THE COMPLEXITY OF MODULE RADICALS

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Abstract. We construct a computable module $\mathcal{M}$ over a computable commutative ring R such that the radical of $\mathcal{M}$, $\mathrm{rad}(\mathcal{M})$, defined as the intersection of all proper maximal submodules, is Π^1_1-complete. This shows that in general such radicals are as (logically) complicated as possible and, unlike many other kinds of ring-theoretic radicals, admit no arithmetical definition.

1. Introduction

The ideal membership problem, i.e. the search for algorithms that decide membership for ideals in computable rings, is one of the oldest problems in Computability Theory and Computable Algebra, and dates as far back as Kronecker [Kro82] who showed that every ideal of a computable presentation of the ring of polynomials with finitely many generators over the integers is computable. Soon after, mathematicians became interested in factoring polynomials over computable fields [vdW03], which mathematicians continued to study in more depth after Turing introduced formal computation [Tur36, FS56, Rab60]. More precise definitions and explanations of some of these concepts can be found in the next section, and also in the first two sections of [Con09].

Beginning with Friedman, Simpson, and Smith's analysis of computable rings and the complexity of their ideals in the context of Reverse Mathematics [FSS83, FSS85, Sim09], as well as the much more recent work of Downey, Lempp, and Mileti [DLM07], mathematicians began to study the ideal membership problem (i.e. the computability complexity) of radicals in computable rings. More specifically, Friedman, Simpson and Smith essentially constructed a computable commutative ring with identity whose prime radical (i.e. the intersection of all prime ideals) is Σ^0_1-complete, while Downey, Lempp, and Mileti constructed a computable commutative ring with identity whose Jacobson radical (i.e. the intersection of all maximal ideals) is Π^0_2-complete. Soon after these results came [Con09], in which the author constructs a computable *noncommutative* ring (with identity) whose prime radical is Π^1_1-complete. Very recently Wu [Wu20] has investigated radicals and socles in the context of Logic and Computability, and the main purpose of the current article is to answer [Wu20, Question 1].

Our one and only theorem constructs a computable commutative ring R with corresponding computable $R-$module $\mathcal{M}$ such that the radical of $\mathcal{M}$ (i.e. the intersection of all maximal $R-$submodules of $\mathcal{M}$), denoted $\mathrm{rad}(\mathcal{M})$ is Π^1_1-complete, and therefore as complex as possible from a logical and (Turing) computational point of view. More precisely we will prove the following theorem.

Theorem 3.1. *There exists a computable module $(R, \mathcal{M})$ with R commutative such that*

$$\mathrm{rad}(\mathcal{M}) = \bigcap_{\mathcal{M}' \in \mathrm{Max}(\mathcal{M})} \mathcal{M}' = \bigcap_{M \in \mathrm{Max}(R)} M \cdot \mathcal{M} \subseteq \mathcal{M}$$

is Π^1_1-complete.

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2. Background

Let $\omega = \{0, 1, 2, \ldots\}$ denote the natural numbers, and let $\omega^+ = \{1, 2, 3, \ldots\} = \omega \setminus \{0\}$ denote the positive natural numbers. Let $\omega^{<\omega}$ denote finite sequences of natural numbers which we will explicitly write with angled brackets, like so

$$\langle a_0, a_1, a_2, \ldots, a_n \rangle \in \omega^{<\omega},\ n \in \omega,\ a_i \in \omega,\ 0 \leq i \leq n.$$

Let $\omega^{+<\omega} = (\omega^+)^{<\omega}$ denote finite sequences of *positive* natural numbers, i.e.

$$\langle a_0, a_1, a_2, \cdots, a_n \rangle \in \omega^{+<\omega} \subset \omega^{<\omega},\ n \in \omega,\ a_i \in \omega^+,\ 0 \leq i \leq n.$$

For any natural number $l \in \omega$, let $\omega^{=l} \subset \omega^{<\omega}$ denote those finite sequences of natural numbers of length l (a similar definition applies to $\omega^{+=l} \subset \omega^{=l}$). Let $\emptyset$ denote the root of $\omega^{<\omega}$, and for all $\sigma \in \omega^{<\omega}$, let $|\sigma| \in \omega$ denote the length of σ and for $\sigma \neq \emptyset$, $|\sigma| > 0$, define σ^- to be the unique prefix of σ of length $|\sigma| - 1$. For any $\sigma, \tau \in \omega^{<\omega}$, we write $\tau \subseteq \sigma$ to denote the fact that τ is a prefix of σ; we write $\tau \subset \sigma$ to denote the fact that τ is a *proper* prefix of σ. We say that $T \subseteq \omega^{<\omega}$ is a *tree* if, for all $\sigma \in T$ and $\tau \subseteq \sigma$, we have that $\tau \in T$. Let ω^ω denote the set of *infinite* sequences of natural numbers, and $\omega^{+\omega}$ denote the set of infinite sequences of positive natural numbers. We write $\sigma \subseteq f$, for $\sigma \in \omega^{<\omega}$, $f \in \omega^\omega$ whenever σ is a finite initial segment of f. For any given tree $T \subseteq \omega^{<\omega}$, let $[T] \subseteq \omega^\omega$ denote the set of infinite $\omega-$sequences, $f \in \omega^\omega$, such that for each $n \in \omega$ we have that the finite initial segment of f of length n is in T, i.e. we have

$$\langle f(0), f(1), f(2), \cdots, f(n) \rangle \in T \subset \omega^{<\omega},$$

where $f(k) \in \omega$ denotes the k^{th} bit of f. We say that a given $\sigma \in T$ is *extendible* whenever there exists $f \in [T] \subset \omega^\omega$ such that $\sigma \subset f$. We say that the tree $T \subset \omega^{<\omega}$ is *well-founded* whenever T has no extendible nodes.

By *computable ring* we mean a commutative ring with identity whose addition, multiplication, and equality relations are all given by computable functions/relations. Recall that $\mathcal{M}$ is an $R-$module if each $x \in R$ defines an $R-$linear transformation $L_x : \mathcal{M} \to \mathcal{M}$ and $L_x L_y = L_y L_x$, for all $x, y \in R$. We will use the well-known fact [Rab60, Section 2.2 Theorem 7] that every computable field has a computable algebraic closure. It follows (via any computable presentation of the rational numbers $\mathbb{Q}$) that there exists a computable presentation of the algebraic numbers, which we shall denote $\mathbb{A}$. We also fix a computable enumeration $\mathbb{A} = \{a_k : k \in \omega^+\}$.

We now give a brief and intuitive review the arithmetical and analytic complexity hierarchies. For more information, including formal definitions, see [Soa16, Rog, Con09]. For our purposes we can fix a (computable commutative) ring (with identity) R. Then we say that $X \subseteq R$ is Σ^0_n, for some $n \in \omega$, if it can be defined via n alternating quantifiers, beginning with $\exists$, that range over the individual elements of R, followed by a computable quantifier-free predicate. On the other hand, we say that $X \subseteq R$ is Π^0_n if it can be defined in a similar way but beginning with a universal quantifier $\forall$. If $X \subseteq R$ is Σ^0_n for some $n \in \omega$ then X is also Π^0_{n+1} and in this case we say that X is *arithmetic*. Not all sets are arithmetic. More specifically, the *analytic* hierarchy extends beyond the arithmetic. We say that $X \subseteq R$ is Π^1_1 if X can be defined by a formula of the form

$$X = \{x \in R : (\forall Z \subseteq R)(\exists z \in R)\varphi(x, Z, z)\} \subseteq R,$$

where φ is a computable quantifier-free predicate.

It is an empirical fact that most ring-theoretic constructions can be carried out arithmetically; i.e. most subsets of rings that algebraists construct are arithmetically definable as defined in the previous paragraph. For example, although the prime radical $N \subset R$ of a commutative ring with identity R (i.e. the intersection of all prime ideals of R) has an obvious Π^1_1 definition that involves quantification over all the prime ideals of R, it is a well-known theorem of Algebra that

$$N = \{x \in R : (\exists n \in \omega)[x^n = 0_R]\} \subset R$$

and hence N is Σ^0_1 (due to the single existential quantifier above) and arithmetic. Such phenomena are fairly common in Algebra, for example it is well-known that the Jacobson radical has a similar (Π^0_2) characterization. However, not all radicals have simple (arithmetic) characterizations, as is shown in [Con09, Theorem 1.7] which classifies the logical and computational complexity of prime radicals of noncommutative rings with identity.

If $\{T_e : e \in \omega\}$ is an effective enumeration of the computable trees, $T_e \subseteq \omega^{<\omega}$, $e \in \omega$, then it is well-known that the set

$$\{e \in \omega : T_e \text{ is well} - \text{founded}\} \subset \omega$$

is non-arithmetic and in fact Π^1_1-complete [Rog, Section 16.3 Theorem XX][1]. By stitching the enumerated trees $\{T_e : e \in \omega\}$ together at a common root, it is possible to construct a *single* tree $T \subset \omega^{<\omega}$ such that $\omega^{=1} \subset T$ and the set

$$\{e \in \omega : \langle e\rangle \in \omega^{=1} \subset \omega^{<\omega}\} \subset \omega$$

is Π^1_1-complete. We will make extensive use of T in proving our main result in the next section.

The main result of this article concerns the algorithmic complexity of the radical of an $R-$module for a commutative ring R with identity. We believe that this construction is not usually covered in standard Algebra courses and so now offer a definition.

Definition 2.1. *Let R be a commutative ring with identity, and $\mathcal{M}$ an $R-$module. Then we define the* <u>radical</u> *of $\mathcal{M}$, denoted* $\text{rad}(\mathcal{M})$*, to be the intersection of all maximal (proper) submodules of $\mathcal{M}$.*

The following alternate characterization of $\text{rad}(\mathcal{M})$ is well-known; see [AF92, Exercise 15.5] for more details. For a commutative ring with identity R and a corresponding $R-$module $\mathcal{M}$, let $\max(\mathcal{M}) \subseteq \mathcal{P}(\mathcal{M})$ denote the set of maximal (proper) $R-$submodules of $\mathcal{M}$, and let $\max(R) \subseteq \mathcal{P}(R)$ denote the set of maximal ideals of R. Here $\mathcal{P}$ denotes the power set operator.

Proposition 2.2. *Let R be a commutative ring with identity, and $\mathcal{M}$ an $R-$module. Then*

$$\text{rad}(\mathcal{M}) = \bigcap_{M' \in \max(\mathcal{M})} M' = \bigcap_{M \in \max(R)} M \cdot \mathcal{M} \subset \mathcal{M}.$$

Here $M' \subset \mathcal{M}$ is a maximal submodule of M, while $M \subset R$ is a maximal ideal of R.

The main idea behind the proof of the proposition is to show that for any given maximal submodule $M' \subset \mathcal{M}$,

$$M = \{r \in R : (\forall m \in \mathcal{M})[r \cdot m \in M']\} \subset R$$

is a maximal ideal of R (otherwise $M' \subset \mathcal{M}$ would not be maximal either). We will use the previous proposition in the next section to prove our main theorem.

For a given field F, we let

$$F[\vec{X}] = F[X_0, X_1, X_2, \cdots]$$

[1]A set $X \subset \omega$ is $\mathcal{X}-$*complete*, for some complexity class X, if *every* set $Y \in \mathcal{X}$ can be computably reduced to X. Or, more generally speaking, X "codes" every member of $\mathcal{X}$. See [Rog] or [Soa16] for more details.

denote the polynomial ring over F with countably infinitely many indeterminates $X_0, X_1, X_2, \ldots$. We say that $m \in F[\vec{X}]$ is a *monomial* if m is a finite product of indeterminates in $F[\vec{X}]$, i.e.

$$m = \prod_{i=0}^{N} X_i^{n_i} \in F[\vec{X}],\ n_i \in \omega,\ N \in \omega.$$

We also say that $1 \in F[\vec{X}]$ is a monomial.

More background on general Commutative and Noncommutative Algebra can be found in [DF99, Lan93, AM69, Lam01, AF92]. For background in Computability and Reverse Mathematics, consult [Soa16, Rog, Sim09]. Finally, for an introduction to Computable Algebra and Computable Ring Theory in the context of the ideal membership problem, see [SHT].

3. Main Results

Theorem 3.1. *There exists a computable module $(R, \mathcal{M})$ with R commutative such that*

$$\mathrm{rad}(\mathcal{M}) = \bigcap_{\mathcal{M}' \in \mathrm{Max}(\mathcal{M})} \mathcal{M}' = \bigcap_{M \in \mathrm{Max}(R)} M \cdot \mathcal{M} \subseteq \mathcal{M}$$

is Π^1_1-complete.

Proof. Recall that $\mathbb{A}$ denotes a computable presentation of the algebraic numbers (any computable algebraically closed field of characteristic zero will do) with a corresopnding fixed computable enumeration

$$\mathbb{A} = \{a_1, a_2, a_3, \ldots\}.$$

Let

$$R = \mathbb{A}[X_1, X_2, X_3, \ldots] = \mathbb{A}[\vec{X}]$$

be the commutative polynomial ring with infinitely many indeterminates over $\mathbb{A}$. Let $T \subseteq \omega^{+<\omega}$ be a computable tree consisting of all $\langle k \rangle \in \omega^{+=1}$, $k \in \omega$, and such that

$$\{k \in \omega : \langle k \rangle \text{ is extendible in } T\}$$

is Π^1_1-complete.

Let

$$\Lambda = \Lambda_T = \{\sigma \in T \subset \omega^{+<\omega} : (\forall \tau \in \omega^{+<\omega},\ \tau \supset \sigma)[\tau \notin T]\}$$

be the set of *leaves* of T, and let

$$\Lambda^+ = \Lambda^+_T = \{\sigma \in \omega^{+<\omega} : \sigma^- \in \Lambda\}$$

be *the length-1 extension of* Λ. Although one cannot, in general, effectively decide whether or not a given $\sigma \in \omega^{+<\omega}$ is in Λ, there is an effective algorithm that decides whether or not a given $\sigma \in \omega^{+<\omega}$ is in Λ^+. Set

$$\Sigma = \omega^{+=1} \cup \Lambda^+ \subset \omega^{+<\omega};$$

we will use Σ to construct $\mathcal{M}$.

We will construct $\mathcal{M}$ as a computable quotient of another computable $R-$module $\mathcal{M}_0$, which we construct now. Let $\mathfrak{X}$ denote the set of monomials of $R = \mathbb{A}[\vec{X}]$ in which X_1 does not appear, and for each $i \in \omega^+$ let $\mathfrak{X}_i$ denote the set of monomials in which neither X_1 nor X_i appear. It follows that $\mathfrak{X} = \mathfrak{X}_1$. Recall that (for us) $1 \in \mathbb{A}$ is a monomial, and thus we have that $1 \in \mathfrak{X}$ and (more generally) $1 \in \mathfrak{X}_i$ for all $i \in \omega^+$. Now, let $\mathcal{M}_0$ be the $\mathbb{A}-$vector space with (symbolic) standard basis generators $G_0 \cup G_1$, where

$$G_0 = \{X m^0_{\langle j \rangle,1} : j \in \omega^+,\ \langle j \rangle \in \omega^{+=1}, X \in \mathfrak{X}\},$$

$$G_1 = \{Z_0 m^0_{\sigma,k},\ Z_1 m^1_{\sigma,k} : \sigma \in \Lambda^+,\ l = |\sigma|,\ 1 \leq k \leq l,\ Z_0 \in \mathfrak{X}_l,\ Z_1 \in \mathfrak{X}\},$$

thus

$$\mathcal{M}_0 = \langle G_0 \cup G_1 \rangle_{\mathbb{A}}.$$

Recall that for $\mathcal{M}_0$ to be an $R-$module it is necessary that each of the $X_i \in R$, $i \in \omega^+$, act like an $\mathbb{A}-$linear operator on $\mathcal{M}_0$.

Definition 3.2. *Fix $\sigma \in \Sigma = \omega^{+=1} \cup \Lambda^+$, $l = |\sigma|$, $1 \le k \le l$, and for each $i \in \{0,1\}$ let*

$$Xm = \begin{cases} Xm^0_{\sigma,1}, X \in \mathfrak{X}, \text{ if } \sigma \in \omega^{+=1}, \\ Xm^0_{\sigma,k}, X \in \mathfrak{X}_k, \text{ if } i = 0, \text{and } \sigma \in \Lambda^+ \\ Xm^1_{\sigma,k}, X \in \mathfrak{X}, \text{ if } i = 1 \text{ and } \sigma \in \Lambda^+. \end{cases}$$

We will call $X \in \mathfrak{X}$ the **monomial part** *of $Xm \in \mathcal{M}_0$, and we will call $m = m^i_{\sigma,k} \in \mathcal{M}_0$ the* **M-part** *of Xm. If $X = 1 \in \mathbb{A}$, then we will call $Xm = m^i_{\sigma,k} \in \mathcal{M}_0$ a* **pure-M** *element of $\mathcal{M}_0$.*

Let $Xm \in G_0 \cup G_1$ be a generator of $\mathcal{M}_0$ with

$$X = \prod_{i=2}^{N} X_i^{n_i} \in \mathfrak{X} \subseteq R, \ n_i \in \omega,$$

and

$$m = m^i_{\sigma,k} \in \mathcal{M}_0$$

for some $\sigma \in \Sigma$, $1 \le k \le |\sigma|$, $i \in \{0,1\}$, $N, n_i \in \omega$. Then we use the suggestive notation and think of the "undotted product" $Xm \in \mathcal{M}_0$ as the resulting $\mathcal{M}_0-$element obtained by applying commuting $\mathbb{A}-$linear operators

$$\underbrace{X_2, X_2, \dots, X_2}_{n_2}, \underbrace{X_3, X_3, \dots, X_3}_{n_3}, \dots, \underbrace{X_N, X_N, \dots, X_N}_{n_N}$$

to the pure-M element $m^i_{\sigma,k} \in \mathcal{M}_0$. We are now ready to describe the action of R on $\mathcal{M}_0$ which will also essentially (eventually) give rise to the action of R on the quotient module $\mathcal{M}$ that we will construct later.

Fix $\sigma \in \Sigma$, $l = |\sigma| \in \omega^+$, $1 \le k \le l$, and let $m = m^0_{\sigma,k} \in \mathcal{M}_0$. We will use the dot operator

$$\_ \cdot \_ : R \times \mathcal{M}_0 \to \mathcal{M}_0$$

to denote the action of R on $\mathcal{M}_0$. Now, let $m = m^0_{\sigma,k} \in \mathcal{M}_0$ be the M-part of a generator $Xm \in G_0 \cup G_1 \subset \mathcal{M}_0$ with monomial-part $X \in \mathfrak{X}_k \subset R$ and define the following "action scheme":

(a) $X_1 \cdot Xm = a_{i_1} Xm \in \mathcal{M}_0$,
(b) $X_j \cdot Xm = X_j Xm \in \mathcal{M}_0$, $j \in \omega^+$, $j \notin \{1, k\}$,
(c) $X_k \cdot Xm = Xm^1_{\sigma,k} + a_{i_k} Xm$, $k \in \omega^+$, $k > 1$,

where $\sigma = \langle i_j : 1 \le j \le l \rangle \in \Sigma \subset \omega^{+<\omega}$, $a_{i_j} \in \mathbb{A}$ for all $j = 1, 2, \dots l = |\sigma|$, and $X_j X \in \mathfrak{X}_k \subset R$, $X_k X \in \mathfrak{X} \subset R$ denote (commutative) monomial products in R. Similarly, if $m = m^1_{\sigma,k}$ denotes the M-part of a generator $Xm \in \mathcal{M}_0$ with corresponding monomial part $X \in R$, we complete our action scheme via:

(d) $X_1 \cdot Xm = a_{i_1} Xm$,
(e) $X_j \cdot Xm = X_j Xm$, for all $j \in \omega^+$, $j \neq 1$,

where $\sigma = \langle i_j : 1 \le j \le l \rangle \in \Lambda^+ \subset \omega^{+<\omega}$, $a_{i_j} \in \mathbb{A}$ for all $j = 1, 2, \dots l = |\sigma|$, and $X_j X \in \mathfrak{X}_k \subset R$ denotes the $R-$monomial (commutative) product of $X_j \in R$ and $X \in R$. Finally, the full action of R on $\mathcal{M}_0$ is obtained via an $\mathbb{A}-$linear extension of the actions of each of the X_i, $i \in \omega^+$, on $\mathcal{M}_0$. Note that each of the X_i, $i \in \omega^+$ defines an $\mathbb{A}-$linear operator on $\mathcal{M}_0$.

The following simple lemmas are actually just a series of simple observations concerning the action of R on $\mathcal{M}_0$ and will be useful later on. We leave the proofs as straightforward exercises for the reader.

Lemma 3.3. *Let $Xm \in G_0 \cup G_1$ be an $\mathcal{M}_0$-generator with M-part either $m^0_{\sigma,1}$, $\sigma \in \omega^{+=1}$, or $m^1_{\sigma,k}$, $\sigma \in \Lambda^+$, $k \in \omega^+$, $1 \leq k \leq |\sigma|$. Then we have that*

$$X_j \cdot Xm = X_j Xm.$$

More specifically we have that:

1. *the monomial-part of $X_j \cdot Xm$ is simply the $R-$product of the indeterminate X_j and the monomial X.*
2. *the M-part of $X_j \cdot Xm$ is the same as the M-part of Xm.*

Lemma 3.4. *Let $Xm \in G_1$ be an M_0-generator with M-part $m^0_{\sigma,k}$, $\sigma \in \Lambda^+$, $k \in \omega^+$, $1 \leq k \leq |\sigma|$. Then, for all $j \in \omega^+$, $j \neq k$, we have that*

$$X_j \cdot Xm = X_j Xm.$$

More specifically, we have that:

1. *the monomial-part of $X_j \cdot Xm$ is simply the $R-$product of the indeterminate X_j and the monomial X.*
2. *the M-part of $X_j \cdot Xm$ is the same as the M-part of Xm.*

We now turn our attention to showing that the actions of the $\mathbb{A}-$linear operators corresponding to $X_i \in R$ and $X_j \in R$ on the $\mathbb{A}-$vector space $\mathcal{M}_0$ commute. Once we have established this it will follow that $\mathcal{M}_0$ is an $R-$module.

Proposition 3.5. *For all $u \neq v$, $u, v \in \omega^+$, we have that*

$$X_u \cdot X_v \cdot m = X_v \cdot X_u \cdot m,$$

for all $m \in \mathcal{M}_0$. It follows that the action of R on $\mathcal{M}_0$ is commutative.

Proof. First of all, note that it suffices to show that the actions of X_u and X_v commute on all generators of $\mathcal{M}_0$.

Now, note that by parts (a) and (d) of our action scheme we have that, relative to the standard $\mathbb{A}-$basis generators

$$\mathcal{M} = \langle G_0 \cup G_1 \rangle_{\mathbb{A}},$$

the action of X_1 is that of a diagonal operator that operates via scalars in $\mathbb{A}$ that only depend upon $\sigma \in \Sigma$ in the $M-$part of a given $\mathbb{A}-$generator, from which it follows that the action of X_1 on $\mathcal{M}_0$ commutes with the actions of all X_i, $i \in \omega^+$, on $\mathcal{M}_0$ (since these actions do not affect σ).

Now, let $Xm \in \mathcal{M}_0$ be a standard basis generator with monomial-part $X \in R$ and M-part $m = m^i_{\sigma,k} \in \mathcal{M}_0$, $\sigma \in \Sigma$, $k \in \omega^+$, $1 \leq k \leq |\sigma|$, $i \in \{0, 1\}$. Note that if $i = 1$ then by part (e) of our action scheme it follows that the actions of all X_u and X_v commute on $Xm \in \mathcal{M}_0$ for all $u, v \in \omega^+$. Furthermore, if $i = 0$ and $u, v \in \omega^+ \setminus \{k\}$ then by part (b) of our action scheme it follows that X_u and X_v commute on $Xm \in \mathcal{M}_0$. Finally, it suffices to show that if $i = 0$ and $u = k$, then for all $v \neq k$, $v \in \omega^+$, we have that

$$X_u \cdot X_v \cdot Xm = X_v \cdot X_u \cdot Xm.$$

In this case we have that

$$X_u \cdot X_v \cdot Xm = X_u \cdot X_v Xm = X_v X m^1_{\sigma,k} + a_{i_k} X_v Xm,$$

and

$$X_v \cdot X_u \cdot Xm = X_v \cdot (Xm^1_{\sigma,k} + a_{i_k} Xm) = X_v X m^1_{\sigma,k} + a_{i_k} X_v X m,^{2}$$

as required. □

We are now ready to construct $\mathcal{M}$ as a quotient of $\mathcal{M}_0$. We must take extra care, however, to ensure that the actions of all X_i, $i \in \omega^+$, remain well-defined on $\mathcal{M}$. We will construct a uniformly computable sequence of $\mathbb{A}-$subspaces

$$S_0 \subset S_1 \subset S_2 \subset \cdots S_t \subset \cdots \subset \mathcal{M},\ t \in \omega,$$

such that

$$S = S_\infty = \bigcup_{t=0}^{\infty} S_t \subset \mathcal{M}_0$$

is computable and

$$\mathcal{M} = \mathcal{M}_0 / S.$$

For each $\sigma \in \Lambda^+$, let

$$s_\sigma = m^0_{\sigma_1,1} - \sum_{k=1}^{|\sigma|} m^1_{\sigma,k}$$

where $\sigma = \langle i_1, i_2, \ldots, i_{|\sigma|} \rangle \in \Lambda^+$, $\sigma_1 = \langle i_1 \rangle \subset \sigma$, $\sigma_1 \in \omega^{+=1}$, and define S_0 to be the $\mathbb{A}-$linear span of s_σ, $\sigma \in \Lambda^+$. In other words,

$$S_0 = \langle s_\sigma : \sigma \in \Lambda^+ \rangle_{\mathbb{A}}.$$

We now show that S_0 is in fact a computable $\mathbb{A}-$subspace of $\mathcal{M}_0$. The argument we give will be useful in eventually showing that $S = \cup_{t\in\omega} S_t$ is also a computable $\mathbb{A}-$subspace of $\mathcal{M}_0$, from which it will then follow that $\mathcal{M} = \mathcal{M}_0/S$ is a computable $R-$module (because the equality relations $=_{\mathcal{M}_0}, =_{\mathcal{M}}$ are computable).

Proposition 3.6. *S_0 is a computable $\mathbb{A}-$subspace of $\mathcal{M}_0$.*

Proof. By definition it is clear that S_0 is an $\mathbb{A}-$subspace of $\mathcal{M}_0$. It suffices to show that S_0 is computable. To do this, we will present an algorithm for deciding whether or not a given $x \in \mathcal{M}_0$ is in S_0. Our algorithm is as follows.

(1) If $x = 0$, then x is in S_0, Otherwise, write $x \neq 0$ as an $\mathbb{A}-$linear combination of generators in $G_0 \cup G_1 \subset \mathcal{M}_0$. Let $\Sigma' \subset \Lambda^+ \subset \omega^{+<\omega}$ be the set of $\sigma \in \omega^{+<\omega}$ for which some generator of the form $m^1_{\sigma,k}$, $1 \leq k \leq |\sigma|$ appears in the linear combination of generators expressing x.

(2) For each $\sigma \in \Sigma'$, $|\sigma| = l$, let $c_\sigma \in \mathbb{A}$ be the coefficient of $m^1_{\sigma,l}$ in the $A-$linear combination for x in (1) above. If any $c_\sigma = 0$, then by our construction of S_0 above it follows that x is not in S_0.

(3) For each $\sigma \in \Sigma'$, $|\sigma| = l$, and $1 \leq k \leq l$, let $c_{\sigma,k} \in \mathbb{A}$ denote the coefficient of $m^1_{\sigma,k}$ in the $\mathbb{A}-$linear coefficient of (1), and let $c_{\sigma,l} = c_\sigma$ be the coeffiecient of part (2) above. If *any* $c_{\sigma,k} \neq c_\sigma$ then by our construction of S_0 above it follows that x is not in S_0. Otherwise all $c_{\sigma,k} = c_\sigma \neq 0$. Now, by our previous remarks we may now assume that for each $\sigma \in \Lambda^+$ such that $m^1_{\sigma,l}$ appears in the linear combination for x in (1) above with coefficient $c_\sigma \in \mathbb{A} \setminus \{0_{\mathbb{A}}\}$, we have that the scalar multiple

$$c_\sigma \sum_{k=1}^{l} m^1_{\sigma,k}$$

appears in the same linear combination for x in (1).

[2]It is important to remember here that undotted products like $X_u X_v$ are monomial products taken in R.

(4) For each $a = a_i \in A$, $i \in \omega^+$, such that there exists at least one $\sigma \in \Sigma'$ such that $\langle i \rangle \subseteq \sigma \in \Lambda^+ \subset \omega^{+<\omega}$, let

$$c_a = \sum_{\substack{\sigma \in \Lambda^+ \\ \langle i \rangle \subset \sigma}} c_\sigma,$$

where $c_\sigma \in \mathbb{A}$ is from (2) and (3) above.

(5) Remove all (scalar multiples of) generators of the form $c_\sigma m^1_{\sigma,k}$, $c_\sigma \in \mathbb{A}$, $1 \leq k \leq |\sigma|$, from the linear combination for x in (1) and then add

$$c_a m^0_{\langle a \rangle,1} = \left(\sum_{\substack{\sigma \supset \langle i \rangle \\ \sigma \in \Lambda^+}} c_\sigma \right) m^0_{\langle a \rangle,1} \in \mathbb{A}G_0,$$

to x, for every $a \in \mathbb{A}$ in (4) above.

(6) Simplify the new linear combination of generators obtained in (5) by summing the $\mathbb{A}-$coefficients for generators repeated more than once. If the sum simplifies to $0_{\mathbb{A}}$ then $x \in S_0$; otherwise not. More specifically, after we remove the generators as described in (5) above, we should only have generators of the form $m^0_{\langle i \rangle,1} \in G_0 \subset \mathcal{M}_0$, $i \in \omega^+$, remaining.

It is not difficult to verify the following facts about our algorithm for computing S_0 above.

- Our algorithm says that every generator

$$s_\sigma = m^0_{\sigma_1,1} - \sum_{k=1}^{|\sigma|} m^1_{\sigma,k} \in S_0 \subset \mathcal{M}_0$$

is in S_0.
- If our algorithm says that m is in S_0, then for all $a \in \mathbb{A}$, our algorithm also says that $a \cdot m \in \mathcal{M}_0$ is in S_0.
- If our algorithm says that $m_0 \in \mathcal{M}_0$ and $m_1 \in \mathcal{M}_0$ are each in S_0, then our algorithm also says that $m_0 + m_1 \in \mathcal{M}_0$ is in S_0.

From which it follows that our algorithm says that all $m \in S_0$ are in S_0.

On the other hand, if our algorithm says that a given $m \in \mathcal{M}_0$ is in S_0, then it follows that

$$m = \sum_{\sigma \in \Sigma'} c_\sigma s_\sigma \in S_0,$$

where Σ' and c_σ are taken from our algorithm (1)-(6) above, and $s_\sigma \in S_0$ is a generator of S_0 that we defined above while constructing S_0. In other words, if our algorithm says that a given $m \in \mathcal{M}_0$ is in S_0, then $m \in S_0$.

It now follows that for all $m \in \mathcal{M}_0$, $m \in S_0$ if and only if our algorithm above says so. Hence, $S_0 \subset \mathcal{M}_0$ is a computable subspace of our computable module $\mathcal{M}_0$.

□

Now, we would like to set $\mathcal{M} = \mathcal{M}_0/S_0$ and show that $\mathcal{M}$ satisfies Theorem 3.1. Although we would like to do so, we will not. The reason why we cannot simply set $\mathcal{M} = \mathcal{M}_0/S_0$ is because, although the actions of each of the indeterminates $X_i \in R$, $i \in \omega^+$, are well-defined on $\mathcal{M}_0$ they are not well-defined on $\mathcal{M}_0/S_0$ because we have, for all $i \in \omega^+$, $i \geq 2$, that

$$X_i \cdot S_0 \nsubseteq S_0.$$

In other words, $\mathcal{M}_0/S_0$ is not an $R-$module. To remedy this important issue, we will enlarge S_0 to another computable subspace $S \supset S_0$ such that

$$X_i \cdot S \subseteq S,$$

for all $i \in \omega^+$. Then we will set $\mathcal{M} = \mathcal{M}_0/S$ and show that $\mathcal{M}$ satisfies Theorem 3.1 above.

We now construct

$$S = \bigcup_{t\in\omega} S_t$$

in stages $t \in \omega$. We have already constructed

$$S_0 = \langle s_\sigma : \sigma \in \Lambda^+\rangle_{\mathbb{A}} = \langle m^0_{\sigma_1,1} - \sum_{k=1}^{|\sigma|} m^1_{\sigma,k} : \sigma \in \Lambda^+\rangle_{\mathbb{A}} \subset \mathcal{M}_0,$$

where $\sigma_1 \subset \sigma \in \Lambda^+$ is the unique length-one initial segment of σ. Before we construct S_t for $t \geq 1$ we make an important observation about the action of indeterminates and monomials on the $\mathcal{M}_0$-generators of the form $m^0_{\sigma_1,1} \in G_0 \subset \mathcal{M}_0$ and $m^1_{\sigma,k} \in G_1 \subset \mathcal{M}_0$, $\sigma \in \Lambda^+$, $\sigma_1 \subset \sigma$, $|\sigma_1| = 1$, $1 \leq k \leq |\sigma|$, $k \in \omega^+$. Let

$$m = \begin{cases} m^0_{\sigma_1,1}, & \text{or} \\ m^1_{\sigma,k} \end{cases}$$

then $m \in \mathcal{M}_0$, and note that we have

- $X_1 \cdot m = a_{i_1} m \in \mathcal{M}_0$, where $\langle i_1\rangle = \sigma_1 \subset \sigma \in \Lambda^+$, and
- $X_i \cdot m = X_i m \in \mathcal{M}_0$.

More generally we have that

- $X_1^{e_1} X \cdot m = a_{i_1}^{e_1} X m \in \mathcal{M}_0$,

where $e_1 \in \omega$ and $X \in \mathfrak{X} \subset R$ is a monomial in which the indeterminate $X_1 \in R$ does not appear as a factor. It now follows that for any generator

$$s_\sigma = m^0_{\sigma_1,1} - \sum_{k=1}^{|\sigma|} m^1_{\sigma,k}$$

of S_0 we have that

$$X_1^{e_1} X \cdot s_\sigma = a_{i_1}^{e_1} X \cdot s_\sigma = a_{i_1}^{e_1} X m^0_{\sigma_1,1} - a_{i_1}^{e_1} \sum_{k=1}^{|\sigma|} X m^1_{\sigma,k} \in \mathcal{M}_0.$$

Now, for each $t = 1, 2, \ldots$ define

$$S_t = \langle X_i \cdot m : m \in S_{t-1}, i \in \omega\rangle_{\mathbb{A}} \subset \mathcal{M}_0$$

and note that the generators of S_t described above are computable uniformly in $t \in \omega$. Moreover, by our construction of S_t we have that $X_i \cdot S_{t-1} \subseteq S_t$, for all $t \in \omega$ and $i \in \omega^+$, from which it follows that $X_i \cdot S \subseteq S$ for

$$S = S_\infty = \bigcup_{t=0}^{\infty} S_t.$$

Moreover, by our previous observation about the action of indeterminates and monomials on the $\mathcal{M}_0$-generators that appear in $s_\sigma \in S_0 \subset M_0$ it follows that $X \cdot s_\sigma \in S_t$ if and only if $t \geq \deg(X)$ where $X \in \mathfrak{X} \subset R$ is a monomial in which X_1 does not appear.

Proposition 3.7. *Let $t_0 \in \omega \cup \{\infty\}$ and set*

$$S = S_\infty = \bigcup_{t \in \omega} S_t.$$

Then S_{t_0} is a computable $\mathbb{A}-$subspace of $\mathcal{M}_0$, uniformly in $t \in \omega$ and $S = S_\infty$ is a computable $R-$submodule of $\mathcal{M}_0$.

Proof. Using our previous algorithm for deciding whether a given $m \in \mathcal{M}_0$ is in $S_0 \subset \mathcal{M}_0$, we will now specify an algorithm that decides whether a given $m \in \mathcal{M}_0$ is in $S = \cup_{t \in \omega} S_t$.

(1) Write a given $m \in \mathcal{M}_0$ in terms of the standard basis generators

$$Z_0 m^0_{\tau,1},\ Z_1 m^0_{\sigma,k},\ Z_2 m^1_{\sigma,k} \in G_0 \cup G_1 \subset \mathcal{M}_0,$$

$\sigma \in \Lambda^+$, $\tau \in \omega^{+=1}$, $1 \leq k \leq |\sigma|$, $Z_0 \in \mathfrak{X}$, $Z_1, Z_2 \in \mathfrak{X}_k$.

(2) If some generator of the form $Zm^0_{\sigma,k}$, $\sigma \in \Lambda^+$, $1 \leq k \leq |\sigma|$, $Z \in \mathfrak{X}_k \subset R$, appears in m with nonzero coefficient then by our construction of S_t, $t \in \omega$, above it follows that $m \notin S_t$ and consequently $m \notin S$.

(3) Given $m \in \mathcal{M}_0$, write

$$m = \sum_X \sum_j c_j X m_j = \sum_X X \cdot \left(\sum_j c_j m_j \right),$$

where

$$m_j = \begin{cases} m^0_{\sigma_1,1}, & \text{or} \\ m^1_{\sigma,k}, & \end{cases}$$

for some $\sigma = \sigma_j \in \Lambda^+$, $\sigma_1 = \sigma_{1,j} \subset \sigma$, $|\sigma_1| = 1$, $1 \leq k = k_j \leq |\sigma|$, $c_j \in \mathbb{A}$, and $X \in \mathfrak{X} \subset R$ is a monomial in which X_1 does not appear. Here $X \in R$ is the monomial-part of $Xm_j \in \mathcal{M}_0$ while $m_j \in \mathcal{M}_0$ is the M-part of $Xm_j \in \mathcal{M}_0$.

(4) For each monomial $X \in R$ in (3), use our previous algorithm (for deciding whether or not a given $x \in \mathcal{M}_0$ is in S_0) to decide whether

$$m_X = \sum_j c_j m_j \in \mathcal{M}_0$$

is in S_0. If our previous algorithm says that each m_X from (3) above is in S_0, then we say that $m \in \mathcal{M}_0$ is in S and $m \in S_t$ for all t greater than or equal to the largest degree of any X in (3) above. Otherwise say that $m \in \mathcal{M}_0$ is not in S_t for all $t \in \omega$ and consequently also say that m is not in $S = \cup_{t \in \omega} S_t \subset \mathcal{M}$.

Fix $t \in \omega \cup \{\infty\}$ and recall that $S = S_\infty = \cup_{t \in \omega} S_t \subset \mathcal{M}_0$; it is easy to verify that:

- If $m = X \cdot s_\sigma \in \mathcal{M}_0$, $\sigma \in \Lambda^+$, is a generator of S_0 and $X \in R$ is a monomial of degree t, if $t \in \omega$, or any finite degree if $t = \infty$, then our algorithm says that $m \in S_t$.
- If our algorithm says that $m \in S_t \subset \mathcal{M}_0$, then for any given $a \in \mathbb{A}$ our algorithm says that $am \in \mathcal{M}_0$ is also in S_t.
- If our algorithm says that m_0 is in S_{t_0} and our algorithm also says that m_1 is in S_{t_1}, for some $t_0, t_1 \in \omega$, then our algorithm says that $m_0 + m_1$ is in $S_{\max(t_0,t_1)}$.

From which it follows that, for any given $m \in S_t$, our algorithm says that m is in S_t.

On the other hand, if our algorithm says that a given $m \in \mathcal{M}_0$ is in S_t, then from parts (3) and (4) of our algorithm it follows that

$$m = \sum_X X \cdot \sum_\sigma c_\sigma s_\sigma = \sum_X \sum_\sigma c_\sigma X \cdot s_\sigma,$$

from which it follows that $m \in S_t$ for all t greater than or equal to the largest degree of any monomial $X \in R$ appearing in the sum above.

It now follows that the $A-$subspaces $S_t \subset \mathcal{M}_0$, $t \in \omega$, and $S \subset \mathcal{M}_0$ are computable, uniformly in $t \in \omega$. By construction we also have that $S = \cup_{t\in\omega} S_t$ is an $R-$submodule of $\mathcal{M}_0$, since it is an $\mathbb{A}-$subspace of $\mathcal{M}_0$ such that

$$X_i \cdot S \subseteq S,$$

for all $i \in \omega^+$. It now follows that $\mathcal{M} = \mathcal{M}_0/S$ is a computable $R-$module. We will now conclude the proof of Theorem 3.1 by showing that the Jacobson radical of the computable module $(R, \mathcal{M})$ is Π^1_1-complete, as required.

□

Definition 3.8. *Let $m \in \mathcal{M}_0$. Then m is a sum of $\mathcal{M}_0-$generators in $G_0 \cup G_1$, each with a monomial part in $\mathfrak{X} \subset R$.*

We will call the set of monomial parts (in R) that appear in some generator appearing in the unique $\mathbb{A}-$linear combination of generators in G_0 and G_1 for m the <u>*monomial parts of m*</u>*.*

Moreover, for any given monomial $X \in \mathfrak{X} \subset R$ in the monomial parts of m, we will call the set of generators in $G_0 \cup G_1$ appearing in m with monomial part X a <u>*monomial slice of m*</u> *or the* <u>*$X-$monomial slice of m*</u>*.*

Proposition 3.9. *Let $m \in \mathcal{M}_0$. Then $m \in S$ if and only if for every monomial $X \in \mathfrak{X}$ appearing in the monomial parts of m, the $X-$monomial slice of m is in S.*

Proof. The proposition follows from parts (3) and (4) of our algorithm for deciding S_t, $t \in \omega \cup \{\infty\}$. □

We will use the notation $\overline{x} \in \mathcal{M}$ to denote the image of $x \in \mathcal{M}_0$ in $\mathcal{M}$ under the canonical homomorphism $\varphi : \mathcal{M}_0 \to \mathcal{M}$, $\varphi(x) = \overline{x}$.

Corollary 3.10. *Let $m_0, m_1 \in \mathcal{M}_0$ be representatives of $\overline{m_0}, \overline{m_1} \in \mathcal{M}$. Then we have that $\overline{m_0} =_{\mathcal{M}} \overline{m_1}$ if and only if the monomial parts of m_0 and m_1 are identical, and moreover for any monomial $X \in \mathfrak{X} \subset R$ in the monomial parts of m_0 and m_1 the $X-$monomial slices of m_0 and m_1 are $\mathcal{M}-$equal.*

Proof. The corollary follows immediately from the preceding Proposition and the fact that $\overline{m_0} =_{\mathcal{M}} \overline{m_1}$ if and only if $m_0 - m_1 \in S$. □

By our construction of the tree $T \subseteq \omega^{+<\omega}$ above, it follows that the following proposition essentially says that the radical of $\mathcal{M}$ is Π^1_1-complete.

Proposition 3.11. *Let*

$$J = \mathrm{rad}(\mathcal{M}) = \bigcap_{M\in \mathrm{Max}(\mathcal{M})} M = \bigcap_{M\in \mathrm{Max}(R)} M \cdot \mathcal{M} \subseteq \mathcal{M}.$$

Then, for each $i \in \omega^+$, $\langle i\rangle \in \omega^{+=1}$,

$$\overline{m^0_{\langle i\rangle,1}} \in J \subset \mathcal{M} \quad \text{if and only if} \quad (\forall f \in [T])[f(1) \neq i],$$

or, equivalently,

$$\overline{m^0_{\langle i\rangle,1}} \notin J \subset \mathcal{M} \quad \text{if and only if} \quad (\exists f \in [T])[f(1) = i].$$

Proof. Recall that

$$\{i \in \omega^+ : (\forall f \in [T])[f(1) \neq i]\}$$

is Π^1_1-complete by our hypothesis on T.

Let $M \subset R$ be a maximal ideal. Then, since $\mathbb{A}$ is algebraically closed, it is well-known that M takes the form

$$M = \langle X_j - a_{i_j} : i_j \in \omega^+ \rangle_R.$$

Let $a = a_j \in \mathbb{A}$, $j \in \omega^+$, be such that $a \neq a_{i_1}$ and consider $m^0_{\langle j \rangle,1} \in G_0$, $m = \overline{m^0_{\langle j \rangle,1}} \in \mathcal{M}$. We claim that $m \in M \cdot \mathcal{M}$. To see why, note that $(X_1 - a_{i_1}) \cdot m \in M \cdot \mathcal{M}$ and

$$(X_1 - a_j) \cdot \overline{m^0_{\langle j \rangle,1}} =_{\mathcal{M}} (a - a_{i_1})\overline{m^0_{\langle j \rangle,1}},$$

and $a = a_j \neq a_{i_1}$. It now follows that $m = \overline{m^0_{\langle j \rangle,1}} \in M \cdot \mathcal{M}$ as required.

Given our remarks in the previous paragraph, to show that

$$\overline{m^0_{\langle i \rangle,1}} \notin J \subset \mathcal{M} \quad \text{if and only if} \quad (\exists f \in [T])[f(1) = i],$$

$i \in \omega^+$, $\langle i \rangle \in \omega^{+=1}$, (as required,) it suffices to show that, for any given $i \in \omega^+$, $\langle i \rangle \in \omega^{+=1}$, we have that

$$\overline{m^0_{\langle i \rangle,1}} \notin J_i \subset \mathcal{M} \quad \text{if and only if} \quad (\exists f \in [T])[f(1) = i],$$

where

$$J_i = \bigcap_{\substack{M \in \max(R), \\ X_1 - a_i \in M}} M \cdot \mathcal{M}.$$

We will now prove this alternate but sufficient property of $\mathcal{M}$.

Fix $i_1 \in \omega^+$ and let

$$M = \langle X_k - a_{i_k} : k \in \omega^+ \rangle_R \subset R$$

be a maximal ideal. It follows that $X_1 - a_{i_1} \in M$. Let $f = \langle i_k : k \in \omega^+ \rangle \in \omega^\omega$. We will show that

$$\overline{m^0_{\langle i_1 \rangle,1}} \notin M \cdot \mathcal{M} \quad \text{if and only if} \quad f \in [T],$$

from which it follows that

$$\overline{m^0_{\langle i_1 \rangle,1}} \notin J_{i_1} \subset \mathcal{M} \quad \text{if and only if} \quad (\exists f \in [T])[f(1) = i_1],$$

and (from our previous remarks) finally it follows that

$$\overline{m^0_{\langle i_1 \rangle,1}} \notin J \subset \mathcal{M} \quad \text{if and only if} \quad (\exists f \in [T])[f(1) = i_1].$$

First, suppose that $f \notin [T]$; we will show that $\overline{m^0_{\langle i_1 \rangle,1}} \in M \cdot \mathcal{M}$. Since $f \notin [T]$, there is a unique $l \in \omega^+$ for which

$$\sigma = \langle f(1) = i_1, f(2) = i_2, \ldots, f(l) = i_l \rangle \in \Lambda^+ \subset \Sigma.$$

We have that

$$X_k - a_{i_k} \in M,$$

for all $1 \leq k \leq l$. Therefore, for each $1 \leq k \leq l$, we have that

$$(X_k - a_{i_k}) \cdot \overline{m^0_{\sigma,k}} = \overline{m^1_{\sigma,k}} \in M \cdot \mathcal{M},$$

from which it follows that

$$\sum_{k=1}^{l} \overline{m^1_{\sigma,k}} = \overline{\sum_{k=1}^{l} m^1_{\sigma,k}} = \overline{m^0_{\langle i_1 \rangle,1}} \in M \cdot \mathcal{M},$$

where the final equality comes from our construction of $\mathcal{M} = \mathcal{M}_0/S$, and the fact that $S_0 \subset S$. It follows that $\overline{m^0_{\langle i_i \rangle,1}} \in J_{i_1}$.

Now, suppose on the other hand that $f \in [T]$. In this case we will show that

$$\overline{m^0_{\langle i_1 \rangle,1}} \notin M \cdot \mathcal{M},$$

where $M = \langle X_k - a_{i_k} : k \in \omega^+ \rangle_R \subset R$ is a maximal ideal. It will then follow that

$$\overline{m^0_{\langle i_1 \rangle, 1}} \notin J \subset \mathcal{M},$$

as required.

Suppose for a contradiction that $\overline{m^0_{\langle i_1 \rangle, 1}} \in M \cdot \mathcal{M}$. Then, by definition we have that

$$\overline{m^0_{\langle i_1 \rangle, 1}} =_{\mathcal{M}} \overline{\sum_{j,k} p_{j,k}(X_k - a_{i_k}) \cdot m_j} \in M \cdot \mathcal{M},$$

where $p_{j,k} \in R = \mathbb{A}[\vec{X}]$ and $m_j \in G_0 \cup G_1 \subset \mathcal{M}_0$. For such an equality to hold, by Corollary 3.10 above, it follows that for each j and k the monomial part of the summand $p_k(X_k - a_{i_k}) \cdot m_j \in \mathcal{M}_0$ is 1 and in this case our algorithm for deciding S says that

$$m^0_{\langle i_1 \rangle, 1} - \sum_{j,k} p_{j,k}(X_k - a_{i_k}) \cdot m_j \in S_0 \subset S \subset \mathcal{M}_0. \tag{$\star$}$$

In turn, by our construction of the action of R on $\mathcal{M}_0$ and our decidability algorithms for both S_0 and S, we claim that the only way for this to hold is if:

(i) no $m^1_{\sigma,k} \in G_1$ or $m^0_{\tau,1} \in G_0$ appears as an m_j in the sum above, unless it is $m = m^0_{\langle i_1 \rangle, 1} \in G_0$, in which case we have that $(X_1 - a_{i_1}) \cdot m = 0$ and can therefore be removed from the sum without affecting its final value.

(ii) for every j, k we have that $p_{j,k} = c_{j,k} \in \mathbb{A}$

(iii) there exists a unique $\sigma \in \Lambda^+ \subset \omega^{+<\omega}$, $\sigma \subset f = \langle a_{i_k} : k \in \omega^+ \rangle \in \omega^{+\omega}$, for which we have that

$$m^0_{\langle i_1 \rangle, 1} - \sum_{k=1}^{|\sigma|} c_k(X_k - a_{i_k}) m^0_{\sigma,k} = m^0_{\langle i_1 \rangle, 1} - \sum_{k=1}^{|\sigma|} m^1_{\sigma,k} = s_\sigma \in S_0 \subset S,$$

and $c_k = 1$ for all $1 \le k \le |\sigma|$.

First of all, (i) and (ii) hold because, as we have already argued for and concluded above, the monomial parts of all generators appearing in ($\star$) are all equal to $1 \in \mathfrak{X} \subset R$ and ($\star$) holds. Furthermore, by our construction of the action of R on $\mathcal{M}_0$, the fact that

$$f = \langle a_{i_k} : k \in \omega^+ \rangle \in \omega^{+\omega},$$

and our decision algorithm for $S_0 \subset \mathcal{M}_0$, in order that ($\star$) holds (from which it follows that the monomial parts of each of the distinct summands $(X_k - a_{i_k}) \cdot m_j$ appearing in ($\star$) are all 1), it follows that we must have

$$(X_k - a_{i_k}) \cdot m_j = m^1_{\sigma,k},$$

for some $\sigma \in \Lambda^+ \subset \Sigma \subset \omega^{+<\omega}$, and $1 \le k \le |\sigma|$. It now follows that, for each j in ($\star$) above, we have that

$$m_j = m^0_{\sigma,k} \in G_1,$$

for some $\sigma \in \Lambda^+ \subset \Sigma \subset \omega^{+<\omega}$, $1 \le k \le |\sigma|$, and $\sigma \subset f$. This is what (iii) says. Now (finally), it is impossible to find such a $\sigma \in \Lambda^+ \subset \omega^{+<\omega}$, $\sigma \subset f$, as in (iii) above, since $f \in [T]$ and so any $\sigma \subset f$, $\sigma \in \omega^{+<\omega}$, cannot also satisfy $\sigma \in \Lambda^+ \subset \Sigma \subset \omega^{+<\omega}$ by our construction of $\Lambda^+ \subset \omega^{+<\omega}$ as the length-1 extensions of the leaves of $T \subset \omega^{+<\omega}$. Thus, we have reached a contradiction, as we promised earlier.

□

This completes the proof of Theorem 3.1.

□

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