A MEASURE-THEORETIC PROOF OF TURING INCOMPARABILITY

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Abstract. We prove that if $\mathcal{S}$ is an ω-model of weak weak König's lemma and $A \in \mathcal{S}$, $A \subseteq \omega$, is incomputable, then there exists $B \in \mathcal{S}$, $B \subseteq \omega$, such that A and B are Turing incomparable. This extends a recent result of Kučera and Slaman who proved that if $\mathcal{S}_0$ is a Scott set (i.e. an ω-model of weak König's lemma) and $A \in \mathcal{S}_0$, $A \subseteq \omega$, is incomputable, then there exists $B \in \mathcal{S}_0$, $B \subseteq \omega$, such that A and B are Turing incomparable.

1. Genericity, Randomness, Logic, and Computability

1.1. **Introduction.** The primary goal of this article is to compare the logical (i.e. foundational) nature of two distinct, but similar mathematical concepts. Both concepts were first introduced in the context of mathematical analysis. Furthermore, they are both over 100 years old, and play a central role in mathematics. The first, called Baire category, was introduced by Baire in his 1899 PhD thesis [Bai99]. The second, called measure, was introduced by Lebesgue in his 1902 PhD thesis [Leb].

Since the introduction of Cohen forcing in the early 1960s, the Baire category theorem has played a central role in mathematical logic and computability theory. However, logicians usually refer to the Baire category theorem as *forcing*, and the objects that the theorem produces as *generics*. Forcing is a widely used tool in mathematical logic and computability theory to construct objects that have various mathematical properties. The simplest construction of this kind in computability theory is cone avoidance or Turing incomparability. Computability theorists typically use forcing constructions to produce sets of natural numbers that are Turing incomparable (for more information on Turing reducibility and basic computability theory, see Section 2 or [Soa]), among other things. More generally, however, one can use forcing to solve the extension of embeddings problem for the Turing degrees $\mathcal{D}$ [Ler, Theorem II.4.11].

Measure theory was first introduced as a tool for proving a converse to the fundamental theorem of calculus. Since then, it has been widely used in many different areas of mathematics, including mathematical logic and computability theory. The earliest well-known measure-theoretic result on computability is a theorem of Sacks and others [Nie, Theorem 5.1.12], which says that $A \subseteq \omega = \{0, 1, 2, \ldots\}$ is incomputable if and only if the set of oracles $f \in 2^\omega$ that compute A has measure zero (for more information on Cantor space 2^ω, consult Section 2). More recently, however,

Date: November 23, 2009.

2000 *Mathematics Subject Classification.* Primary 03D28, 03F35;
Secondary 03F60.

Key words and phrases. Computability theory, reverse mathematics, measure theory, randomness, Baire category theorem, forcing.

The author was partially supported by NSERC grant PDF 373817-2009 and a Fields-Ontario Fellowship from the Fields Institute. Furthermore, he would like to thank Professor T.A. Slaman for helpful conversations which lead to this research. This research was done during the author's stay at the Fields Institute from July to December 2009.

the study of effective measure theory (i.e. computable measure theory) has seen significant growth in scope and depth. In particular, the subfield of effective measure theory called algorithmic randomness has seen a tremendous growth spirt over the last 10 years, and many interesting connections between measure, randomness, and computability have been established in that time.

1.1.1. *Comparing Genericity with Randomness.* We have already mentioned that the canonical example of forcing in computability theory is the construction of Turing incomparable sets $A, B \subseteq \omega$ (and, more generally, the solution of the extension of embeddings problem [Ler, Theorem II.4.11]). Therefore, a natural way to compare genericity and randomness is to see whether or not there is a proof of Turing incomparability via algorithmic randomness. This is the content of the main theorem of this article (Theorem 4.1). It gives a proof of Turing incomparability via randomness (i.e. measure theory). There are reasons for thinking that genericity and randomness are similar concepts, and reasons for thinking of them as distinct. The main similarity between these two notions is that they both talk about "big sets" and "small sets." In particular, both of these concepts say that "big sets" (where the definition of "big" depends on the concept) are nonempty. On the other hand, the Baire category theorem (i.e. genericity) builds a set that *meets* a "big set" (i.e. a comeager set), while randomness builds a set that *avoids* every "small set" (i.e. set of measure zero of a certain low complexity). Thus, while the overall philosophy behind the Baire category theorem is the same as that of randomness (both notions say that sets that satisfy a type of largeness requirement are nonempty), the philosophies behind their proofs are different. It is well-known that, from the point of view of reverse mathematics and ω-models (which is the point of view that we take in this article), these concepts are distinct [BS93, Theorem 3.2].

1.2. **The Main Theorem.** The main theorem of this article (Theorem 4.1) says that a particular subsystem of second order arithmetic proves the sentence

$$(\forall A)(\exists B)[\emptyset <_T A \Rightarrow A \not\leq_T B \ \ \& \ B \not\leq_T A]$$

in the context of ω-models (for more information on reverse mathematics and subsystems of second order arithmetic including ω-models, consult Section 2 or [Sim]). This subsystem of second order arithmetic is called weak weak König's lemma (WWKL) and is related to effective randomness. WWKL says that every Π^0_1-class (i.e. effectively closed set) of positive measure is nonempty, and it is equivalent to saying that for every set $A \subseteq \omega$, there exists a set $B \subseteq \omega$ such that B is random relative to A (for more information on randomness and Π^0_1-classes, consult either Section 2 or [Nie]).

1.3. **The Kučera/Slaman Theorem.** Recently, Kučera and Slaman [KS07] proved that if $\mathcal{S}$ is a Scott set (i.e. an ω-model of WKL weak König's lemma; see Section 2 for more details) then for every incomputable set $A \in \mathcal{S}$, $A \subseteq \omega$, there exists $B \in \mathcal{S}$, $B \subseteq \omega$, such that $A \not\leq_T B$ and $B \not\leq_T A$. This problem was originally posed by Friedman and McAllister [CJ00, Problems 3.2,3.3] and remained unsolved for many years. In particular, it was advances in the theory of algorithmic randomness that eventually yielded a solution to the problem. More specifically, the recent work of Hirschfeldt, Nies, and Stephan [HNS07] and Nies [Nie05] on K-triviality (for more information on K-trivials, consult either Section 2 or [Nie, Chapter 5]) were key steps in solving this problem.

To prove the main theorem [KS07, Theorem 2.1], Kučera and Slaman divided the proof into two parts. The first part handles the case when the incomputable set $A \subseteq \omega$ of the previous paragraph is not K-trivial, while the second part deals with the case when A is K-trivial. Thus, the proof of [KS07, Theorem 2.1] is nonuniform. This nonuniformity is the source of some serious obstacles when one tries to extend or generalize [KS07, Theorem 2.1]. We give a brief overview of the method of the proof of [KS07, Theorem 2.1] in Section 4 below.

The main significance of the Kučera/Slaman theorem [KS07, Theorem 2.1] is that it provides a (nonuniform) proof of Turing incomparability via the axiom WKL (weak König's lemma). In other words, if we denote the axiom given by the Baire category theorem by BCT (i.e. BCT says that for every $A \subseteq \omega$ there exists a $B \subseteq \omega$ such that B is 1-generic relative to A; for the precise definition of 1-genericity, see [Soa, Exercise VI.3.6]), then in the context of ω-models BCT and WKL both prove the sentence

$$(\forall A \subseteq \omega)(\exists B \subseteq \omega)[\emptyset <_T A \Rightarrow A \not\leq_T B \ \&\ B \not\leq_T A]. \tag{1}$$

However, via BCT the proof is uniform, while via WKL the proof is nonuniform, since it is divided up into cases (based upon randomness considerations). In addition to exploring the similarities between BCT and WKL, we feel that our new randomness-theoretic (i.e. measure-theoretic) proof of Turing incomparability (Theorem 4.1) directly relates effective randomness to the Turing incomparability problem, thus making the randomness considerations of [KS07, Theorem 2.1] somewhat less mysterious. We also note that our proof is nonuniform, in exactly the same way as [KS07, Theorem 2.1].

1.4. **The Plan of the Paper.** The next section (i.e. Section 2) introduces the main ideas from computability theory, randomness, and reverse mathematics that we shall need to prove our main result. Section 3 reviews the relevant theorems that will help us to prove the main theorem of this article. In the final section (i.e. Section 4), we state and prove the main theorem of this article (Theorem 4.1). The author is thankful to the anonymous referee for helpful comments and for helping to streamline the exposition of this article.

2. Preliminaries and Notation

Our computability-theoretic terminology and notation follows that of Soare [Soa], and our randomness-theoretic terminology and notation follows that of Nies [Nie].

We refer to elements of 2^ω (i.e. the set of all infinite binary strings) as *reals* or *sets* and identify each real $A \in 2^\omega$ with the set of natural numbers given by $A^{-1}(1) \subseteq \omega$. Also, $2^{<\omega}$ denotes the set of all finite binary strings. For any given $A \in 2^\omega$, $n \in \omega$, let $A\restriction n \in 2^{<\omega}$ denote the first n bits of A. By *tree* we refer to a downwards closed subset of $2^{<\omega}$. Recall that a Σ^0_1-class is a collection of reals that can be computably enumerated, and that any such class can be represented as the union of a prefix-free computably enumerable (c.e.) set of finite binary strings $\sigma \in 2^{<\omega}$. The complement of a Σ^0_1-class (in 2^ω) is called a Π^0_1-class. A Π^0_1-class can be represented as the set of infinite paths through a computable binary tree. We will also use relativized versions, i.e. $\Sigma^{0,A}_1$-classes and $\Pi^{0,A}_1$-classes, for some given set $A \in 2^\omega$. Π^0_1-classes play a prominent role in logic, reverse mathematics, and algorithmic randomness.

The following definition is due to Martin-Löf.

Definition 2.1. Let $A \in 2^\omega$ be given. A *Martin-Löf test relative to* A is a uniformly c.e. in A sequence of $\Sigma^{0,A}_1$-classes $\{U^X_n\}_{n\in\omega}$ such that $\mu(U^X_n) \leq 2^{-n}$, where μ denotes the standard (i.e. Lebesgue) measure on 2^ω. Any subset of $\cap_{n\in\omega} U^X_n$ is called a *Martin-Löf null set relative to* A. When $X = \emptyset$ we say *Martin-Löf test* and *Martin-Löf null set*, respectively. A real $X \in 2^\omega$ is *Martin-Löf random (1-random) relative to* A whenever X is not contained in any Martin-Löf null set relative to X. If

$X = \emptyset$ we say that X is Martin-Löf random (1-random). Let MLR^A denote the set of Martin-Löf random reals relative to A, and set $\mathsf{MLR}^\emptyset = \mathsf{MLR}$.

Martin-Löf constructed a universal Martin-Löf test, $\{U_n\}_{n\in\omega}$, with the special property that for all $X \in 2^\omega$, X is 1-random if and only if $X \notin \cap_{n\in\omega} U_n$. This construction relativizes to all oracles $A \in 2^\omega$.

We will use $K(\sigma)$ to denote the prefix-free Kolmogorov complexity of $\sigma \in 2^{<\omega}$, and similarly $K^A(\sigma)$ to denote the prefix-free Kolmogorov complexity relative to the given oracle $A \in 2^\omega$. Schnorr [Sch71] proved that $X \in 2^\omega$ is 1-random if and only if there is a constant $c \in \omega$ such that for every $n \in \omega$ we have that $K(X\restriction n) \geq n + c$.

Definition 2.2. Fix $A \in 2^\omega$. The following properties describe various kinds of computational weakness associated with 1-randomness.

(1) MLR^A=MLR.
(2) $(\exists c)(\forall n)[K(A\restriction n) \leq K(n) + c]$.
(3) $(\exists c)(\forall \sigma)[K(\sigma) \leq K^A(\sigma) + c]$.
(4) $A \leq_T Z$ for some $Z \in 2^\omega$ that is 1-random relative to A.

We say that A is *low for 1-randomness* if A satisfies (1); we say that A is *K-trivial* if A satisfies (2); we say that A is *low for K* if A satisfies (3); we say that A is a *basis for 1-randomness* if A satisfies (4). Property (1) was first introduced by Zambella [Zam90]; property (2) was first introduced by Chaitin [Cha76]; property (3) was first introduced by Muchnik (unpublished, see [Nie, page 165]); property (4) was first introduced by Kučera [Kuc93]. It is well-known that properties (1)-(4) above are equivalent [HNS07, Nie05], and that every K-trivial set is low [Nie05], i.e. if $A \in 2^\omega$ is K-trivial then $A' \equiv_T \emptyset'$.

2.1. **Reverse Mathematics and Subsystems of Second Order Arithmetic.** In this section we introduce three subsystems of second order arithmetic: RCA, WWKL, and WKL[1]. RCA and WKL were introduced by H. Friedman [Fri75], while WWKL was first introduced by Simpson and Xu [SY90]. It is known that RCA is strictly weaker than WWKL, which in turn is strictly weaker than WKL (i.e. in terms of strength we have that $\mathsf{RCA} < \mathsf{WWKL} < \mathsf{WKL}$) [SY90]. For more information on reverse mathematics and subsystems of second order arithmetic, consult [Sim]. Recall that if T is a theory, and P is a sentence in the language of T, then to show that T proves P it suffices to show (via Gödel's completeness theorem) that every model of T is also a model of P. Throughout this article we work *exclusively* with ω-models. That is, we work with models whose first-order parts are the standard natural numbers $\omega = \{0, 1, 2, \ldots\}$, thus restricting the second order parts of our models to subsets of the power set of ω (that satisfy various computability-theoretic closure properties as described below). As usual, we identify ω-models with their second order parts.

2.1.1. *RCA.* RCA stands for recursive comprehension axiom. It asserts that whenever $A \subseteq \omega$ exists, and $B \leq_T A$, then B also exists. It is known that the ω-models of RCA are simply the Turing ideals. In other words, the models of RCA are the subsets of the power set of ω that are closed under $\oplus$ and $\leq_T$.

[1]Normally, these subsystems of second order arithmetic appear with a subscript 0 that indicates a restricted induction scheme (restricted to Σ^0_1 formulas only). Since we are working exclusively within ω-models, we are implicitly assuming unrestricted induction for all formulas, and therefore omit subscripts since for our purposes they hold no meaning.

2.1.2. *WWKL.* WWKL stands for weak weak König's lemma. It asserts that RCA holds, plus the axiom that says for every $A \subseteq \omega$ and every $\Pi^{0,A}_1$-class $X \subseteq 2^\omega$ such that $\mu(X) > 0$, we have that $X \neq \emptyset$ (i.e. there is some $f \in X$). It is well-known (via a theorem of Kučera [Nie, Proposition 3.2.24] and the existence of a universal Martin-Löf test) that WWKL is equivalent to the assertion of RCA plus the axiom that says for every set $A \subseteq \omega$ we have that $\mathsf{MLR}^A \neq \emptyset$. To prove our main theorem, we shall use the latter equivalent characterization of WWKL in place of the original definition.

2.1.3. *WKL.* WKL stands for weak König's lemma. WKL consists of RCA, plus the axiom that asserts that for every set $A \subseteq \omega$, and every infinite computable tree $T \subseteq 2^{<\omega}$ relative to the oracle A, there exists an infinite path through T. WKL is at least as strong as WWKL since it is well-known that every $\Pi^{0,A}_1$-class of positive measure can be represented as the set of paths through an infinite A-computable tree. An ω-model of WKL is sometimes called a *Scott set* or *Scott class*.

3. Some Known Results

In this section we collect the definitions and known results that will help us to prove the main theorem in the next section. Most of this material can be found in [Nie, Chapter 5].

The first theorem that we require is an old result of Sacks and others.

Theorem 3.1. [dLMSS, Sac] *If $A \subseteq \omega$ is incomputable and Φ is an oracle Turing machine. Then the $\Pi^{0,A}_2$-class $\{X \in 2^\omega : \Phi^X = A\}$ has (Lebesgue) measure zero.*

The next fact that we will need is a recent result of Kjos-Hanssen, Miller, and Solomon [KHMS].

Definition 3.2. Let $A, B \subseteq \omega$. We say that A is *LR*-reducible to B, and write $A \leq_{LR} B$, if $\mathsf{MLR}^B \subseteq \mathsf{MLR}^A$.

Theorem 3.3. [KHMS, Theorem 3.2] *The following are equivalent for given sets $A, B \subseteq \omega$.*

1. *$A \leq_{LR} B$ and $A \leq_T B'$.*
2. *Every $\Pi^{0,A}_1$-class contains a $\Sigma^{0,B}_2$-class of equal measure.*
3. *Every $\Sigma^{0,A}_2$-class contains a $\Sigma^{0,B}_2$-class of equal measure.*

Recall (via Definition 2.2) that $A \in 2^\omega$ is K-trivial if and only if $A \leq_{LR} \emptyset$ and $A \leq_T \emptyset'$. Also note that item (3) of Theorem 3.3 above can be (equivalently) restated as saying that every $\Pi^{0,A}_2$-class is contained inside a $\Pi^{0,B}_2$-class of equal measure. Thus, if $A \in 2^\omega$ is K trivial, then by Theorem 3.3 it follows that every $\Pi^{0,A}_2$-class is contained inside a Π^0_2-class of equal measure. We will use this fact in the proof of Theorem 4.1 below.

Next, we present work of Hirschfeldt and Miller (unpublished, see [Nie, Theorem 5.3.15]) on the set of randoms contained within a Π^0_2-class of measure zero. More specifically, the proof of [Nie, Theorem 5.3.15] yields the following theorem.

Theorem 3.4. [Nie, Theorem 5.3.15] *Let R be a Π^0_2-class of measure zero. Then there is a cost function $c_R(x, s)$ that satisfies the limit condition and such that every $A \subseteq \omega,\ A \in \Delta^0_2$, that possesses a computable approximation obeying c is computable relative to every random set $X \in R$. In other words, A is a uniform Turing lower bound for the set of randoms in R.*

The current paragraph gives a brief introduction to cost functions. Roughly speaking a *cost function* is a computable function $c : \omega^2 \to \omega$ that assigns a cost to every pair of natural numbers $\langle x, s\rangle \in \omega^2$. Cost functions are used when constructing Δ^0_2 sets via computable approximations (i.e. the limit lemma). More specifically, the cost associated with the construction of $A = \lim_s f(x, s)$, $A \in \Delta^0_2$, at stage $s \in \omega$ is equal to $c(x, s)$, where $x \in \omega$ is least such that $f(x, s) \neq f(x, s-1)$. We say that the approximation $f(x, s)$ *obeys* the cost function c if the sum of the costs over all stages is finite. It follows that approximations that obey a given cost function must change infrequently, and therefore obeying a cost function is a notion of computational weakness for Δ^0_2 sets. A cost function satisfies the *limit condition* if (roughly speaking) the limit of the costs tends to zero as s tends to infinity. For more information on cost functions, including precise definitions, consult [Nie, Section 5.3].

We will also need to know that the c.e. K-trivials are the most powerful amongst the K-trivials. This is the content of [Nie, Corollary 5.5.3] and was first proven in [Nie05].

Theorem 3.5. [Nie05] *For any given K-trivial set $A \subseteq \omega$, there is a c.e. K-trivial set $B \subseteq \omega$ such that $A \leq_T B$.*

Next, we shall need to know that obeying cost functions is compatible with (lower) cone avoidance for low c.e. sets. A special case of the following result was first proven by Nies in [Nie02]. The general case is proven in [Nie05, Theorem 5.3.22].

Theorem 3.6. [Nie, Theorem 5.3.22] *Let c be a cost function that satisfies the limit condition. Then for every low c.e. set B there is a c.e. set A that obeys c and such that $A \not\leq_T B$.*

The final result that we need is the well-known Sacks splitting theorem with (upper) cone avoidance for incomputable Δ^0_2 sets. It was first proven by Sacks.

Theorem 3.7. [Soa, Proposition VII.3.3, Exercise VII.3.9] *Let $B, C \subseteq \omega$ be such that B is c.e. and $C \in \Delta^0_2$ is incomputable. Then there exist c.e. sets $A_0, A_1 \subseteq \omega$ such that*

(1) *$A_0 \cup A_1 = B$ and $A_0 \cap A_1 = \emptyset$.*
(2) *$C \not\leq_T A_i$, for $i \in \{0, 1\}$.*

Furthermore, we have that $A_0 \oplus A_1 \equiv_T B$.

4. A Measure-Theoretic Proof of Turing Incomparability

The goal of this section is to prove the main theorem of this article, which we now state.

Theorem 4.1. *Let $\mathcal{S}$ be an ω-model of* WWKL. *Then for every incomputable $A \subseteq \omega$ such that $A \in \mathcal{S}$, there exists $B \subseteq \omega$ such that $B \in \mathcal{S}$ and $B|_T A$ (i.e. $B \not\leq_T A$ and $A \not\leq_T B$).*

Before we give the proof Theorem 4.1, we wish to briefly review the proof of [KS07, Theorem 2.1], which says that for any Scott set $\mathcal{S}$ and any incomputable set $A \in \mathcal{S}$, there is a $B \in \mathcal{S}$ such that $A|_T B$.

To prove [KS07, Theorem 2.1], the authors break the proof up into two parts. The first part of the proof deals with the case where $A \subseteq \omega$ is not K-trivial, and is valid in WWKL (as well as WKL), and is therefore applicable in the context of this article. The proof of the first part is quite simple, and uses item (4) of the characterization of K-trivials that we gave in Section 2.2. If A is not K-trivial, then use WWKL to produce a set $B \subseteq \omega$, $B \in \mathcal{S}$, such that B is random relative to A. Then, by item (4) in Section 2.2, we have that $A \not\leq_T B$. Furthermore, since B is random relative to A, it follows that B is not K-trivial and thus $B \not\leq_T A$. Therefore, we have that $A|_T B$.

The second part of the proof deals with the case where A is K-trivial. In this case the authors construct a Π^0_1-class $X \subseteq 2^\omega$ such that every element of X is Turing incomparable with A. We will not give all the details here, but we do point out that to achieve $(\forall f \in X)[A \not\leq_T f]$ the authors employ the Sacks preservation strategy for avoiding upper cones of Δ^0_2 sets (recall that if A is K-trivial, then A is Δ^0_2). The reason why the authors' proof does not go through in WWKL is that the Sacks preservation strategy enumerates many basic clopen sets out of the Π^0_1-class $X \subseteq 2^\omega$, thus thinning X down to a set of measure zero. Therefore, WWKL is unable to conclude that $X \neq \emptyset$. In other words, the main obstruction in getting the proof of [KS07, Theorem 2.1] to go through in WWKL is its use of the Sacks preservation strategy.

We point out that our proof of Theorem 4.1 below also employs the Sacks preservation strategy, because the proof of Theorem 3.7 above uses the Sacks preservation strategy to avoid the cone above the incomputable Δ^0_2 set $C \subseteq \omega$. In other words, our proof finds a way to use the Sacks preservation strategy without thinning out our Π^0_1-class, thus avoiding the obstruction associated with the proof of [KS07, Theorem 2.1]. We also note that the second part of our proof uses the full hypothesis that $A \subseteq \omega$ is K-trivial (via Theorem 3.3 above), whereas the second part of [KS07, Theorem 2.1] can be easily modified so that is valid for any Δ^0_2 set of effective packing dimension zero.

Proof of Theorem 4.1. Suppose that we are given an ω-model $\mathcal{S}$ of WWKL, and $A \in \mathcal{S},\ A \subseteq \omega$, such that $\emptyset <_T A$. Using WWKL, we must construct a set $B \in \mathcal{S},\ B \subseteq \omega$, such that $B \not\leq_T A$ and $A \not\leq_T B$. To achieve this goal, we use the theorems listed in the previous section. By previous remarks in this section, we may assume that A is K-trivial.

First, using the axiom WWKL, construct a set $B_0 \subseteq \omega$, $B_0 \in \mathcal{S}$, such that B_0 is random relative to A (for our purposes we could also take B_0 random relative to $\emptyset$). Now, since B_0 is random relative to A, it follows that B_0 is also random (relative to $\emptyset$), from which it follows that B_0 is not K-trivial, and therefore $B_0 \not\leq_T A$. Hence, if $A \not\leq_T B_0$ then we have proven the theorem, so assume that $A \leq_T B_0$. Thus, we have that $A \leq_T B_0$ for some random $B_0 \in \mathcal{S}$.

Fix an oracle Turing machine Φ such that $\Phi^{B_0} = A$. Recall that, by Theorem 3.1, the $\Pi^{0,A}_2$-class given by

$$R_0 = \{f \in 2^\omega : \Phi^f = A\} \subseteq 2^\omega$$

satisfies $\mu(R_0) = 0$ and $B_0 \in R_0$. By our remarks following the statement of Theorem 3.3 in the previous section, we have that $R_0 \subseteq R$ for some Π^0_2-class R such that $\mu(R) = 0$. Therefore, there exists a Π^0_2-class $R \subseteq 2^\omega$ such that $B_0 \in R$, for some random set $B_0 \in \mathcal{S}$, $B_0 \subseteq \omega$, and $\mu(R) = 0$.

Now, Theorem 3.4 produces a cost function $c_R(n, s),\ n, s \in \omega$, that satisfies the limit condition and such that if $X \subseteq \omega$ is any Δ^0_2 set that possesses a computable approximation that obeys c, then *every* random set $C \in R$ computes X. Therefore, in particular, we may set $C = B_0$ since we know that $B_0 \in \mathcal{S},\ B \subseteq \omega$, is random and $B_0 \in R$.

Using Theorem 3.5, let $A_0 \subseteq \omega$, $A \leq_T A_0$, be a c.e. K-trivial set (A_0 need not be in the ω-model $\mathcal{S}$). Recall that A_0 is low. Therefore, we may apply Theorem 3.6 to construct a c.e. set $B_1 \subseteq \omega$ that obeys the cost function $c_R(n, s)$ from the previous paragraph, and such that $B_1 \not\leq_T A_0$ (and thus $B_1 \not\leq_T A$). Note that $B_1 \in \mathcal{S}$, since B_1 obeys c_R, and therefore $B_1 \leq_T B_0$. If we have that $A \not\leq_T B_1$, then we have proven Theorem 4.1, so assume that $A \leq_T B_1$. Since $B_1 \not\leq_T A$, we have that $A <_T B_1$.

Recall that $A \subseteq \omega$ is Δ^0_2 and incomputable. This enables us to apply Theorem 3.7 to produce c.e. sets $B_2, B_3 \subseteq \omega,\ B_2, B_3 \in \mathcal{S}$, such that $B_2 \oplus B_3 \equiv_T B_1$ and $A \not\leq_T B_2, B_3$. Now, since $A <_T B_1$, it follows that at least one of $B_2, B_3 \subseteq \omega$ satisfies $B_i \not\leq_T A$ (otherwise we would have that $B_1 \equiv_T B_2 \oplus B_3 \leq_T A$, a contradiction), $i \in \{2, 3\}$. Furthermore, by our construction of B_i, we also have that $A \not\leq_T B_i$. Therefore, setting $B = B_i$ yields a set $B \in \mathcal{S},\ B \subseteq \omega$, that is Turing incomparable with A. □

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