

Suppose ψ contains no negations, and let σ be a sequence of moves in the game $G_{\mathcal{S}_{\mathrm{I}}(\psi)}$. Then σ determines a subsentence of ψ: at a disjunction, player I's next move chooses a disjunct (if it is player II's turn, player II's move is irrelevant); at a conjunction, player II's next move chooses a conjunct (if it player I's turn, player I's move is irrelevant).

We formalize this with the following definition, which is effective in ψ. For $\sigma \in \omega^{<\omega}$:

- If σ is the empty string, $\sigma(\psi) = \psi$.

- If $\sigma(\psi) = \texttt{True}$, $(\sigma\hat{\ }d)(\psi) = \texttt{True}$ for all $d \in \omega$.
- If $\sigma(\psi) = \texttt{False}$, $(\sigma\hat{\ }d)(\psi) = \texttt{False}$ for all $d \in \omega$.
- If $\sigma(\psi) = \bigvee_n \theta_n$ and $|\sigma|$ is even, $(\sigma\hat{\ }d)(\psi) = \theta_d$ for all $d \in \omega$.
- If $\sigma(\psi) = \bigvee_n \theta_n$ and $|\sigma|$ is odd, $(\sigma\hat{\ }d)(\psi) = \sigma(\psi)$ for all $d \in \omega$.
- If $\sigma(\psi) = \bigwedge_n \theta_n$ and $|\sigma|$ is odd, $(\sigma\hat{\ }d)(\psi) = \theta_d$ for all $d \in \omega$.
- If $\sigma(\psi) = \bigwedge_n \theta_n$ and $|\sigma|$ is even, $(\sigma\hat{\ }d)(\psi) = \sigma(\psi)$ for all $d \in \omega$.

Now, for a sentence ψ, let $\langle\psi_\beta\rangle_{\beta\leqslant\alpha}$ be the sequence of subsentences of ψ. IHC implies that the set of $\beta \leqslant \alpha$ such that ψ_β is strategically true exists. So we can define the strategy "always choose the satisfied subsentence" for player I in $G_{\mathcal{S}_{\mathrm{I}}(\psi)}$. More formally, if σ is the (possibly empty) sequence of moves which have been played so far, the strategy instructs us thus:

- If $\sigma(\psi) = \texttt{True}$ or $\texttt{False}$, play 0.
- If $\sigma(\psi) = \bigwedge_n \theta_n$, play 0.
- If $\sigma(\psi) = \bigvee_n \theta_n$ and $\sigma(\psi)$ is strategically true, play the least d such that θ_d is strategically true. If $\sigma(\psi)$ is not strategically true, play 0.

We call this strategy the *satisfaction strategy for ψ*. Note that this definition requires $\mathtt{RCA}_0$ + IHC. We can extend this to sentences with negations by letting the satisfaction strategy for ψ be the satisfaction strategy for $P(\psi)$, the sentence which is equivalent to ψ but contains no negations.

Lemma 3.3 ($\mathtt{RCA}_0$ + IHC)**.** *For any infinitary propositional sentence ψ, if ψ is strategically satisfied, then the satisfaction strategy for ψ is a winning strategy for $G_{\mathcal{S}_{\mathrm{I}}(\psi)}$.*

Proof. By Lemma 1.9, we may assume that ψ contains no negations.

Let $\langle\psi_\beta\rangle_{\beta\leqslant\alpha}$ be the sequence of subsentences of ψ. We claim that for all $\beta \leqslant \alpha$, if ψ_β is strategically satisfied, then the satisfaction strategy for ψ is a winning strategy for $G_{\mathcal{S}_{\mathrm{I}}(\psi_\beta)}$. Note that the set $X = \{\beta \leqslant \alpha \mid \psi_\beta \text{ is strategically satisfied}\}$ exists by IHC, and the satisfaction strategy for ψ_β is computable from $\langle\psi_\beta\rangle$ and X uniformly in β. Thus the sequence of satisfaction strategies exists by recursive comprehension, and so by Observation 1.8, the set

$$Y = \{\beta \leqslant \alpha \mid \text{the satisfaction strategy for } \psi_\beta \text{ is not a winning strategy for } G_{\mathcal{S}_{\mathrm{I}}(\psi_\beta)}\}$$

exists by arithmetic comprehension. Then $X \cap Y$ exists and is the set of $\beta \leqslant \alpha$ at which the claim fails. Since this set exists, we may proceed by induction.

Fix β. If ψ_β is not strategically satisfied, the claim is trivially true. So henceforth, we assume that ψ_β is strategically satisfied.

If $\psi_\beta = \texttt{True}$, the result is immediate.

It cannot be that $\psi_\beta = \texttt{False}$, since ψ_β is strategically satisfied.

If $\psi_\beta = \bigvee_n \theta_n$, then the first play of the satisfaction strategy for ψ_β will be a d with θ_d strategically satisfied. There are now several cases:

- If $\theta_d = \texttt{True}$, then $\mathcal{S}_{\mathrm{II}}(\theta_d) = \omega^\omega$. So $d\hat{\ }\omega^\omega \subseteq \mathcal{S}_{\mathrm{I}}(\psi_\beta)$, and thus the satisfaction strategy for ψ_β will always produce an element of $\mathcal{S}_{\mathrm{I}}(\psi_\beta)$.
- It cannot be that $\theta_d = \texttt{False}$, since θ_d is strategically satisfied.
- If θ_d is a disjunction, then $\mathcal{S}_{\mathrm{II}}(\theta_d) = (\mathcal{S}_{\mathrm{I}}(\theta_d))^*$. So $d\hat{\ }a\hat{\ }\mathcal{S}_{\mathrm{I}}(\theta_d) \subseteq \mathcal{S}_{\mathrm{I}}(\psi_\beta)$ for any $a \in \omega$. Further, the satisfaction strategy for ψ_β above $d\hat{\ }a$ is the same as the satisfaction strategy for θ_d, and by hypothesis the later strategy always produces an element of $\mathcal{S}_{\mathrm{I}}(\theta_d)$. So the satisfaction strategy for ψ_β always produces an element of $\mathcal{S}_{\mathrm{I}}(\psi_\beta)$.

- If $\theta_d = \bigwedge_n \chi_n$ is a conjunction, then $a\hat{\ }\mathcal{S}_{\mathrm{I}}(\chi_a) \subseteq \mathcal{S}_{\mathrm{II}}(\theta_d)$ for any $a \in \omega$. So $d\hat{\ }a\hat{\ }\mathcal{S}_{\mathrm{I}}(\chi_a) \subseteq \mathcal{S}_{\mathrm{I}}(\psi_\beta)$. Further, the satisfaction strategy for ψ_β above $d\hat{\ }a$ is the same as the satisfaction strategy for χ_a, and by hypothesis the later strategy always produces an element of $\mathcal{S}_{\mathrm{I}}(\chi_a)$. So the satisfactoin strategy for ψ_β always produces an element of $\mathcal{S}_{\mathrm{I}}(\psi_\beta)$.

If $\psi_\beta = \bigwedge \theta_n$, then the first play of the satisfaction strategy for ψ_β will be 0. Then for any a which player II might play, θ_a is strategically satisfied. So by hypothesis, the satisfaction strategy for θ_a always produces an element of $\mathcal{S}_{\mathrm{I}}(\theta_a)$. But the satisfaction strategy for θ_a is the same as the satisfaction strategy for ψ_β above $0\hat{\ }a$, and $0\hat{\ }a\hat{\ }\mathcal{S}_{\mathrm{I}}(\theta_a) \subseteq \mathcal{S}_{\mathrm{I}}(\psi_\beta)$. So the play of the satisfaction strategy for ψ_β always produces an element of $\mathcal{S}_I(\psi_\beta)$. □

Lemma 3.4 ($\mathtt{RCA}_0$)**.** `IHC` *implies* `CI`*.*

Proof. Suppose $\bigwedge_n \theta_n$ is an infinitary propositional sentence with each θ_n strategically satisfied. If each $\theta_n = \langle \theta_{\beta,n} \rangle_{\beta \leqslant \alpha_n}$, then by `IHC` the set $X = \{(\beta, n) \mid \theta_{\beta,n}$ is strategically satisfied$\}$ exists, and the satisfaction strategy for θ_n is computable from $\bigwedge_n \theta_n$ and X, uniformly in n. So the sequence of satisfaction strategies exists by recursive comprehension, and by Lemma 3.3, this is a sequence of winning strategies. So by Proposition 1.12(3), we know that $\bigwedge_n \theta_n$ is strategically satisfied. □

Finally, we will need `LEM`.

Theorem 3.5 ($\mathtt{RCA}_0$)**.** `IHC` *implies* `LEM`*.*

Proof. Fix an infinitary propositional sentence ψ, and let $\langle \psi_\beta \rangle_{\beta \leqslant \alpha}$ be its sequence of subsentences. We claim that for all $\beta \leqslant \alpha$, either ψ or $\neg\psi$ is strategically satisfied. The set of β for which this fails exists by `IHC`. The claim then follows by induction (using `CI` at the appropriate step). □

3.2. **A digression.** Although our intention is to directly show that `IHC` implies $\mathtt{ATR}_0$, we derail the progression a moment to show that `IHC` implies clopen determinacy, since the proof is straightforward.

Lemma 3.6 ($\mathtt{ACA}_0$)**.** *Every clopen set is of the form $\mathcal{S}_{\mathrm{I}}(\psi)$, for some infinitary propositional sentence ψ. Moreover, ψ can be obtained uniformly from a representation of the clopen set.*

Proof. Fix a clopen set $\mathcal{U}$, and let U_1 and U_2 be disjoint anti-chains representing $\mathcal{U}$ and $\omega^\omega \backslash \mathcal{U}$, respectively. Consider the tree

$$T = \{\sigma \in \omega^{<\omega} \mid \sigma \in (U_1 \cup U_2)^{\subseteq}\}.$$

Here again $U^{\subseteq}$ indicates the closure of U under the taking of initial segments. As argued in Theorem 2.1, T exists by recursive comprehension.

Totally order T with the Kleene-Brouwer ordering. $\mathtt{ACA}_0$ proves that the Kleene-Brouwer ordering is a well-ordering since T is well-founded [Sim09]. Define infinitary propositional sentences ψ_σ for $\sigma \in T$ as follows:

- If $\sigma \in U_1$, $\psi_\sigma = \mathtt{True}$.
- If $\sigma \in U_2$, $\psi_\sigma = \mathtt{False}$.
- If $\sigma \notin (U_1 \cup U_2)$ and $|\sigma|$ is even, $\psi_\sigma = \bigvee_n \psi_{\sigma\hat{\ }d}$.
- If $\sigma \notin (U_1 \cup U_2)$ and $|\sigma|$ is odd, $\psi_\sigma = \bigwedge_n \psi_{\sigma\hat{\ }d}$.

We claim that for $\sigma \in T$, $\{f \in \omega^\omega \mid \sigma\hat{\ } f \in \mathcal{U}\}$ is precisely $G_{\mathcal{S}_j(\psi_\sigma)}$, for $j = \mathrm{I}$ if $|\sigma|$ is even, and $j = \mathrm{II}$ if $|\sigma|$ is odd. Since equality of clopen sets is a Π^0_1 statement, the set of σ for which this fails exists by arithmetic comprehension. The claim then follows by induction.

Thus $\mathcal{U} = \mathcal{S}_\mathrm{I}(\psi_\lambda)$, where λ is the empty string. □

Theorem 3.7 ($\mathtt{RCA}_0$)**.** $\mathtt{IHC}$ *implies clopen determinacy.*

Proof. Fix a clopen set $\mathcal{U}$. Since $\mathtt{IHC}$ implies $\mathtt{ACA}_0$, $\mathcal{U} = \mathcal{S}_\mathrm{I}(\psi)$ for some infinitary propositional sentence ψ. Since $\mathtt{IHC}$ implies $\mathtt{LEM}$, either ψ is strategically true, and thus player I has a winning strategy for $G_{\mathcal{S}_\mathrm{I}(\psi)} = G_\mathcal{U}$, or $\neg\psi$ is strategically true, and thus player II has a winning strategy for $G_{\omega^\omega \setminus \mathcal{S}_\mathrm{II}(\neg\psi)} = G_{\mathcal{S}_\mathrm{I}(\psi)} = G_\mathcal{U}$. □

We can also use Lemma 3.6 to analyze $\mathtt{CI}$. Montalbán introduced the following axiom, which we express in our own notation:

Definition 3.8 (Montalbán [Mon06])**.** The *axiom of choice for determined games*, denoted by $\mathtt{DG\text{-}AC}$, states that if $\mathcal{U} = \bigcup_n n\hat{\ }\mathcal{U}_n$ is a clopen set, and for every n, one of the players has a winning strategy for $G_{\omega^\omega \setminus \mathcal{U}_n}$, then one of the players has a winning strategy for $G_\mathcal{U}$.

Montalbán showed that over $\mathtt{RCA}_0$, Σ^1_1-$\mathtt{AC}$ implied $\mathtt{DG\text{-}AC}$ and $\mathtt{DG\text{-}AC}$ implied Δ^1_1-comprehension. We shall show that $\mathtt{DG\text{-}AC}$ is equivalent to $\mathtt{CI}$. First we require $\mathtt{ACA}_0$.

Lemma 3.9 ($\mathtt{RCA}_0$)**.** $\mathtt{CI}$ *implies* $\mathtt{ACA}_0$*.*

Proof. This is similar to the proof of Lemma 3.2. Again, fix X and let $\psi(e)$ be as before. Note that $\psi(e)$ is strategically satisfied precisely if $e \in X'$, and $\neg\psi(e)$ is strategically satisfied precisely if $e \notin X'$. So $\psi(e) \vee \neg\psi(e)$ is strategically satisfied for all e. By $\mathtt{CI}$, $\psi = \bigwedge_e(\psi(e) \vee \neg\psi(e))$ is strategically satisfied. So fix a strategy for player II to force into $\mathcal{S}_\mathrm{II}(\psi)$.

If player I begins by playing e, player II must either play 0 or 1 (choosing $\psi(e)$ or $\neg\psi(e)$). If player II chooses $\psi(e)$, then the strategy above $\langle e0\rangle$ is a strategy for player II to force into $\mathcal{S}_\mathrm{II}(\psi(e))$. Similarly, if player II choses $\neg(\psi(e))$, then the strategy above $\langle e1\rangle$ is a strategy for player II to force into $\mathcal{S}_\mathrm{II}(\neg\psi(e))$. So X' is precisely the set of e such that player II plays 0 in response to player I beginning with e. Thus X' exists by recursive comprehension. □

Theorem 3.10 ($\mathtt{RCA}_0$)**.** $\mathtt{DG\text{-}AC}$ *is equivalent to* $\mathtt{CI}$*.*

Proof. First we show that $\mathtt{CI}$ implies $\mathtt{DG\text{-}AC}$. Fix a clopen set $\mathcal{U} = \bigcup_n n\hat{\ }\mathcal{U}_n$ such that for every n, one of the players has a winning strategy for $G_{\omega^\omega \setminus \mathcal{U}_n}$. If for some n, player II has a winning strategy for $G_{\omega^\omega \setminus \mathcal{U}_n}$, then player II has a strategy to force into $\mathcal{U}_n$. Then a winning strategy for player I in $G_\mathcal{U}$ is straightforward: play n, then follow player II's strategy to force into $\mathcal{U}_n$.

So suppose that for every n, player I has the winning strategy for $G_{\omega^\omega \setminus \mathcal{U}_n}$. Since $\mathtt{CI}$ proves $\mathtt{ACA}_0$, for each n there is a ψ_n such that $\mathcal{S}_\mathrm{I}(\psi_n) = \omega^\omega \setminus \mathcal{U}_n$. Then player I can force into $\mathcal{S}_\mathrm{I}(\psi_n)$, and so ψ_n is strategically satisfied. Moreover, the sequence $\langle\psi_n\rangle_{n\in\omega}$ exists by uniformity, and so $\psi = \bigwedge_n \psi_n$ exists. Also note that $\mathcal{S}_\mathrm{II}(\psi) = \omega^\omega \setminus \mathcal{U}$. By $\mathtt{CI}$, $\bigwedge_n \psi_n$ is strategically satisfied, and so player II can force into $\omega^\omega \setminus \mathcal{U}$. Thus player II has a winning strategy for $G_\mathcal{U}$, and $\mathtt{DG\text{-}AC}$ follows.

Now we show that $\mathtt{DG\text{-}AC}$ implies $\mathtt{CI}$. Fix an infinitary propositional sentence $\bigwedge_n \psi_n$, such that every ψ_n is strategically satisfied. Let $\mathcal{U}_n = \mathcal{S}_\mathrm{II}(\neg\psi_n) = \omega^\omega \setminus \mathcal{S}_\mathrm{I}(\psi_n)$.

Then for every n, player I has a winning strategy for $G_{\omega^\omega \setminus \mathcal{U}_n} = G_{\mathcal{S}_{\mathrm{I}}(\psi_n)}$. Let $\mathcal{U} = \bigcup_n n\hat{\ }\mathcal{U}_n$. By DG-AC, some player has a winning strategy for $G_{\mathcal{U}}$. If player I has the winning strategy, then player I can force into $n\hat{\ }\mathcal{U}_n$ for some n. Considering the strategy for this above n, player II can force into $\mathcal{U}_n = \mathcal{S}_{\mathrm{II}}(\neg\psi_n)$. But this contradicts ψ_n being strategically satisfied, and so is impossible.

So it must be that player II has the winning strategy. But

$$\mathcal{U} = \bigcup_n n\hat{\ }(\omega^\omega \setminus \mathcal{S}_{\mathrm{I}}(\psi_n)) = \omega^\omega \setminus \left(\bigcup_n n\hat{\ }\mathcal{S}_{\mathrm{I}}(\psi_n) \right) = \omega^\omega \setminus \mathcal{S}_{\mathrm{II}}(\psi),$$

and so player II can force into $\mathcal{S}_{\mathrm{II}}(\psi)$. Thus ψ is strategically satisfied, and CI follows. □

3.3. IHC implies ATR_0.

As promised, we show that IHC implies ATR_0 directly.

Theorem 3.11 (RCA_0). IHC *implies* ATR_0.

Proof. Given an ordinal α and a set X, we must show that $X^{(\alpha)}$ exists. Here $X^{(\alpha)}$ is any set Z satisfying:

- $Z^{[0]} = X$; and
- For all $\beta < \alpha$ with $\beta > 1$, $Z^{[\beta]} = (Z^{[<\beta]})'$.

RCA_0 proves that any such Z is unique.

Note that, by arithmetic comprehension, the sets

$$C(\beta, e) = \{\sigma \in \omega^{<\omega} \mid \sigma \subset \omega^{[<\beta]} \ \& \ \{e\}^\sigma(e)\downarrow\}$$

exist uniformly in β and e. We construct sentences $\psi(\beta, e)$ for all $\beta < \alpha$ and $e \in \omega$ as follows:

- If $e \in X$, $\psi(0, e) = \mathtt{True}$.
- If $e \notin X$, $\psi(0, e) = \mathtt{False}$.
- For $\beta > 0$, $\psi(\beta, e) = \bigvee_{\sigma \in C(\beta, e)} \left[\left(\bigwedge_{\sigma(\langle \gamma, y \rangle) = 1} \psi(\gamma, y) \right) \wedge \left(\bigwedge_{\sigma(\langle \gamma, y \rangle) = 0} \neg\psi(\gamma, y) \right) \right]$.

These are constructed from X and α by effective transfinite recursion.

By IHC, the set $Z = \{\langle \beta, e \rangle \mid \psi(\beta, e) \text{ is strategically satisfied}\}$ exists. We claim that Z satisfies the criteria to be $X^{(\alpha)}$. By arithmetic comprehension, the set of β at which it fails to meet the criteria exists. So we may proceed by induction.

If $e \in Z^{[\beta]}$, then $\psi(\beta, e)$ is strategically satisfied. Thus, by Proposition 1.12(2), for some $\sigma \in C(\beta, e)$,

$$\left(\bigwedge_{\sigma(\langle \gamma, y \rangle) = 1} \psi(\gamma, y) \right) \wedge \left(\bigwedge_{\sigma(\langle \gamma, y \rangle) = 0} \neg\psi(\gamma, y) \right)$$

is strategically satisfied. By Proposition 1.12(3), each $\psi(\gamma, y)$ with $\sigma(\langle \gamma, y \rangle) = 1$ is strategically satisfied, as is every $\neg\psi(\gamma, y)$ with $\sigma(\langle \gamma, y \rangle) = 0$. Thus, by definition of Z, $y \in Z^{[\gamma]}$ for every $\sigma(\langle \gamma, y \rangle) = 1$, and $y \notin Z^{[\gamma]}$ for every $\sigma(\langle \gamma, y \rangle) = 0$. Thus σ is an initial segment of $Z^{[<\beta]}$, and so $e \in (Z^{[<\beta]})'$.

Conversely, if $e \in (Z^{[<\beta]})'$, there is some $\sigma \in C(\beta, e)$ with σ an initial segment of $Z^{[<\beta]}$. By definition of Z, $\psi(\gamma, y)$ is strategically satisfied for every $\sigma(\langle \gamma, y \rangle) = 1$, and $\psi(\gamma, y)$ is not strategically satisfied for every $\sigma(\langle \gamma, y \rangle) = 0$. By LEM, $\neg\psi(\gamma, y)$

is strategically satisfied for every $\sigma(\langle\gamma, y\rangle) = 0$. By $\mathtt{CI}$,

$$\left(\bigwedge_{\sigma(\langle\gamma,y\rangle)=1} \psi(\gamma, y)\right) \wedge \left(\bigwedge_{\sigma(\langle\gamma,y\rangle)=0} \neg\psi(\gamma, y)\right)$$

is strategically satisfied. By Proposition 1.12(2), $\psi(\beta, e)$ is strategically satisfied, and thus $e \in Z^{[\beta]}$. □

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