EFFECTIVELY APPROXIMATING MEASURABLE SETS BY OPEN SETS

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Abstract. We examine an effective version of the standard fact from analysis which says that, for any $\varepsilon > 0$ and any Lebesgue-measurable subset of Cantor space, $X \subseteq 2^\omega$, there is an open set $U_\varepsilon \subseteq 2^\omega$, $U_\varepsilon \supseteq X$, such that $\mu(U_\varepsilon) \leq \mu(X)+\varepsilon$, where $\mu(Z)$ denotes the Lebesgue measure of $Z \subseteq 2^\omega$, that arises naturally in the context of algorithmic randomness.

More specifically, our main result shows that for any given rational numbers $0 \leq \varepsilon < \varepsilon' \leq 1$, and uniformly computably enumerable sequence $\{U_n\}_{n\in\omega}$ of Σ^0_1-classes such that $(\forall n)[\mu(U_n) \leq \varepsilon]$, there exists a $\Sigma^{0,\emptyset'}_1$-class, Y, such that $Y \supseteq \liminf_n U_n$, and $\mu(Y) \leq \varepsilon'$. Moreover, Y can be obtained uniformly from ε, ε', and a u.c.e. index for $\{U_n\}_{n\in\omega}$. This answers a recent question of Bienvenu, Muchnik, Shen, and Vereshchagin. We also determine the truth-values of several modifications of our main result, showing that several similar, but stronger, statements are false.

1. Introduction

Recently, there has been much interest in the subfield of effective measure theory that examines randomness properties from the algorithmic viewpoint. The main goal of this line of research is to better understand the nature of algorithmic randomness by relating randomness properties to computability-theoretic properties, such as Turing reducibility. For an introduction to algorithmic randomness and Kolmogorov complexity, consult [DH10, DHNT06, Nie09]; for an introduction to computability theory, consult [Rog87, Soa87].

Some of the most recent results in algorithmic randomness relate the algorithmic randomness properties of a set $A \subseteq \omega$ to its ability to effectively (i.e. computably) approximate Borel sets with respect to (Lebesgue) measure. For example, in [KH07] it is shown that $A \subseteq \omega$ is "randomly feeble" (i.e. K-trivial) if and only if every effectively closed set relative to A of positive measure contains an effectively closed set of positive measure (relative to $\emptyset$), or, equivalently, every effectively open set relative to A of measure strictly less than 1 is contained within an effectively open set of measure strictly less than 1. The author also characterizes this property in terms of a domination condition. Furthermore, [KH07] and [Nie09, Theorem 5.6.9] also characterize various instances of a reducibility notion based on randomness properties (called LR-reducibility) in terms of approximating Borel sets by open sets.

In this article we examine the effective content of the related, standard, well-known fact from classical mathematical analysis, which says that for every $\varepsilon > 0$ and (Lebesgue) measurable $X \subseteq 2^\omega$, there exists an open set U_ε such that

$$\mu(U_\varepsilon) \leq \mu(X) + \varepsilon \quad \text{and} \quad U_\varepsilon \supseteq X,$$

where $\mu(Z)$ denotes the Lebesgue measure of $Z \subseteq 2^\omega$. In other words, every measurable set can be covered by an open set of arbitrarily close measure. Our main result is an analogue of several other well-known results in the same vein, including that result in effective measure theory which plays a significant role in effective randomness, and says that every uniform

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sequence of Σ^0_n-classes can be uniformly approximated (i.e. covered) by $\Sigma_1^{0,\emptyset^{(n-1)}}$-classes of arbitrarily close measure [Kau91, Kur81]. One important and immediate consequence of this result says that being $(n+1)$-random is no different than being 1-random relative to $\emptyset^{(n)}$. This consequence allows one to apply arguments and techniques involving open sets to higher randomness notions, such as n-randomness, $n \in \omega,\ n > 1$. Questions regarding approximating Borel sets (with respect to Lebesgue measure) via effectively open and closed sets have been considered by various mathematicians in recent years, including [BMSV10, KH07] and others.

Before we state our main theorem (Theorem 3.1), we wish to introduce some of the main concepts used in its statement. Given a sequence of subsets of Cantor space, $\{U_n\}_{n\in\omega}$, we define $\liminf_n U_n$ as follows

$$\liminf_n U_n = \bigcup_{n\in\omega} \bigcap_{k\geq n} U_k.$$

In other words, for every $f \in 2^\omega$ we have that $f \in \liminf_n U_n$ if and only if $f \in U_k$, for cofinitely many $k \in \omega$. It follows that if $(\forall n)[\mu(U_n) \leq \varepsilon]$, for some $\varepsilon \in \mathbb{R}$, then we have that $\mu(\liminf_n U_n) \leq \varepsilon$; more generally, we have that $\mu(\liminf_n U_n) \leq \liminf_n \mu(U_n)$. Roughly speaking, our main theorem says that if for every $n \in \omega$ we have that $U_n \subseteq 2^\omega$ is a sufficiently simple subset of Cantor space such that $\mu(U_n) \leq \varepsilon$, then, for any given $\varepsilon' > \varepsilon$, there exists a sufficiently simple set $Y \subseteq 2^\omega$ such that

$$\liminf_n U_n \subseteq Y \quad \text{and} \quad \mu(Y) \leq \varepsilon'.$$

Moreover, $Y \subseteq 2^\omega$ can be obtained uniformly from $\varepsilon,\ \varepsilon'$, and a *u.c.e.* index the sequence $\{U_n\}_{n\in\omega}$.

Our main theorem (Theorem 3.1) answers an outstanding question of Bienvenu, Muchnik, Shen, and Vereshchagin [BMSV10]. More specifically, [BMSV10] asks if (the first part of) the following theorem holds.

Theorem 3.1. *Let $0 \leq \varepsilon < \varepsilon' \leq 1$ be rational numbers, and let $\{U_n\}_{n\in\omega}$ be a sequence of uniformly Σ^0_1-classes (in Cantor space) such that $\mu(U_n) \leq \varepsilon$ for every $n \in \omega$. Then there exists a $\Sigma_1^{0,\emptyset'}$-class $Y \subseteq 2^\omega$ such that $\mu(Y) \leq \varepsilon'$ and $U = \liminf_n U_n \subseteq Y$, where*

$$U = \liminf_n U_n = \bigcup_{n\in\omega} \bigcap_{k\geq n} U_k.$$

Furthermore, a $\Sigma_1^{0,\emptyset'}$ index for $Y \subseteq 2^{<\omega}$ can be obtained uniformly from $\varepsilon, \varepsilon'$, and a u.c.e. index for the sequence of sets $U_n,\ n \in \omega$.

The main goal of [BMSV10] is to simplify the proofs of several theorems from algorithmic randomness, by putting them in a common perspective. One of the general results that the authors establish is a weaker version of Theorem 3.1 which is essentially identical to Theorem 3.1, except that $U = \liminf_n U_n$ is replaced by $U_0 = \bigcup_{n\in\omega}(\bigcap_{k\geq n} U_k)^o$, where Z^o denotes the interior of $Z \subseteq 2^\omega$. This is [BMSV10, Theorem 6]. The authors then use this weaker theorem to prove the following result of Miller, Nies, Stephan, Terwijn [Mil04, NST05].

Theorem 1.1. [NST05, Theorem 2.8][Mil04, Corollary 2] *For all $f \in 2^\omega$, we have that f is 2-random if and only if*

$$(\exists c)(\exists^\infty n)[C(f\restriction n) \geq n - c],$$

where $C(\sigma)$ denotes the plain Kolmogorov complexity of $\sigma \in 2^{<\omega}$. In other words, $f \in 2^\omega$ is 2-random if and only if f is infinitely often C-maximizing.

Recently, J. Miller has proved the following theorem [Mil10, Theorem 4.1], which is analogous to [NST05, Theorem 2.8] (above), but with prefix-free Kolmogorov complexity (i.e. $K(\sigma),\ \sigma \in 2^{<\omega}$) replacing plain Kolmogorov complexity (i.e. $C(\sigma),\ \sigma \in 2^{<\omega}$). The converse to [Mil10, Theorem 4.1] was shown by Yu, Ding, and Downey [YDD04].

Theorem 1.2. [Mil10, Theorem 4.1] *Suppose that $f \in 2^\omega$ is 2-random. Then we have that*

$$(\exists c)(\exists^\infty n)[K(f \upharpoonright n) \geq n + K(n) + c].$$

In other words, if $f \in 2^\omega$ is 2-random, then f is infinitely often K-maximizing.

This raises the following (somewhat vague) question, to which we do not know the answer.

Question 1.3 (J. Miller)**.** *Is there a "direct" proof of Theorem 1.2 from Theorem 3.1?*

In Section 4, we prove a partial converse to Theorem 3.1. It essentially says that our construction of $Y \subseteq 2^\omega$ in Theorem 3.1 is optimal, since it is uniform in ε' ($> \varepsilon$). More precisely, we have the following theorem.

Theorem 4.1. *Let $D \subseteq \omega$ be such that Theorem 3.1 holds with D in place of $\emptyset'$, uniformly in $\varepsilon'(> \varepsilon)$. Then $\emptyset' \leq_T D$.*

In particular, there is a set $U \subseteq 2^\omega$ of the form $U = \liminf_n [U_n]$, for some u.c.e. collection of sets $U_n \subseteq 2^{<\omega}$, $n \in \omega$, such that if $D \subseteq \omega$ satisfies Theorem 3.1 in place of $\emptyset'$, uniformly in ε', for this particular U, then $\emptyset' \leq_T D$.

Basically, Theorem 4.1 says that Theorem 3.1 is optimal in the sense that any set $D \subseteq \omega$ that satisfies Theorem 3.1 in place of $\emptyset'$, and uniformly in ε', must compute $\emptyset'$. Therefore, $\emptyset'$ is the weakest set that satisfies Theorem 3.1. In other words, Theorem 4.1 says that the class of sets that satisfy Theorem 3.1 in place of $\emptyset'$, and uniformly in ε', is equal to the cone above $\emptyset'$.

In Section 5 we show that we cannot relax the uniformity hypothesis in Theorem 4.1, because if we did then Theorem 4.1 would fail due to a cone avoidance property. In particular, we prove the following.

Theorem 5.2. *Let $C \subseteq \omega$ be any incomputable set. Then the class of sets $X \subseteq \omega$ such that for any given $0 < \varepsilon < \varepsilon' < 1$, $\varepsilon, \varepsilon' \in \mathbb{Q}$, and uniformly Σ^0_1-classes $\{U_n\}_{n\in\omega}$ such that $\mu(U_n) \leq \varepsilon$, $n \in \omega$, there is a $\Sigma^{0,X}_1$-class $[W_X]$ such that*

$$\mu([W_X]) \leq \varepsilon' \quad \& \quad [W_X] \supseteq \liminf_n U_n$$

contains a member $X_0 \subseteq \omega$ such that $C \not\leq_T X_0$.

In other words, the class of sets X that satisfy Theorem 4.1 above without the uniformity condition (with respect to ε') has the (upper) cone avoidance property.

In Section 6, we show that if the hypothesis of Theorem 3.1 that says $(\forall n)[\mu([U_n]) \leq \varepsilon]$ is weakened to say that $(\exists^\infty n)[\mu([U_n]) \leq \varepsilon]$, then the resulting statement is false. In particular, we prove Theorem 6.2 below, which implies Theorem 6.1 below. Theorem 6.1 answers a question of J. Miller, and A. Shen. It was originally thought that if one replaced Theorem 3.1 with the negation of Theorem 6.2 (if it were true) in Question 1.3 above, then one could use the machinery of Solovay functions to give a positive answer to the resulting question. However, Theorem 6.1 suggests that this approach will not work.

Theorem 6.1. *Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \frac{3}{4}$ (note that $0 \leq \varepsilon < \varepsilon' \leq 1$ and $\varepsilon, \varepsilon' \in \mathbb{Q}$). There exists a sequence of uniformly Σ^0_1-classes (in Cantor space), $\{[U_n]\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that $\mu([U_n]) \leq \varepsilon$ for infinitely many $n \in \omega$ and for all $\Sigma^{0,\emptyset'}_1$-classes, $[Y] \subseteq 2^\omega$, $Y \subseteq 2^{<\omega}$, such that $\mu([Y]) \leq \varepsilon'$ we have that $U = \liminf_n [U_n] \nsubseteq [Y]$, where*

$$U = \liminf_n [U_n] = \bigcup_{n\in\omega} \bigcap_{k\geq n} [U_k].$$

Theorem 6.2. *Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \frac{3}{4}$ (note that $0 \leq \varepsilon < \varepsilon' \leq 1$ and $\varepsilon, \varepsilon' \in \mathbb{Q}$). There exists a sequence of uniformly Σ^0_1-classes (in Cantor space), $\{[U_n]\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that*

$\mu([U_n]) \leq \varepsilon$ for infinitely many $n \in \omega$ and for all $\Sigma_1^{0,\emptyset'}$-classes, $[Y] \subseteq 2^\omega$, $Y \subseteq 2^{<\omega}$, such that $\mu([Y]) \leq \varepsilon'$ we have that

$$U_0 = \bigcup_{n\in\omega}(\bigcap_{k\geq n}[U_k])^o \nsubseteq [Y],$$

where $Z^o \subseteq 2^\omega$ denotes the interior of $Z \subseteq 2^\omega$.

2. Basic Definitions and Notation

Let $2^{<\omega}$ denote the full binary tree (i.e. the set of finite binary sequences), and let 2^ω denote Cantor space (i.e. the set of infinite binary sequences).

For every $\sigma \in 2^{<\omega}$, let $[\sigma] \subseteq 2^\omega$ denote the basic clopen set

$$[\sigma] = \{f \in 2^\omega : \sigma \subset f\}.$$

The sets $[\sigma]$, $\sigma \in 2^{<\omega}$, form a basis for the topology of 2^ω. More generally, if $A \subseteq 2^{<\omega}$, let

$$[A] = \{f \in 2^\omega : (\exists \sigma \in A)[\sigma \subset f]\} = \bigcup_{\sigma\in A}[\sigma].$$

Finally, for every (Lebesgue measurable) $X \subseteq 2^\omega$, let $\mu(X)$ denote the Lebesgue measure of X. Note that μ is computable in the sense that the function that assigns to every $\sigma \in 2^{<\omega}$ the value

$$\mu([\sigma]) = 2^{-|\sigma|} \in \mathbb{Q}$$

is a computable function.

Throughout this article we will mostly employ the computability-theoretic notation and conventions found in [Soa87]. In particular, the reader should note that for a given computably enumerable set, $U \subseteq \omega$, we will use U_s, $s \in \omega$, to denote the (finite) set of elements enumerated into U by stage s. Also, we use $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ to denote a fixed computable pairing function. For more information on basic computability theory consult [Soa87].

3. Our Main Theorem

In this section we will use [BMSV10, Theorem 6], along with the Lebesgue Density Theorem, to prove the following theorem.

Theorem 3.1. *Let $0 \leq \varepsilon < \varepsilon' \leq 1$ be rational numbers, and let $\{U_n\}_{n\in\omega}$ be a sequence of uniformly Σ_1^0-classes (in Cantor space) such that $\mu(U_n) \leq \varepsilon$ for every $n \in \omega$. Then there exists a $\Sigma_1^{0,\emptyset'}$-class $Y \subseteq 2^\omega$ such that $\mu(Y) \leq \varepsilon'$ and $U = \liminf_n U_n \subseteq Y$, where*

$$\liminf_n U_n = \bigcup_{n\in\omega}\bigcap_{k\geq n} U_k.$$

Furthermore, a $\Sigma_1^{0,\emptyset'}$ index for $Y \subseteq 2^{<\omega}$ can be obtained uniformly from $\varepsilon, \varepsilon'$, and a u.c.e. index for the sequence of sets U_n, $n \in \omega$.

Recall that [BMSV10, Theorem 6] is essenitally the same as Theorem 3.1 above, except that $U = \liminf_n U_n$ is replaced by $U_0 = \bigcup_{n\in\omega}(\bigcap_{k\geq n} U_k)^o$, where Z^o denotes the interior of $Z \subseteq 2^\omega$.

We now state the Lebesgue Density Theorem.

Theorem 3.2 (Lebesgue Density Theorem)**.** *Let $X \subseteq 2^\omega$ be such that $\mu(X) > 0$. Then, for any given $0 \leq \varepsilon < 1$, there exists $\sigma \in 2^{<\omega}$ such that*

$$\frac{\mu([\sigma] \cap X)}{\mu([\sigma])} \geq \varepsilon.$$

Proof of Theorem 3.1. First of all, for any given set $X \subseteq 2^\omega$ and $\delta \in \mathbb{Q}$, $0 < \delta < 1$, we make the following definition.

Definition 3.3. *Let $Int_\delta(X) \subseteq 2^\omega$ denote the union of all $[\sigma] \subseteq 2^\omega$, $\sigma \in 2^{<\omega}$, such that*

$$\frac{\mu(X \cap [\sigma])}{\mu([\sigma])} > 1 - \delta.$$

The following lemma collects several basic but important properties about $Int_\delta(X)$.

Lemma 3.4. *Fix $X, X_n \subseteq 2^\omega$, $n \in \omega$, and $\delta \in \mathbb{Q}$, $0 < \delta < 1$.*

(i) *$Int_\delta(X)$ is an open set. Moreover, $Int_\delta(X)$ is effectively open whenever X is effectively open and an effective index for $Int_\delta(X)$ can be uniformly obtained from effective indices for X and δ.*

(ii) *$Int_\delta(X)$ covers X up to a set of measure zero.*

(iii)
$$\mu(Int_\delta(X)) \leq \frac{1}{1-\delta}\mu(X).$$

(iv)
$$Int_\delta\left(\bigcap_{i\in\omega} X_i\right) \subseteq \bigcap_{i\in\omega} Int_\delta\left(X_i\right).$$

Proof. The proof of (i) is easy and follows directly from the definitions; we therefore leave it to the reader.

To prove (ii), assume the contrary, i.e. suppose that $\mu(X \setminus Int_\delta(X)) > 0$. Then, by the Lebesgue Density Theorem (above) it follows that there exists $\sigma \in 2^{<\omega}$ for which

$$\frac{\mu([\sigma] \cap (X \setminus Int_\delta(X))}{\mu([\sigma])} > 1 - \delta,$$

from which it follows that

$$\frac{\mu([\sigma] \cap X)}{\mu([\sigma])} > 1 - \delta,$$

and so $\sigma \in Int_\delta(X)$, a contradiction. This proves (ii).

To prove (iii), first write $Int_\delta(X)$ as a countable disjoint union of basic open sets (in 2^ω) as follows:

$$Int_\delta(X) = \bigcup_{i\in\omega} [\sigma_i],$$

such that for each $i \in \omega$ we have that

$$\frac{\mu(X \cap [\sigma_i])}{\mu([\sigma_i])} > 1 - \delta, \text{ i.e. } \frac{\mu(X \cap [\sigma_i])}{1-\delta} > \mu([\sigma_i]).$$

By (ii) above we must have that $\sum_{i\in\omega} \mu(X \cap [\sigma_i]) = \mu(X)$. Therefore, summing the last displayed inequality above over all $i \in \omega$ yields (iii).

The proof of (iv) follows directly from the definitions, and is left to the reader. □

We now continue with the proof of Theorem 3.1 above. Suppose that we are given

$$U = \bigcup_{n\in\omega} \bigcap_{k\geq n} U_k$$

as in the statement of the theorem. It follows that

$$U = \bigcup_{n\in\omega} \bigcap_{k\geq n} U_k \subseteq^* \bigcup_{n\in\omega} Int_\delta\left(\bigcap_{k\geq n} U_k\right) = \bigcup_{n\in\omega}\left[Int_\delta\left(\bigcap_{k\geq n} U_k\right)\right]^o \subseteq \bigcup_{n\in\omega}\left[\bigcap_{k\geq n} Int_\delta\left(U_k\right)\right]^o,$$

where $X_0 \subseteq^* X_1$, $X_0, X_1 \subseteq 2^\omega$, denotes the fact that $\mu(X_1 \setminus X_0) = 0$, and Z^o denotes the interior of $Z \subseteq 2^\omega$. The first step $\subseteq^*$ displayed above follows from Lemma 3.4 (ii); the second step = follows from Lemma 3.4 (i); and the third step $\subseteq$ follows from Lemma 3.4 (iv). For each $n \in \omega$, let

$$V_n = Int_\delta(U_k).$$

For now we want to cover

$$\bigcup_{n\in\omega}[\cap_{k\geq n}V_k]^o$$

with a $\Sigma_1^{0,\emptyset'}$-class $[W] \subseteq 2^\omega$ such that

$$\mu([W]) \leq \frac{\varepsilon' + \varepsilon}{2}.$$

By our previous remarks displayed above it will then follow that $[W]$ covers U up to a set of measure zero.

It follows from Lemma 3.4 (iii) above and our hypothesis on $\{U_k\}_{k\in\omega}$ that for each $\delta \in \mathbb{Q}$, $0 < \delta < 1$, and $n \in \omega$, we have that $\mu(V_n) \leq \frac{\varepsilon}{1-\delta}$. Furthermore, by our construction of V_n, $n \in \omega$, and Lemma 3.4 (i) above it follows that $\{V_k\}_{k\in\omega}$ is a uniformly computable sequence of open sets in Cantor space. Therefore, the sequence $\{V_n\}_{n\in\omega}$ satisfies the hypotheses of [BMSV10, Theorem 6] and by choosing δ small enough it follows that the class $[W] \subseteq 2^\omega$ mentioned in the previous paragraph exists.

We now turn our attention to finishing the proof of Theorem 3.1 by constructing a $\Sigma_1^{0,\emptyset'}$-class $V \subseteq 2^\omega$ of measure at most $\frac{\varepsilon'-\varepsilon}{2}$ that covers $U \setminus [W] \subseteq 2^\omega$.

3.0.1. *Constructing* $V \subseteq 2^{<\omega}$. The existence of V is a corollary of Lemma 3.6 (below), which follows directly from the following result of Kautz and Kurtz [Kau91, Kur81]. We omit the proof of Lemma 3.6, which follows directly from the following theorem.

Theorem 3.5. [Kau91, Kur81][DHNT06, Theorem 12.5(iv)] *From the index of a Π_n^0-class T and $q \in \mathbb{Q}$, one can $\emptyset^{(n)}$-compute the index of an open Σ_{n-1}^0-class (i.e. a $\Sigma_1^{0,\emptyset^{(n-2)}}$-class) $U \supseteq T$ such that $\mu(U) - \mu(T) < q$. Moreover, if $\mu(T)$ is computable from $\emptyset^{(n-1)}$, then the index of U can be found computably from $\emptyset^{(n-1)}$.*

Lemma 3.6. *Let $V_n \subseteq 2^\omega$, $n \in \omega$, be a uniformly computable collection of Π_2^0-classes, all of measure zero. Then, for any given $\varepsilon > 0$, there exists a $\Sigma_1^{0,\emptyset'}$-class $V \subseteq 2^\omega$ such that*

$$\mu(V) \leq \varepsilon \quad \text{and} \quad \bigcup_{i=0}^{\infty} V_n \subseteq V.$$

Moreover, a $\Sigma_1^{0,\emptyset'}$ index for V can be obtained uniformly and effectively from a u.c.e. index for the sequence of sets V_n, $n \in \omega$.

Corollary 3.7. *There exists a $\Sigma_1^{0,\emptyset'}$-class, $[V] \subseteq 2^\omega$, $V \subseteq 2^{<\omega}$, such that*

$$\mu([V]) \leq \frac{\varepsilon' - \varepsilon}{2} \quad \text{and} \quad U \setminus [W] \subseteq [V].$$

Moreover, a $\Sigma_1^{0,\emptyset'}$ index for $V \subseteq 2^{<\omega}$ can be obtained uniformly and effectively from a u.c.e. index for the sequence of sets V_n, $n \in \omega$.

Proof. Apply Lemma 3.6 to the uniformly computable sequence of Π_2^0-classes given by

$$[V_n] = (\bigcap_{k=n}^{\infty} U_k) \setminus [W] \subseteq 2^\omega.$$

□

We leave it to the reader to check that our $\Sigma_1^{0,\emptyset'}$ index for $Y = [W] \cup [V] \subseteq 2^{<\omega}$ in the statement of Theorem 3.1 is uniform in $\varepsilon, \varepsilon'$, and the u.c.e. index for the sequence of sets U_n, $n \in \omega$. This completes the proof of Theorem 3.1. □

4. Characterizing the oracles that satisfy Theorem 3.1 in place of $\emptyset'$

In this section we prove a sort of converse to Theorem 3.1. It essentially says that, because Theorem 3.1 is uniform in $\varepsilon' \in \mathbb{Q}$, our construction of $Y \subseteq 2^{<\omega}$ in Theorem 3.1 is optimal.

Theorem 4.1. *Suppose that $D \subseteq \omega$ is such that Theorem 3.1 holds with D in place of $\emptyset'$, uniformly in $\varepsilon'(> \varepsilon)$. Then we have that $\emptyset' \leq_T D$.*

In particular, there is a set $U \subseteq 2^\omega$ of the form $U = \liminf_n U_n$, for some u.c.e. collection of Σ^0_1-classes, $[U_n] \subseteq 2^\omega$, $U_n \subseteq 2^{<\omega}$, $n \in \omega$, such that if $D \subseteq \omega$ satisfies Theorem 3.1, uniformly in ε', for this particular U, then $\emptyset' \leq_T D$.

Proof. Let $\varepsilon = \frac{1}{6} < 1$, and let ε'_n, $n \in \omega$, be a computable sequence of rational numbers such that $\varepsilon'_n > \varepsilon$ for all $n \in \omega$, and $\lim_n \varepsilon'_n = \varepsilon$. We define a uniformly c.e. collection of sets U_n, $n \in \omega$, such that $(\forall n)[\mu([U_n]) \leq \varepsilon]$ as follows.

Let $\emptyset'_s$, $s \in \omega$, be a computable approximation to $\emptyset'$, and for all $k \in \omega$, let $\sigma_k = 0^k 1 \in 2^{<\omega}$. Now, for all $n \in \omega$ we enumerate every $\tau \supseteq \sigma_{2k+2}, \tau \in 2^{<\omega}$, into U_n if and only if $k - 1 \in \emptyset'_n$. Otherwise, if $k - 1 \notin \emptyset'_n$, we enumerate all $\tau \supseteq \sigma_{2k+1}1$ into U_n.

It is not difficult to check that for every $n \in \omega$, we have that

$$\mu([U_n]) = \sum_{i=0}^{\infty} 2^{-3-2i} = \frac{1}{6} = \varepsilon.$$

Also, since $\lim_s \emptyset'_s(n)$ exists for every $n \in \omega$, it follows that for every $\sigma \in 2^{<\omega}$, $\lim_n U_n(\sigma)$ exists. Therefore, if we set $U = \liminf_n [U_n]$, $U \subseteq 2^\omega$, then for every $k \in \omega$, $k \geq 1$, exactly one of the following two conditions holds:

(1) $[\sigma_{2k+2}] \subseteq U$, or
(2) $[\sigma_{2k+1}1] \subseteq U$.

Moreover, condition (1) holds if and only if $k - 1 \in \emptyset'$, and (2) holds otherwise. In this way, we have coded $\emptyset'$ into $\liminf_n U_n$. Next, we show how to extract this information via $D \subseteq \omega$.

One can compute $\emptyset'$ from $D \subseteq \omega$ as follows. To decide whether or not $x \in \omega$ is in $\emptyset'$, first choose $N \in \omega$ large enough so that $\varepsilon'_N - \varepsilon < \frac{1}{2^{2x+5}}$, and take a set $X_N \subseteq 2^{<\omega}$, $X_N \in \Sigma^{0,D}_1$, such that $\mu([X_N]) \leq \varepsilon'_N$ and $U \subseteq [X_N]$. Furthermore, suppose that $X_{N,s}$ is a D-computable c.e. approximation to X_N. Now, it follows from the construction of U_n, $n \in \omega$, and our definition of $N \in \omega$, that (relative to D) we will eventually witness *exactly* one of the following two things:

(1) $(\exists s)[\sigma_{2x+4} \in X_{N,s}]$, or
(2) $(\exists s)[\sigma_{2x+3}1 \in X_{N,s}]$.

If we witness (1), then it follows (by the construction of U_n, $n \in \omega$) that $x \in \emptyset'$. Otherwise, if we witness condition (2), then it follows (by the construction of U_n, $n \in \omega$) that $x \notin \emptyset'$. □

5. Cone Avoidance

The main goal of Section 5 is the proof of Theorem 5.2 below. Generally speaking, Theorem 5.2 says that, if we do not require the uniformity condition (with respect to ε') in Theorem 4.1, then Theorem 4.1 fails because of a cone avoidance property. In particular we will show that if we do not require the uniformity condition with respect to ε then every uniformly almost everywhere dominating Turing degree satisfies the conclusion of Theorem 3.1 in place of $\emptyset'$. For more information on uniformly almost everywhere dominating degrees see [Nie09, pages 234-7] or [CGM06]. In particular, it is known that the set of uniformly almost everywhere dominating degrees coincides with the set of Turing degrees $\mathbf{d}$ such that $\mathbf{0}' \leq_{LR} \mathbf{d}$ (for more information on $\leq_{LR}$ consult [Nie09]), and that for all $C \subseteq \omega$ there exists a uniformly almost everywhere dominating degree $\mathbf{d}$ such that $\mathbf{d}$ does not compute C. In other words, the class of almost everywhere dominating Turing degrees has the upper cone avoidance property. See [CGM06, Lemma 4.8] for more information.

We will also use the following lemma of Kjos-Hannssen, Miller, and Solomon [KHMS].

Lemma 5.1. [KHMS, Theorem 3.2] *For any $A, B \subseteq \omega$, the following are equivalent:*

(1) $A \leq_{LR} B$ *and* $A \leq_T B'$;
(2) *Every* $\Pi^{0,A}_1$*-class has a* $\Sigma^{0,B}_2$*-subclass of the same measure;*
(3) *Every* $\Sigma^{0,A}_2$*-class has a* $\Sigma^{0,B}_2$*-subclass of the same measure.*

Theorem 5.2. *Let $C \subseteq \omega$ be any incomputable set. Then the class of sets $X \subseteq \omega$ such that for any given $0 < \varepsilon < \varepsilon' < 1$, $\varepsilon, \varepsilon' \in \mathbb{Q}$, and uniformly Σ^0_1-classes $\{U_n\}_{n\in\omega}$ such that $\mu(U_n) \leq \varepsilon$, $n \in \omega$, there is a $\Sigma^{0,X}_1$-class $[W_X]$ such that*

$$\mu([W_X]) \leq \varepsilon' \;\;\&\;\; [W_X] \supseteq \liminf_n U_n = U$$

contains a member $X_0 \subseteq \omega$ such that $C \not\leq_T X_0$.

In other words, the class of sets X that satisfy Theorem 4.1 above without the uniformity condition (with respect to ε) has the (upper) cone avoidance property.

Proof. Let $0 < \varepsilon < \varepsilon' < 1$, $\varepsilon, \varepsilon' \in \mathbb{Q}$, and (for now) let D_X be any uniformly almost everywhere dominating set (i.e. a set of uniformly almost everywhere dominating Turing degree). Let $\{U_n\}_{n\in\omega}$ and U be as in the statement of the current theorem. Now, by Theorem 3.1 above there is a $\Sigma^{0,\emptyset'}_1$-class, $[W] \subseteq 2^\omega$, $W \subseteq 2^{<\omega}$, $W \leq_T \emptyset'$, such that $U \subseteq [W]$ and $\mu([W]) < \varepsilon'$. Furthermore, it is well-known that if D is uniformly almost everywhere dominating then $\emptyset' \leq_T D'$ and $\emptyset' \leq_{LR} D$; see [Nie09] for more details. Now, by [KHMS, Theorem 3.2] it follows that $[W]$ is contained in a Π^{0,D_X}_2-class of measure strictly less than ε', and, since every Π^{0,D_X}_2-class is the intersection of Σ^{0,D_X}_1-classes, it follows that there is a Σ^{0,D_X}_1-class of measure strictly less than ε' that covers $[W]$, and hence also covers U. We have shown that for every uniformly almost everywhere dominating set $D_X \subseteq \omega$ there is a Σ^{0,D_X}_1-class of measure strictly less than ε' that covers U.

Now, since the class of uniformly almost everywhere dominating degrees has the cone avoidance property(see [CGM06, Lemma 4.12] for more details), it follows that we can choose $X_0 = D_X \subseteq \omega$ as in the conclusion of the theorem (i.e. $C \not\leq_T X_0$). □

6. A stronger version of Theorem 3.1 that fails

In this section we prove the following theorem.

Theorem 6.1. *Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \frac{3}{4}$ (note that $0 \leq \varepsilon < \varepsilon' \leq 1$ and $\varepsilon, \varepsilon' \in \mathbb{Q}$). There exists a sequence of uniformly Σ^0_1-classes (in Cantor space), $\{[U_n]\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that $\mu([U_n]) \leq \varepsilon$ for infinitely many $n \in \omega$ and for all $\Sigma^{0,\emptyset'}_1$-classes, $[Y] \subseteq 2^\omega$, $Y \subseteq 2^{<\omega}$, such that $\mu([Y]) \leq \varepsilon'$ we have that $U = \liminf_n [U_n] \not\subseteq [Y]$, where*

$$U = \liminf_n [U_n] = \bigcup_{n\in\omega} \bigcap_{k\geq n} [U_k].$$

Theorem 6.1 says that if, in Theorem 3.1, we replace the condition $(\forall n)[\mu(U_n) \leq \varepsilon]$ by the condition $(\exists^\infty n)[\mu(U_n) \leq \varepsilon]$, then the resulting statement is false. Note that if $(\exists^\infty n)[\mu(U_n) \leq \varepsilon]$, then it follows that $\mu(U) = \mu(\liminf_n U_n) \leq \varepsilon$. Hence, classically, there exists an open set that covers U, but Theorem 6.2 says that in general this open set is not a Σ^0_1-class relative to $\emptyset'$.

To prove Theorem 6.1, we will actually prove the following (stronger) statement, which is analogous to Theorem 6.1 in the case where we are considering the theorem of [BMSV10] in place of Theorem 3.1. Recall that the theorem of [BMSV10] is the same as that of Theorem 3.1, except that it replaces $U = \liminf_n U_n$ by $U_0 = \bigcup_{n\in\omega}(\bigcap_{k\geq n} U_k)^o$, where $X^o \subseteq 2^\omega$ denotes the *interior* of $X \subseteq 2^\omega$.

Theorem 6.2. *Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \frac{3}{4}$ (note that $0 \le \varepsilon < \varepsilon' \le 1$ and $\varepsilon, \varepsilon' \in \mathbb{Q}$). There exists a sequence of uniformly Σ^0_1-classes (in Cantor space), $\{[U_n]\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that $\mu([U_n]) \le \varepsilon$ for infinitely many $n \in \omega$ and for all $\Sigma^{0,\emptyset'}_1$-classes, $[Y] \subseteq 2^\omega$, $Y \subseteq 2^{<\omega}$, such that $\mu([Y]) \le \varepsilon'$ we have that*

$$U_0 = \bigcup_{n\in\omega}(\bigcap_{k\ge n}[U_k])^o \nsubseteq [Y].$$

Proof of Theorem 6.2. Before we give the complete proof of Theorem 6.2, which diagonalizes against all possible $\Sigma^{0,\emptyset'}_1$-classes, we will give the basic module for diagonalizing against a *single* $\Sigma^{0,\emptyset'}_1$-class $[Y_0] \subseteq 2^\omega$, $Y_0 \subseteq 2^{<\omega}$. Afterwards, we will show how to put two of these modules together to diagonalize against a *pair* of $\Sigma^{0,\emptyset'}_1$-classes $[Y_0], [Y_1] \subseteq 2^\omega$, $Y_0, Y_1 \subseteq 2^{<\omega}$. Then, finally, we will show how to put infinitely many such modules together to diagonalize against all $\Sigma^{0,\emptyset'}_1$-classes $[Y_0], [Y_1], [Y_2], \ldots, [Y_n], \ldots \subseteq 2^\omega$, $Y_0, Y_1, Y_2, \ldots, Y_n, \ldots \subseteq 2^{<\omega}$. The construction and verification of the latter procedure is an application of the (well-known) infinite injury priority method.

Before we begin the proof of Theorem 6.2, we wish to point out to the reader that, to prove Theorem 6.2, we will construct a u.c.e. sequence of sets $U_n \subseteq 2^{<\omega}$, $n \in \omega$, such that for every $\Sigma^{0,\emptyset'}_1$-class $[Y] \subseteq 2^\omega$ that satisfies $\mu([Y]) \le \varepsilon' = \frac{3}{4}$, there exists some $\sigma \in 2^{<\omega}$ such that $[\sigma] \subseteq \liminf_n [U_n]$, but $[\sigma] \nsubseteq [Y]$. Therefore, we can replace U_0 in Theorem 6.2 by U in Theorem 6.1 if we so choose.

6.1. **Diagonalizing against a *single* $\Sigma^{0,\emptyset'}_1$-class $[Y_0] \subseteq 2^\omega$.** Let $Y_{0,s} \subseteq 2^{<\omega}$, $s \in \omega$, be a computable approximation to $Y_0 \subseteq 2^{<\omega}$. In other words, $Y_{0,s}$ is such that for every $\sigma \in 2^{<\omega}$ we have that $\sigma \in Y_0$ if and only if $\sigma \in Y_{0,s}$ for cofinitely many $s \in \omega$.

Lemma 6.3. *Without any loss of generality we can assume that $\mu([Y_{0,s}]) \le \frac{3}{4} = \varepsilon'$ for all $s \in \omega$.*

Proof. Let $Y \subset 2^{<\omega}$ be given such that $[Y] \subset 2^\omega$ is $\Sigma^{\emptyset',0}_1$, $\mu([Y]) \le \frac{3}{4} = \varepsilon'$, and let $Y_{0,s} \subseteq 2^{<\omega}$ be a computable approximation to Y. Note that, by the compactness of Cantor space 2^ω, we can assume without any loss of generality that $\sigma \in Y$ if and only if $\sigma \in Y_{0,s}$ for cofinitely many $s \in \omega$. Now, let $\hat{Y}_{0,s}$ be the computable approximation obtained by restricting $Y_{0,s}$ to a set of measure $\frac{3}{4}$ – i.e. if $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k, \ldots$, $k \in \omega$, is a fixed effective listing of the elements of $2^{<\omega}$, then for all $k \in \omega$ we let $\{\sigma_0, \ldots, \sigma_k\} \cap Y_{0,s} \subseteq \hat{Y}_{0,s}$ if and only if

$$\mu([\{\sigma_0, \ldots, \sigma_k\} \cap Y_{0,s}]) \le \frac{3}{4}.$$

First of all note that $\hat{Y}_{0,s}$ is obtained uniformly and effectively from $Y_{0,s}$ and the fixed rational parameter $\frac{3}{4}$.

We claim that $\hat{Y}_{0,s}$ is also a computable approximation to Y. For suppose not. Then, since $Y \in \Sigma^{\emptyset',0}_1$ and $[Y]$ is an open subset of Cantor space, it follows that there exists $\rho \in 2^{<\omega}$ such that for cofinitely many $s \in \omega$, $\rho \in Y_{0,s} \setminus \hat{Y}_{0,s}$. But then it follows (from our construction of $\hat{Y}_{0,s}$, $s \in \omega$) that $\mu([Y_{0,s}])$ for cofinitely many $s \in \omega$, and therefore $\mu([Y]) > \frac{3}{4}$ (or else we would have included ρ in $\hat{Y}_{0,s}$ for cofinitely many $s \in \omega$), a contradiction. □

Our construction proceeds as follows. Recall that we are trying to construct a u.c.e. sequence of sets $\{U_n\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that $(\exists^\infty n)[\mu([U_n]) \le \frac{1}{2} = \varepsilon]$, and, if $U = \liminf_n [U_n]$, then either $\mu([Y_0]) > \frac{3}{4} = \varepsilon'$, or else $U \nsubseteq [Y_0]$.

We will construct $\{U_n\}_{n\in\omega}$ u.c.e. such that $U = \liminf_n [U_n] \nsubseteq [Y_0]$. Our construction proceeds (in stages) as follows. Let $\sigma_0 = 0 \in 2^{<\omega}$ and $\sigma_1 = 1 \in 2^{<\omega}$ be the binary strings of length 1. At stage $s = 0$ we define $U_{n,0} = \emptyset$ for all $n \in \omega$. At stage $s > 0$, we check to see if $[\sigma_0] \nsubseteq [Y_{0,s}]$. If so, then we enumerate σ_0 into $U_{n,s}$ for all $n \le s$. Otherwise, we enumerate

σ_1 into $U_{s,s}$. For each $n \in \omega$ set $U_n = \cup_{s\in\omega} U_{n,s}$. This ends the construction of the uniformly computable sequence of effectively open sets $\{[U_n]\}_{n\in\omega}$.

To verify that $U = \liminf_n [U_n] \nsubseteq [Y_0]$ and $(\exists^\infty n)[\mu([U_n]) \leq \frac{1}{2}]$, consider the following two cases. Case 1 says that there are infinitely many stages $s \in \omega$ for which we have that $[\sigma_0] \nsubseteq [Y_{0,s}]$. By compactness (of 2^ω), it follows that $[\sigma_0] \nsubseteq [Y_0]$. We claim that $[\sigma_0] \subset U$. In fact, we have that $[\sigma_0] \subset [U_n]$, for every $n \in \omega$. To see why this is the case, let $n \in \omega$ be given. Then, since we are in case 1, it follows that there is some stage $s_0 \in \omega,\ s_0 > n$, such that $[\sigma_0] \nsubseteq [Y_{0,s_0}]$, at which point the construction above enumerates σ_0 into U_n at stage s_0. Now, since $[\sigma_0] \subseteq U = \liminf_n U_n$, but $[\sigma_0] \nsubseteq [Y_0]$, it follows that $U \nsubseteq [Y_0]$, as required. Note that, by the construction of $\{U_n\}_{n\in\omega}$ above, it follows that if $s \in \omega$ is a stage at which $[\sigma_0] \nsubseteq [Y_{0,s}]$, then $U_s = \{\sigma_0\}$ and $\mu([U_s]) = \frac{1}{2}$, since (by our construction of $\{U_n\}_{n\in\omega}$ above) at no later stage do we enumerate $\sigma_1 \in U_s$. Hence, since we are in case 1, there are infinitely many $n \in \omega$ such that $\mu([U_n]) = \frac{1}{2}$. We now move on to case 2.

Case 2 says that for cofinitely many stages $s \in \omega$, we have that $[\sigma_0] \subseteq [Y_{0,s}]$. In this case, since $\mu([Y_0]) \leq \frac{3}{4} < 1$, it follows that $[\sigma_1] \nsubseteq [Y_0]$ (or else by compactness it would follow that $\sigma_1 \in Y_{0,s}$ for cofinitely many $s \in \omega$, from which it would follow that for some $s \in \omega$ we have that $2^\omega \subseteq [Y_{0,s}]$, and hence $\mu([Y_{0,s}]) = 1$, a contradiction). We claim that $[\sigma_1] \subseteq U = \liminf_n [U_n]$, so that $U \nsubseteq [Y_0]$, as required. Let $s_0 \in \omega$ be such that for all $t \geq s_0$ we have that $[\sigma_0] \subseteq [Y_{0,t}]$. Now, by our construction of $\{U_n\}_{n\in\omega}$ above, it follows that for all $t \geq s_0$, we have that $U_t = \{\sigma_1\}$. Hence, $[\sigma_1] \subseteq U$, and there exist infinitely many $t \in \omega$ such that $\mu([U_t]) = \frac{1}{2}$. This ends the verification of our construction of $\{U_n\}_{n\in\omega}$, and completes the proof of our claim that it is possible to diagonalize against a *single* $\Sigma_1^{0,\emptyset'}$-class, $[Y_0]$.

6.2. **Diagonalizing against a *pair* of $\Sigma_1^{0,\emptyset'}$-classes $[Y_0], [Y_1] \subseteq 2^\omega$.** Now that we have given the basic module of our construction, we aim to give the reader an idea of how two of our modules fit together to construct the u.c.e sequence of sets $\{U_n\}_{n\in\omega}$, $U_n = \cup_{s\in\omega} U_{n,s}$. In the next subsection, we will give the complete construction of $\{U_n\}_{n\in\omega}$, which employs infinitely many of our basic modules in an infinite injury priority argument. Let $Y_{0,s}, Y_{1,s} \subseteq 2^{<\omega}$ be computable approximations to $Y_0, Y_1 \subseteq 2^{<\omega}$, as defined in the previous subsection.

Assume, for now, that we wish to diagonalize against a pair of $\Sigma_1^{0,\emptyset'}$-classes, $[Y_0] \subseteq 2^\omega$ and $[Y_1] \subseteq 2^\omega$. To do this, we employ two of our basic modules outlined in the previous subsection. Before we give the construction, however, we require some basic definitions and notation that will be used in the next subsection as well.

First, we construct a (finite) tree of strategies $\mathcal{T} \subseteq \omega^\omega$, as follows. Every node $\rho \in \mathcal{T}$ satisfies $|\rho| \leq 2$. Furthermore, $\mathcal{T}$ has exactly 4 nodes of length 1, and every node of length 1 has exactly 16 successor nodes of length 2. The nodes of $\mathcal{T}$ of length 1 correspond to the four nodes of $2^{<\omega}$ of length 2; we label these nodes $\tau_1, \tau_2, \tau_3, \tau_4 \in 2^{<\omega}$, listed in lexicographic order. Similarly, if $\rho_{\tau_i} \in \mathcal{T},\ 1 \leq i \leq 4$, is the node of length 1 corresponding to $\tau_i \in 2^{<\omega}$, then the successor nodes of $\rho_{\tau_i} \in \mathcal{T}$ correspond to the 16 nodes $\sigma_1, \sigma_2, \ldots, \sigma_{16} \in 2^{<\omega}$ of length 4 (listed in lexicographic order). For any node $\rho \in \mathcal{T}$, of length 2, we associate to $\rho = \langle i, j\rangle$ the pair of nodes $\langle \tau_i, \sigma_j\rangle$ defined above. We will also associate to every $\rho \in \mathcal{T}$ of length 2 a number, $N_{\rho,s} \in \omega$, that varies nondecreasingly with respect to the stages of our construction, $s \in \omega$.

Let $\rho \in \mathcal{T}$ be a node of length 2 on our tree of strategies, such that $\rho = \langle i, j\rangle,\ 1 \leq i \leq 4, 1 \leq j \leq 16$. We associate to $\rho \in \mathcal{T}$ the following strategy. Strategy ρ attempts to enumerate the clopen sets $[\tau_i], [\sigma_j] \subseteq 2^\omega$ into $[U_n]$ for all $n \geq N_{\rho,s}$, and may be injured by other strategies on $\mathcal{T}$ that redefine $N_{\rho,s}$ to be strictly larger at a later stage. If this happens infinitely often then our strategy $\rho \in \mathcal{T}$ fails to achieve its goal. Exactly how the strategy $\rho \in \mathcal{T},\ |\rho| = 2$, achieves its goal will be described in detail later on in this subsection.

For any two incomparable nodes $\rho_1, \rho_2 \in \mathcal{T}$, we say that ρ_1 is *to the left of* ρ_2 if we have that $\rho_1(l) < \rho_2(l)$, where $l \in \omega$ is least such that $\rho_1(l) \neq \rho_2(l)$. In the case that $\rho_1(l) > \rho_2(l)$, we say that ρ_1 is *to the right of* ρ_2.

We are now ready to give our construction of the u.c.e. sequence of sets $\{U_n\}_{n\in\omega}$, $U_n = \cup_{s\in\omega} U_{n,s}$, which diagonalizes against a pair of $\Sigma_1^{0,\emptyset'}$-classes, $[Y_0], [Y_1] \subseteq 2^\omega$. Our construction proceeds as follows.

At stage $s = 0$, set $U_{n,s} = \emptyset$ for all $n \in \omega$, and $N_{\rho,s} = 0 \in \omega$ for all $\rho \in \mathcal{T}$, $|\rho| = 2$. We say that strategy $\rho \in \mathcal{T}$, $|\rho| = 2$, *requires attention* at stage $s > 0$ if $\rho = \langle i, j\rangle$, $1 \leq i \leq 4, 1 \leq j \leq 16$, and we have that

$$[\tau_i] \nsubseteq [Y_{0,s}] \text{ and } [\sigma_j] \nsubseteq [Y_{1,s}].$$

At stage $s > 0$, we act as follows.

Fix a stage $s > 0$. Let $\rho = \langle i, j\rangle \in \mathcal{T}$, $|\rho| = 2$, $1 \leq i \leq 4, 1 \leq j \leq 16$, be the *least* node on $\mathcal{T}$ that requires attention at stage s. In other words, $\rho \in \mathcal{T}$ is such that there is no $\rho' \in \mathcal{T}$, $|\rho'| = 2$, to the left of ρ that requires attention at stage s (note that such a ρ must exist, since we may assume without any loss of generality, as we did in the previous subsection, that for all $s \in \omega$, we have that $\mu([Y_{0,s}]), \mu([Y_{1,s}]) \leq \frac{3}{4} = \varepsilon'$). In this case, we enumerate $\tau_i, \sigma_j \in 2^{<\omega}$ into $U_{n,s}$, for all $N_{\rho,s-1} \leq n \leq s$. We also set $N_{\rho',s} = s + 1$, for all $\rho' \in \mathcal{T}$, $|\rho'| = 2$, to the right of ρ, and set $N_{\rho',s} = N_{\rho',s-1}$, $\rho \in \mathcal{T}$, $|\rho| = 2$, otherwise. This ends our construction of $\{U_n\}_{n\in\omega}$, $U_n = \cup_{s\in\omega} U_{n,s}$. We now verify that our construction succeeds in producing a u.c.e. sequence of sets, $\{U_n\}_{n\in\omega}$, $U_n \subseteq 2^{<\omega}$, such that for infinitely many $n \in \omega$ we have that $\mu([U_n]) \leq \frac{1}{2} = \varepsilon$ and we also have that $U = \liminf_n [U_n] \nsubseteq [Y_0]$, $U = \liminf_n [U_n] \nsubseteq [Y_1]$.

To verify that our construction has indeed succeeded, we must consider the lim inf of the nodes of length 2 in $\mathcal{T}$ that require attention at some stage $s \in \omega$. In other words, we would like to consider the unique node $\rho = \langle i, j\rangle \in \mathcal{T}$, $|\rho| = 2$, $1 \leq i \leq 4, 1 \leq j \leq 16$, such that ρ requires attention at infinitely many stages $s \in \omega$, but all nodes of length 2 to the left of ρ require attention at only finitely many stages. It is not difficult to verify that such a ρ exists. By definition of ρ, fix a stage $s_0 \in \omega$ large enough such that at all subsequent stages $t \geq s_0$ no node to the left of ρ requires attention.

Note that in this case we have that $(\forall t \geq s_0)[N_{\rho,t} = N_{\rho,s_0}]$, in other words our construction of $\{U_n\}_{n\in\omega}$ above never resets the value of N_{ρ,s_0} after stage s_0. We claim that $(\forall n \geq N_{\rho,s_0})[\tau_i, \sigma_j \in U_n]$, and hence $[\tau_i], [\sigma_j] \subseteq U = \liminf_n [U_n]$. Note that, since during our construction of $\{U_n\}_{n\in\omega}$, $\rho = \langle i, j\rangle \in \mathcal{T}$ required attention infinitely often (by definition of ρ), then by compactness of 2^ω it follows that $[\tau_i] \nsubseteq [Y_0]$ and $[\sigma_j] \nsubseteq [Y_1]$. Therefore, we have that $U = \liminf_n [U_n]$ satisfies $U \nsubseteq [Y_0]$ and $U \nsubseteq [Y_1]$. Let $s_0 < s_1 < s_2 < s_3 < \cdots < s_n < \cdots$ be an infinite sequence of stages such that for all $l > 0$ we have that $\rho \in \mathcal{T}$ requires attention at stage s_l. Fix $n \in \omega$, $n > s_0$, and let $s_l > n$. Then, by our construction of $\{U_n\}_{n\in\omega}$ above, and the fact that $N_{\rho,s_l} = N_{\rho,s_0}$, by our construction of $\{U_n\}_{n\in\omega}$ above we have that $\tau_i, \sigma_j \in U_{n,s_l}$, and thus our claim is valid. Next, we show that for all $l > 0$ we have that $\mu([U_{s_l}]) \leq \frac{1}{2}$.

We shall show that for every $l > 0$, the measure of $[U_{s_l}] \subseteq 2^\omega$ is *exactly* $\frac{1}{4} + \frac{1}{16} = \frac{5}{16} < \frac{1}{2} = \varepsilon$. To do this, it suffices to show that for every $l > 0$ we have that $U_l = \{\tau_i, \sigma_j\}$. To prove the latter claim, let $l > 0$ be given. Note that, by our construction of $\{U_n\}_{n\in\omega}$ above, we do not enumerate anything into U_n before stage $s = n$. Therefore, we have that $U_{s_l,s_l-1} = \emptyset$. By our construction above, and the definition of s_l, we know that at stage $s = s_l$ we will enumerate τ_i, σ_j into $U_{s_l,s_l} \subseteq 2^{<\omega}$. However, during stage $s = s_l$, we also set $N_{\rho',s_l} = s_l + 1$ for all $\rho' \in \mathcal{T}$ to the right of $\rho \in \mathcal{T}$. This means that no strategy to the right of ρ can enumerate anything into $U_{s_l,t}$, for any stage $t \geq s_l$. Furthermore, by definition of s_0, we know that no strategy to the left of ρ will require attention at any stage $t \geq s_l > s_0$. Therefore, it follows that $U_{s_l} = \langle \tau_i, \sigma_j\rangle$, as claimed.

This completes our demonstration of diagonalizing against two $\Sigma_1^{0,\emptyset'}$-classes $[Y_0], [Y_1] \subseteq 2^\omega$. In the next subsection, we move on to the general case, and give a complete proof of Theorem 6.2 above.

6.3. **The general case: Diagonalizing against all $\Sigma_1^{0,\emptyset'}$-classes of measure less than or equal to $\frac{3}{4}$.** Let $Y_{0,s}, Y_{1,s}, \ldots, Y_{n,s}, \ldots \subseteq 2^{<\omega}$ be a uniformly computable sequence of computable approximations to the (complete list of) $\Sigma_1^{0,\emptyset'}$-classes $Y_0, Y_1, \ldots, Y_n, \ldots \subseteq 2^{<\omega}$, as defined in the previous two subsections.

The proof of Theorem 6.2 is similar to the construction and verification given in the previous subsection. As before, we shall construct a tree of strategies $\mathcal{T} \subseteq \omega^{<\omega}$, however, now our tree of strategies shall be finitely branching, as opposed to finite. We construct $\mathcal{T}$ as follows. $\mathcal{T}$ has exactly 4 strings of length 1, and for every $n \in \omega$, if $\sigma \in \mathcal{T}$ is a string of length $k \in \omega$, then σ has exactly $2^{2(k+1)}$ successor nodes on $\mathcal{T}$. From our construction of $\mathcal{T}$, it follows that our tree of strategies in the previous subsection lives inside our current definition of $\mathcal{T}$. The main difference now is that our current tree of strategies is infinite.

We interpret nodes on $\mathcal{T}$ as in the previous subsection. For every $k \in \omega$, and $\sigma \in \mathcal{T}$ of length k, the $2^{2(k+1)}$ successor nodes of σ on $\mathcal{T}$ correspond to the $2^{2(k+1)}$ nodes of length $2(k+1)$ in $2^{<\omega}$; label these nodes $\sigma_1^{k+1}, \sigma_2^{k+1}, \ldots, \sigma_{2(k+1)}^{k+1}$ in lexicographic order. Now, if $\rho \in \mathcal{T}$ is of length $l \in \omega$, $\rho = \langle r_1, r_2, \ldots, r_l \rangle$, then the strategy associated with ρ attempts to ensure that $[\sigma_{r_1}^1], [\sigma_{r_2}^2], \ldots, [\sigma_{r_l}^l] \subseteq U = \liminf_n [U_n]$, by enumerating these clopen sets into the sequence $\{[U_n]\}_{n\in\omega}$. The precise way in which this is done will be described later; it is similar to that given in the previous subsection, when we diagonalized against a pair of $\Sigma_1^{0,\emptyset'}$-classes $[Y_0], [Y_1] \subseteq 2^\omega$.

For all $\rho = \langle r_1, r_2, \ldots, r_l \rangle \in \mathcal{T}$, we say that ρ *requires attention at stage* $s \in \omega$ if for every $1 \leq k \leq l$ we have that $[\sigma_{r_k}^k] \nsubseteq [Y_{k,s}]$. Note that if $\rho \in \mathcal{T}$ requires attention at infinitely many stages $s \in \omega$ and the strategy ρ succeeds (as described in the previous paragraph), then we have successfully diagonalized against the first l-many $\Sigma_1^{0,\emptyset'}$-classes $[Y_1], [Y_2], \ldots, [Y_l] \subseteq 2^\omega$. Also,note that (by our definition above) if $\rho \in \mathcal{T}$ requires attention at stage $s \in \omega$, then all $\tau \subseteq \rho$ also require attention at stage s.

As in the previous subsection, we also introduce the numbers $N_{\rho,s}$, $\rho \in \mathcal{T}$, $s \in \omega$. Thus, for every $\rho \in \mathcal{T}$, we think of $N_{\rho,s}$ as assigning a nondecreasing sequence of numbers (in stages $s \in \omega$) to ρ. Moreover, if the strategy $\rho = \langle r_1, r_2, \ldots, r_l \rangle \in \mathcal{T}$ is to succeed, then we will have that $\lim_s N_{\rho,s} = N$ exists, and for all $s \geq N$ we have that $[\sigma_{r_1}^1], [\sigma_{r_2}^2], \ldots, [\sigma_{r_l}^l] \subseteq \cap_{k\geq N}[U_k]$. Therefore, $[\sigma_{r_1}^1], [\sigma_{r_2}^2], \ldots, [\sigma_{r_l}^l] \subseteq U = \liminf_n [U_n]$. Again, this is similar to our construction in the previous subsection.

We are now ready to proceed with the construction and verification of $\{U_n\}_{n\in\omega}$, $U_n = \cup_{s\in\omega} U_{n,s}$, in Theorem 6.2. The main difference between this proof and those of the previous two subsections is that now we are required to diagonalize against *all* $\Sigma_1^{0,\emptyset'}$-classes $[Y_0], [Y_1], \ldots, [Y_n], \ldots \subseteq 2^\omega$ of measure less than or equal to $\varepsilon' = \frac{3}{4}$. Recall that $Y_{0,s}, Y_{1,s}, \ldots, Y_{n,s}, \ldots \subseteq 2^{<\omega}$ is a uniformly computable sequence of computable approximations to the generating sets $Y_0, Y_1, \ldots, Y_n, \ldots \subseteq 2^{<\omega}$, respectively, and that for every $n, s \in \omega$ we have that $\mu([Y_{n,s}]) \leq \frac{3}{4}$. Our construction proceeds as follows.

At stage $s = 0$, we set $U_{n,0} = \emptyset$ for all $n \in \omega$, and $N_{\rho,0} = 0$, for all $\rho \in \mathcal{T}$. At stage $s > 0$, we let $\rho_s = \langle r_1, r_2, \ldots, r_s \rangle \in \mathcal{T}$ be the leftmost (as defined in the previous subsection) node of length s that requires attention (ρ_s exists by our assumptions on the uniform sequence of computable approximations $\{Y_{n,s}\}_{n,s\in\omega}$). Now, we say that strategy ρ_s *receives attention* as follows. First, enumerate $\sigma_{r_k}^k \in 2^{<\omega}$ into $U_{n,s}$, for all $1 \leq k \leq s$ and $N_{\rho\restriction(k-1),s-1} \leq n \leq s$. Finally, we set $N_{\rho',s} = s+1$, for all $\rho' \in \mathcal{T}$ to the right of ρ_s or extending ρ_s (where "to the right" is as defined in the previous subsection), and $N_{\rho',s} = N_{\rho',s-1}$ otherwise. This ends our construction of $\{U_n\}_{n\in\omega}$, $U_n = \cup_{s\in\omega} U_{n,s}$. We now verify that, indeed, we have that $(\forall m)[\liminf_n [U_n] \nsubseteq [Y_m]]$, and $(\exists^\infty m)[\mu([U_m]) \leq \frac{1}{2} = \varepsilon]$.

First, we verify that $(\forall m)[\liminf_n [U_n] \nsubseteq [Y_m]]$. To do this, we must consider $\liminf_s \rho_s = f \in \omega^\omega$. That is, $f \in \omega^\omega$ is the unique infinite path through $\mathcal{T}$ such that for every $k \in \omega$, we have that $\rho' = f\restriction k \in \mathcal{T}$ receives attention at infinitely many stages, and every $\tau' \in \mathcal{T}$ to the left of f receives attention finitely often. It is not difficult to show that $f \in \omega^\omega$ exists. We claim that every strategy along f succeeds, and because of this we succeed in diagonalizing against all $\Sigma_1^{0,\emptyset'}$-classes $\{[Y_n]\}_{n\in\omega}$.

To prove this, let $m \in \omega$ be given. We will show that we succeed in diagonalizing against $[Y_m] \subseteq 2^\omega$ via strategy $f\restriction m = \rho = \langle r_1, r_2, \ldots, r_m\rangle \in \mathcal{T}$. Let $s_0 \in \omega,\ s_0 > m$, be large enough such that for all $t \geq s_0$, no requirement to the left of $f\restriction m$ receives attention at stage t (s_0 exists by our definition of f above). Note that, by our construction of $\{U_n\}_{n\in\omega}$ above, and our definition of $s_0 \in \omega$, we have that $N_{\rho,t} = N_{\rho,s_0}$, for all $t \geq s_0$.

We now claim that $(\forall k \geq N_{\rho,s_0})[\sigma^m_{r_m} \in U_k]$ (and hence $[\sigma^m_{r_m}] \subseteq \liminf_n [U_n]$), but $[\sigma^m_{r_m}] \nsubseteq [Y_m]$. The latter part of our claim follows from the fact that 2^ω is compact, and for infinitely many stages $s \in \omega$, we have that $\rho \in \mathcal{T}$ receives attention at stage s. On the other hand, by our definition of f we have that for every stage $s_1 \geq s_0$, there is a stage $t \geq s_1$ at which some strategy $\rho' \supseteq \rho$ receives attention. Moreover, it follows from our construction of $\{U_n\}_{n\in\omega}$ that at stage t we enumerate $[\sigma^m_{r_m}] \subseteq 2^\omega$ into U_n for all $N_{\rho,s_0} = N_{\rho,t} \leq n \leq t$. It now follows that $(\forall k \geq N_{\rho,s_0})[[\sigma^m_{r_m}] \subseteq [U_k]]$, and therefore we may conclude that $[\sigma^m_{r_m}] \subseteq \liminf_n [U_n]$, as required. Next, we verify that there are infinitely many $n \in \omega$ such that $\mu([U_n]) \leq \frac{1}{2} = \varepsilon$.

Let $n_0 \in \omega$ be given. We must show that there exists some $n \geq n_0$, $n \in \omega$, such that $\mu([U_n]) \leq \frac{1}{2}$. We proceed as follows. First, let $s_0 \in \omega, s_0 \geq n_0$, be any stage such that for all stages $t \geq s_0$, we have that ρ_t either extends $\rho = \rho_{s_0}$, or is to the right of ρ_{s_0} (it is not difficult to show that such an $s_0 \in \omega$ exists). Now, we claim that $\mu([U_{s_0}]) \leq \frac{1}{2}$. To see why this is the case, note that (by our construction of $\{U_n\}_{n\in\omega}$) we have that $U_{s_0,s_0-1} = \emptyset$. Also note that (by our construction of $\{U_n\}_{n\in\omega}$) at stage s_0 the measure of $[U_{s_0}]$ increases by at most

$$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{2^{2(s_0+1)}} < \frac{1}{3} < \frac{1}{2}.$$

Now, by our definition of $s_0 \in \omega$, and by the way we defined $N_{\tau,s_0} \in \omega$, for all $\tau \in \mathcal{T}$, we have that $(\forall t \geq s_0)[N_{\rho,t} = N_{\rho,s_0} \leq s_0]$, and for all $\rho' \in \mathcal{T}$ that either extend ρ, or lie to the right of ρ, we have that $N_{\rho',s_0} > s_0$. Therefore, by our construction of $\{U_n\}_{n\in\omega}$ above, we will not enumerate any new elements into $U_{s_0,t} \subseteq 2^{<\omega}$ at any stage $t > s_0$ (because, by our construction of $\{U_n\}_{n\in\omega}$ and the way we defined N_{τ,s_0}, $\tau \in \mathcal{T}$ at stage s_0, the only way we could enumerate a new element into U_{s_0} at some stage $t > s_0$ is if some strategy *to the left* of ρ_{s_0} received attention at stage t, which cannot happen by definition of $s_0 \in \omega$). Therefore, we have that $\mu([U_{s_0,t}]) \leq \frac{1}{2}$ for all $t \geq s_0$, from which it follows that $\mu([U_{s_0}]) \leq \frac{1}{2} = \varepsilon$, as required. □

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