INFINITE DIMENSIONAL PROPER SUBSPACES OF COMPUTABLE VECTOR SPACES

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ABSTRACT. This article examines and distinguishes different techniques for coding incomputable information into infinite dimensional proper subspaces of a computable vector space, and is divided into two main parts. In the first part we describe different methods for coding into infinite dimensional subspaces. More specifically, we construct several computable infinite dimensional vector spaces each of which satisfies one of the following:

(1) Every infinite/coinfinite dimensional subspace computes Turing's Halting Set $\emptyset'$;
(2) Every infinite/cofinite dimensional proper subspace computes Turing's Halting Set $\emptyset'$;
(3) There exists $x \in V$ such that every infinite dimensional proper subspace not containing x computes Turing's Halting Set $\emptyset'$;
(4) Every infinite dimensional proper subspace computes Turing's Halting Set $\emptyset'$.

Vector space (4) generalizes vector spaces (1) and (2), and its construction is more complicated. The same simple and natural technique is used to construct vector spaces (1)-(3). Finally, we examine the reverse mathematical implications of our constructions (1)-(4).

In the second part we examine the limitations of our simple and natural method for coding into infinite dimensional subspaces described in the previous paragraph. In particular, we prove that our simple and natural coding technique cannot produce a vector space of type (4) above, and that any vector space of type (4) must have "densely many" (from a certain point of view) finite dimensional computable subspaces. In other words, the construction of a vector space of type (4) is *necessarily* more complicated than the construction of vector spaces of types (1)-(3). We also introduce a new statement (in second order arithmetic) about the existence of infinite dimensional proper subspaces in a restricted class of vector spaces related to (1)-(3) above and show that it is implied by weak König's lemma in the context of reverse mathematics. In the context of reverse mathematics this gives rise to two statements from effective algebra about the existence of infinite dimensional proper subspaces (for a certain class of vector spaces) of the form $(\forall V)[X(V) \to A(V)]$ and $(\forall V)[X(V) \to B(V)]$, that each imply $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$, but such that the seemingly weaker statement $(\forall V)[X(V) \to A(V) \vee B(V)]$ is provable via $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. Furthermore, we highlight some general similarities between constructing of infinite dimensional proper subspaces of computable vector spaces and constructing solutions to computable instances of various combinatorial principles such as Ramsey's Theorem for pairs.

1. INTRODUCTION

Computable algebra is the branch of mathematical logic that deals with the algorithmic properties of algebraic structures, and dates back to the works of early mathematicians including Euclid, Gauss, and others. More recently the subject was formalized by Turing

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and others, leading to the well-known solutions of the word problem for groups by Novikov and Boone, and Hilbert's tenth problem by Matiyasevich and others.

This main theorem of this article answers a problem of Downey and others who asked about the proof-theoretic strength of the statement "every infinite dimensional vector space contains a proper infinite dimensional subspace" in second order arithmetic. Moreover this problem grew out of an attempt to classify the proof-theoretic strength of the well-known theorem from Commutative Algebra that says every Artinian ring is Noetherian. The latter problem was recently solved by the author.

More specifically, this article is a sequel to [6, 10, 13, 14] in which the author and others attempted to determine the reverse mathematical strengths of the statements "every Artinian ring is Noetherian," "every ring that is not a field contains a nontrivial ideal," and "every vector space of dimension at least 2 has a nontrivial subspace," over RCA_0. More information on commutative algebra including Artinian and Noetherian rings can be found in [1, 15, 25, 26]. We assume that the reader is familiar with reverse mathematics [39] as well as [6, 13, 14], although we will briefly review most of what we require from these sources. Recall that RCA_0 (recursive comprehension axiom) is the subsystem of second order arithmetic corresponding to the axiom that says Δ_1-definable sets (with parameters) exist (i.e. computable sets and Turing reductions exist); WKL_0 is the subsystem of second order arithmetic corresponding to the axiom of weak König's lemma which is RCA_0 conjuncted with the statement "every infinite binary branching tree has an infinite path;" ACA_0 (arithmetic comprehension axiom) is the subsystem of second order arithmetic corresponding to the axiom that says all arithmetically definable sets exist. It is known that ACA_0 is equivalent to saying that for every set A, the Halting Set relative to A, A', exists (for more information on the Halting Set and its relativization consult [40, 41, 46, 47]). More information on subsystems of second order arithmetic and the program of reverse mathematics, including RCA_0, WKL_0, and ACA_0, can be found in [39]. In [6] the author showed that the statement "every Artinian ring is of finite length" is equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$ ($\mathsf{B}\Sigma_2$ is a bounding principle for Σ_2 formulas; for more information see [19, 31]), and that the statement "every Artinian integral domain is Noetherian" is equivalent to WKL_0 over RCA_0. In [29], Montalbán calls a theorem of mathematics *nonrobust* whenever there exists another "similar" theorem that is not equivalent to the first theorem over RCA_0. Montalbán also points out that usually nonrobustness leads to theorems of mathematics that are not equivalent to any of the "big five" subsystems of second-order arithmetic: RCA_0 (recursive comprehension axiom), WKL_0 (weak König's lemma), ACA_0 (arithmetic comprehension), ATR_0 (arithmetic transfinite induction), and $\mathsf{\Pi}^1_1{-}\mathsf{CA}_0$ (Π^1_1-comprehension) in the context of ω-models. Recall that an ω-model is a model of second order arithmetic whose first order part is the standard natural numbers $\omega = \{0, 1, 2, \ldots\}$. More information on the "big five" subsystems of second-order arithmetic as well as ω-models can be found in [39]. Recently, finding theorems of mathematics that are not equivalent to the "big five" in the context of ω-models has become a topic of great interest among computability theorists (examples of recent articles include [21, 22, 28, 9]) because the proofs of these theorems correspond to nonstandard (i.e. interesting) mathematical arguments. However, computability theorists have yet to discover an *algebraic*[1] theorem that is not equivalent to one of the "big five" in the context of ω-models. Since it is nonrobust it could very well be the case that the statement "every Artinian ring is Noetherian" is not equivalent to any of the "big five" subsystems of second order arithmetic in the context of ω-models.

To classify the reverse mathematical strength of a theorem of second-order arithmetic (see [39] for more details) a mathematician essentially has to determine how much information

[1]Throughout this article our use of the term *algebra* is restricted to groups, rings, fields, and their actions.

can be coded into that theorem. In the context of ω-models[2] this intuition is made precise by Shore's notion of *computable entailment* [37]. More specifically, in order to show that one theorem of mathematics implies another in the context of reverse mathematics (and ω-models), one must essentially[3] code solutions of instances of the latter theorem into solutions of instances of the former theorem. On the other hand, to show that one given theorem does not imply another given theorem in the context of reverse mathematics (and ω-models), one must essentially show that solutions to instances of the latter theorem cannot be coded into solutions to instances of the former theorem. In other words, generally speaking, to prove a nonimplication one must establish some sort of limitation on the coding that can be done by the former theorem. For more information see [37].

To prove that the statement "every Artinian ring is of finite length" implies ACA_0 [6, Section 6] the author essentially constructed a computable ring R such that every infinite strictly descending chain of ideals in R codes Turing's Halting Set $\emptyset'$. More specifically, however, the ring R was a quotient of the ring Q generated by elements $\langle 1, X_n : n \in \omega\rangle$ such that $X_m X_n = 0$ for all $m, n \in \omega$. It is not difficult to see that Q resembles an infinite dimensional vector space, with all the ideals and quotients of Q (as a ring) corresponding to subspaces and quotient spaces of Q (as a vector space). The ring R was essentially an infinite dimensional quotient space of Q, modulo an infinite dimensional subspace. Thus, coding information into infinite dimensional proper subspaces of computable infinite dimensional vector spaces arises naturally in the context of determining the reverse mathematical strength of the theorem "every Artinian ring is Noetherian." This lead some mathematicians, including R. G. Downey and S. Lempp, (and others) to ask about the reverse mathematical strength of the statement "every infinite dimensional vector space has an infinite dimensional proper subspace." Another reason for examining vector spaces in the context of Artinian rings is that the proof of the theorem "every Artinian ring is Noetherian" [26], very roughly speaking, divides an Artinian ring into finitely many finite dimensional vector spaces, and uses the fact that the theorem holds for the vector spaces (i.e. any chain of strictly increasing subspaces in a finite dimensional vector space eventually stabilizes) to show that the theorem holds for the ring. Thus, the theory of vector spaces plays an important role in the proof that every Artinian ring is Noetherian. Yet another reason for asking these types of questions about infinite dimensional vector spaces is that it relates to [14].

This article is divided into two main parts. Both parts examine coding into infinite dimensional proper subspaces of infinite dimensional vector spaces. The first part consists of Sections 4 through 7 and examines the positive side of things. More specifically, we prove the following theorems in the system RCA_0.

Theorem 1.1 (RCA_0)**.** *There exists a computable infinite dimensional vector space V such that every infinite/coinfinite dimensional subspace of V computes the Halting Set $\emptyset'$.*

Theorem 1.2 (RCA_0)**.** *There exists a computable infinite dimensional vector space V such that every infinite/cofinite dimensional proper subspace of V computes the Halting Set $\emptyset'$.*

Theorem 1.3 (RCA_0)**.** *There exists a computable infinite dimensional vector space V, and $0 \neq x \in V$, such that every infinite dimensional subspace of V not containing x computes the Halting Set $\emptyset'$.*

Theorem 1.4 (RCA_0)**.** *There exists a computable infinite dimensional vector space V such that every infinite dimensional proper subspace of V computes Halting Set $\emptyset'$.*

[2]An ω-model is a model of RCA_0 whose first-order part is the standard natural numbers $\omega = \{0, 1, 2, \ldots\}$. These models are usually identified with their second-order parts, and it is known that $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-model iff $\mathcal{M}$ is closed under join $\oplus$ and Turing reducibility $\leq_T$.

[3]More precisely, one must code solutions of the latter theorem into finite iterations of solutions of the former theorem. For more information see [37].

The proof of Theorem 1.3 above is based on techniques developed in the proof of Theorem 1.1 above. The proof of Theorem 1.4 is different and more complex than the others. Theorems 1.1 and 1.2 were proven independently by Downey and Turetsky, and Downey, Greenberg, Kach, Lempp, Miller, Ng, and Turetsky, [DGKLMNT] respectively. Our proof of Theorem 1.1 is similar to that of Downey and Turetsky. We will present both our proof of Theorem 1.2, as well as the (different) proof of [DGKLMNT] which is based heavily on the results and constructions of [14]. Our proof of Theorem 1.2 is more direct and complicated, but constructs a "simpler" vector space (as we shall see later on), while [DGKLMNT] found a much simpler and cleaner proof using a "more complicated" vector space.

To examine our theorems above in the context of reverse mathematics, we now introduce five subsystems of second-order arithmetic, all of which we take to imply RCA_0 and the following axioms:

COINF_0: Every infinite dimensional vector space contains an infinite/coinfinite dimensional subspace.

COFIN_0: Every infinite dimensional vector space contains an infinite/cofinite dimensional proper subspace.

$\mathsf{x}-\mathsf{INF}_0$: For every infinite dimensional vector space V, and nonzero vector $x \in V$, there exists an infinite dimensional subspace of V that does not contain x.

INF_0: Every infinite dimensional vector space contains an infinite dimensional proper subspace.

Here we interpret the phrase "infinite dimensional" to mean "of arbitrarily large finite dimension." We will do this throughout the rest of this article. Later on in Section 4 below we will introduce another subsystem of second order arithmetic, $\mathsf{INF}_0^{\mathsf{2B}}$, which generally speaking says that one can always find proper infinite dimensional subspaces of vector spaces belonging to a certain class of vector spaces, called $2-$based vector spaces, that arise naturally in the contexts of computable and reverse algebra[4] and this article, and that we will define in Section 4 below. More precisely, let

$\mathsf{COINF}_0^{\mathsf{2B}}$: Every infinite dimensional 2-based vector space contains an infinite/coinfinite dimensional subspace.

$\mathsf{COFIN}_0^{\mathsf{2B}}$: Every infinite dimensional 2-based vector space contains an infinite/cofinite dimensional proper subspace.

$\mathsf{INF}_0^{\mathsf{2B}}$: Every infinite dimensional 2-based vector space contains an infinite dimensional proper subspace.

Now, if $X(V)$ is the predicate that says "V is a 2-based vector space," $A(V)$ is the predicate that says that "V contains a proper infinite/coinfinite dimensional subspace," and $B(V)$ is the predicate that says that "V contains a proper infinite cofinite dimensional subspace," then we have that

$$\mathsf{COINF}_0^{\mathsf{2B}} \text{ is of the form } (\forall V)[X(V) \to A(V)];$$

$$\mathsf{COFIN}_0^{\mathsf{2B}} \text{ is of the form } (\forall V)[X(V) \to B(V)];$$

and finally

$$\mathsf{INF}_0^{\mathsf{2B}} \text{ is of the form } (\forall V)[X(V) \to A(V) \vee B(V)].$$

We will show that $\mathsf{COINF}_0^{\mathsf{2B}}$ and $\mathsf{COFIN}_0^{\mathsf{2B}}$ are each equivalent to ACA_0 over RCA_0, but, interestingly, we have that $\mathsf{INF}_0^{\mathsf{2B}}$ is provable in the strictly weaker system of WKL_0. The precise reverse mathematical strength of $\mathsf{INF}_0^{\mathsf{2B}}$ is still open.

Upon interpreting the above theorems about general (i.e. not necessarily 2-based) vector spaces in the context of reverse mathematics we get the following corresponding results in terms of the "big five."

[4]Many of the computable vector spaces used to establish nontrivial lower bounds in the context of reverse mathematics have been $2-$based vector spaces.

Theorem 1.5. COINF_0 *is equivalent to* $\mathsf{ACA_0}$ *over* $\mathsf{RCA_0}$.

Theorem 1.6. COFIN_0 *is equivalent to* $\mathsf{ACA_0}$ *over* $\mathsf{RCA_0}$.

Theorem 1.7. $\mathsf{x}-\mathsf{INF}_0$ *is equivalent to* $\mathsf{ACA_0}$ *over* $\mathsf{RCA_0}$.

Theorem 1.8. INF_0 *is equivalent to* $\mathsf{ACA_0}$ *over* $\mathsf{RCA_0}$.

To prove that $\mathsf{ACA_0}$ implies each of the statements listed above, use the fact that $\mathsf{ACA_0}$ implies the existence of a basis B [39, III.4.3] and the fact that for any given subset of basis vectors $B_0 \subseteq B$, $\mathsf{WKL_0}$ proves that the existence of a subspace containing B_0 and not containing $B \setminus B_0$ (the basic idea behind this argument can be found in [14, Section 3]). Recall also that $\mathsf{ACA_0}$ implies $\mathsf{WKL_0}$. Throughout the rest of this article we will only consider the reversals in Theorems 1.5-1.8 above.

The proofs of Theorems 1.1, 1.3, 1.5, and 1.7 are contained in Section 5 below. The proofs of Theorems 1.2 and 1.6 are contained in Section 6 below. The proofs of Theorems 1.4 and 1.8 can be found in Section 7 below. Theorem 1.4 is the main theorem of this article.

Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [14] already proved that the statement "every vector space contains a nontrivial subspace" is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$, making INF_0 nonrobust. Furthermore, the author has (very recently) proven that the statement "every Artinian ring is Noetherian" ($\mathsf{ART_0}$) is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0} + \mathsf{I\Sigma_2}$ [8] ($\mathsf{I\Sigma_2}$ is the induction scheme for Σ_2 formulas; for more information see [19, 31]). It is interesting that the reverse mathematical strengths of $\mathsf{ART_0}$ and $\mathsf{INF_0}$ differ, since the study of $\mathsf{INF_0}$ began, in part, as a way of better understanding $\mathsf{ART_0}$. It is even more interesting that both $\mathsf{ART_0}$ and $\mathsf{INF_0}$ are nonrobust, and yet each equivalent to one of the "big five" systems in the context of ω-models. The question of whether or not there exists a natural theorem from algebra that is not equivalent to one of the "big five" in the context of ω-models is still open.

In the second part of this article, which consists solely of Section 8 below, we establish limitations on the coding methods used to prove Theorems 1.1, 1.2, and 1.3 above. In particular, we use weak König's lemma and the Jockusch-Soare Low Basis Theorem [23] to show that any "simple" vector space constructed via the general and natural procedure used to prove Theorems 1.1, 1.2, and 1.3 above, i.e. any infinite dimensional computable $2-$based vector space, contains a *low* infinite dimensional proper subspace (we will define "lowness" in Section 3.1 below), and has "densely many" finite dimensional computable subspaces (we will explain ourselves more precisely in Section 8 below). We will use these facts to derive some interesting consequences about the algebraic/computability-theoretic structure of infinite dimensional computable vector spaces in which no finite dimensional subspace is computable. Finally, we also highlight some general similarities between constructing an infinite dimensional proper subspace of a given "simple" (i.e. $2-$based) vector space and constructing an infinite homogeneous set in the context of Ramsey's theorem for pairs (see [5] for more details on the computability theory of Ramsey's theorem for pairs). More details are given in Section 8 below.

Our main goal in the first part of this article is to examine different methods for coding information into infinite dimensional proper subspaces of computable infinite dimensional vector spaces. Our main theorems in part one are Theorems 1.4 and 1.8, which say that the statement "every infinite dimensional vector space contains an infinite dimensional proper subspace" is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$. We will always reason in $\mathsf{RCA_0}$.

2. Computable algebra and algebraic reverse mathematics: A general overview

Computable algebra was first studied by algebraists in the 1800s and early 1900s [24, 20, 48], although the subject was formally introduced by Fröhlich and Shepherdson [18] after the invention of computability theory by Turing [46, 47] and others. Much work has been done in computable algebra after [18], and in particular the computability of rings, fields, and vector spaces, by: Rabin [32], Baur [2], Metakides and Nerode [27], Shore [38], Remmel [33], and others. Later on Friedman, Simpson, and Smith [16, 17] investigated the computability theory and reverse mathematics of groups, rings, and fields. Afterwards Solomon investigated the reverse mathematics of ordered groups [42, 43, 44]. Much more recently, however, the program of effective algebra and algebraic reverse mathematics was taken up by Downey, Lempp, and Mileti in [13], as well as Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán in [14] and the author in [6, 7].

Recent developments in effective and reverse mathematics [36, 5, 21, 22] have lead to the discovery of many mathematical statements whose reverse mathematical strength is not one of the "big five" subsystems of second order arithmetic: RCA_0, WKL_0, ACA_0, ATR_0, $\Pi^1_1-\mathsf{CA}_0$ (see [39] for more details). However, almost all of these statements are combinatorial in nature, and, more specifically, none of them is algebraic. In fact the existence of an algebraic theorem whose reverse mathematical strength is not equivalent to one of the "big five" in the context of ω-models is still unresolved. Until very recently the statements "every Artinian ring is Noetherian" and "every infinite dimensional vector space contains an infinite dimensional proper subspace" seemed like good candidates for algebraic statements not equivalent to any of the "big five" in the context of ω-models since they are nonrobust. However, the main theorem of this article (Theorem 1.4) says that the latter statement is equivalent to ACA_0 in the context of ω-models, which is one of the "big five." Very recently [8] the author has also shown that the former statement is equivalent to WKL_0 in the context of ω-models, a different member of the "big five."

2.1. **The plan of the paper.** In Section 3 we introduce the basic definitions and notation that we will use in part one. Then, in Section 4 we prove a key lemma that we will use for coding in the proofs of Theorems 1.1, 1.2, and 1.3 below. In Section 5 we use the key lemma (i.e. Lemma 4.1) to code the Halting Set $\emptyset'$ into proper subspaces of infinite/coinfinite dimension and prove Theorem 1.1. We also prove Theorem 1.3. In Section 6 we use the key lemma to code into proper subspaces of infinite/cofinite dimension and prove Theorem 1.2. Finally, we abandon the key lemma of Section 4 and use a more complicated coding technique to prove Theorem 1.4 (the main theorem of this article), generalizing all of our results in Sections 5 and 6. In Section 8 (i.e. in part two of this article) we show that a certain statement in second order arithmetic, which we denote by INF^{2B}_0 and introduce in Section 4 below, is implied by WKL_0 over RCA_0. We then discuss the interesting implications that this has in the context of our earlier constructions in Sections 5, 6, and 7.

3. Preliminaries and notation

3.1. **Computability Theory in RCA_0.** We assume that the reader is familiar with the basics of computability theory and reverse mathematics. For an introduction to these subjects, consult [40, 41, 39]. We use $\omega = \{0, 1, 2, \ldots\}$ to denote the set of (standard) natural numbers. We will use $\mathbb{N}$ to denote the set of (possibly nonstandard) natural numbers in a given model of RCA_0. All of our definitions are in RCA_0, and we use the term "computable" to mean Δ_1-definable, and "computable in an oracle $A \subseteq \mathbb{N}$" means Δ_1-definable in the parameter A. See [39] for more details on reverse mathematics, RCA_0, and Δ_1-definability. Our standard computability-theoretic notation will follow that of [40, 41]. In particular, we will write $X \leq_T Y$, $X, Y \subseteq \mathbb{N}$, to mean that X is computable in Y and we will write A' to denote the Halting Set relative to (i.e. the Turing jump of) $A \subseteq \mathbb{N}$. It is known that A never computes A'. We write ϕ_e, $e \in \mathbb{N}$, to denote the standard effective listing of the partial computable functions and ϕ^A_e denotes the standard effective listing of partial computable functions relative to the oracle/parameter $A \subseteq \mathbb{N}$. The Halting Set $\emptyset'$ is then equal to the set of $x \in \mathbb{N}$ such that the ϕ_x halts on input x, and for any parameter $A \subseteq \mathbb{N}$, A' is defined similarly. We call a set $A \subseteq \mathbb{N}$ *low* whenever $A' \leq_T \emptyset'$ (we always have that $\emptyset' \leq_T A'$). It follows that if A is low then A cannot compute the Halting Set $\emptyset'$ (in fact the computability strength of A is much less than those sets that compute $\emptyset'$). Recall that an infinite set $A \subseteq \mathbb{N}$ is *computably enumerable* (c.e.) iff A is Σ_1-definable iff A is the range of a 1-1 computable function. A set $A \subseteq \mathbb{N}$ is computable if and only if both A and A^c (the complement of A) are computably enumerable.

Let $A_0 \subseteq \mathbb{N}$ be a computably enumerable set constructed via a "movable marker construction" (see [40, 41] for more details) whose complement $A_0^c = \{0 = a_0^c < a_1^c < a_2^c < \cdots\}$ dominates the modulus (i.e. settling time) of the Halting Set $\emptyset'$ -i.e. for all $n \in \mathbb{N}$, a_n^c is larger than the settling time of the first n bits of $\emptyset'$.[5] It is well-known that $\mathsf{RCA_0}$ suffices to prove that every finite initial segment of the Halting Set exists, from which it follows that every marker settles (i.e. every marker comes to a limit) and A_0^c is infinite (i.e. unbounded). $\mathsf{RCA_0}$ also suffices to prove that for every $n \in \mathbb{N}$ the finite set $\{a_0^c, a_1^c, \dots, a_n^c\} \subset \mathbb{N}$ exists (more generally $\mathsf{RCA_0}$ suffices to show that every finite initial segment of every c.e. set exists; one can prove this directly via the pigeonhole principle or via the strong Σ_1-bounding principle [39, Exercise II.3.14]). Let A_0^s, $s \in \mathbb{N}$, be an effective enumeration of A_0 with the property that at each stage $s \in \mathbb{N}$ there is *exactly one* $x \in \mathbb{N}$ such that $x \in A_0^{s+1} \setminus A_0^s$ and define $A_0^{-1} = \emptyset$. Let $a_n^{c,s} \in \mathbb{N}$, $n, s, \in \mathbb{N}$, be a nondecreasing computable approximation to $a_n^c \in \mathbb{N}$ – i.e. for all $n \in \mathbb{N}$ we have that $a_n^{c,s} \leq a_n^{c,s+1}$, $s \in \mathbb{N}$, and $\lim_s a_n^{c,s} = a_n^c$. It is well-known and easy to see that if $f : \mathbb{N} \to \mathbb{N}$ is a function such that for every $x \in \mathbb{N}$ we have that $f(x) \geq a_x^c$ then f computes the Halting Set $\emptyset'$ since for almost all $x \in \mathbb{N}$ we have that $f(x)$ is larger than the stage at which x is enumerated into $\emptyset'$, if this ever happens. We will use this fact repeatedly in what follows. Let $a_{-1}^c = -1$ and define the n^{th} *component of* A_0 to be the interval $A_{0,n} = \{a_{n-1}^c + 1, a_{n-1}^c + 2, \dots, a_n^c\} \subset \mathbb{N}$. Note that the components of A_0 partition $\mathbb{N}$.

3.2. **Linear Algebra.** We assume that the reader is familiar with the basics of linear algebra and vector spaces at the level of most introductory undergraduate courses for mathematicians. Let $\mathbb{Q}$ denote a fixed computable representation of the rational numbers, and let $q_0, q_1, q_2, \dots$ be an effective (i.e. computable) listing of the elements of $\mathbb{Q}$. All of the vector spaces that we will consider will be $\mathbb{Q}$-vector spaces. By computable vector space we mean a countable vector space over $\mathbb{Q}$ (coded as a computable subset of natural numbers representing vectors) such that the addition and scalar multiplication operations are given by computable functions (on the natural numbers that represent vectors). For more information on the basics of computable vector spaces consult [14]. Let $\mathbb{Q}_\infty = \mathbb{Q}[v_0, v_1, v_2, \dots]$ be a fixed computable representation of the unique vector space over $\mathbb{Q}$ with standard basis vectors $v_0, v_1, v_2, \dots$ and let $u_0, u_1, u_2, \dots \in \mathbb{Q}_\infty$ be a fixed computable listing of the elements of $\mathbb{Q}_\infty$. We will use the term "standard representation" of $x \in \mathbb{Q}_\infty$ to mean the unique linear combination of standard basis vectors $v_0, v_1, v_2, \dots$ that equals x.

All of the vector spaces that we construct in this article will be quotients of the form $\mathbb{Q}_\infty/S$, for some computable proper subspace $S \subset \mathbb{Q}_\infty$. If $S \subset \mathbb{Q}_\infty$ is a subspace and $x \in \mathbb{Q}_\infty$, then we use the notation $\overline{x}$ to denote the image of x in the quotient space $\mathbb{Q}_\infty/S$. Whenever we consider more than one quotient we will always specify the particular quotient that a vector $\overline{x}$ belongs to by writing $\overline{x} \in \mathbb{Q}_\infty/S$ for the appropriate subspace $S \subset \mathbb{Q}_\infty$. Similarly, we will write $\overline{S}_0 \subseteq \mathbb{Q}_\infty/S$ to denote the image of the subspace $S_0 \subseteq \mathbb{Q}_\infty$ in the quotient $\mathbb{Q}_\infty/S$. So long as S is computable, one can always pass uniformly and effectively between $\overline{S}_0 \subseteq \mathbb{Q}_\infty/S$ and $S_0 \subseteq \mathbb{Q}_\infty$, as well as $\overline{x} \in \mathbb{Q}_\infty/S$ and $x \in \mathbb{Q}_\infty$. If V is a $\mathbb{Q}$-vector space and $V_0 \subseteq V$ is a collection of vectors in V, then we write $\langle V_0 \rangle$ to denote the subspace of V generated (i.e. spanned) by V_0. Similarly, for all $v_0, v_1, \dots, v_n \in V$ we write $\langle v_0, v_1, \dots, v_n \rangle$ to denote the subspace of V generated by $\{v_0, v_1, \dots, v_n\}$ and $\langle V_0, v_0, \dots, v_n \rangle$ to denote the

[5] The author is especially grateful to S. Lempp and K.M. Ng for suggesting a simplified version of A_0 based solely on the Halting Set. Originally, the construction of A_0 was more complicated, and the author required $\mathsf{B\Sigma_2}$ to show that A_0^c is infinite.

subspace of V generated by $V_0 \cup \{v_0, v_1, \ldots, v_n\}$. For convenience we say that the trivial subspace $\{0\}$ is spanned by $\emptyset$. Note that the span and quotient operations commute, and most problems in finite-dimensional linear algebra (such as finding the subspace spanned by a finite set of vectors) have a computable solution and so these solutions exist in $\mathsf{RCA_0}$. We will always use the term "linear combination" to mean "finite linear combination with nonzero $\mathbb{Q}$-coefficients." We say that a set of vectors is *linearly independent* if every nontrivial linear combination of those vectors is nonzero. Recall that we say a vector space is *infinite dimensional* whenever it is of arbitrarily large finite dimension – i.e. whenever it contains linearly independent subsets of arbitrarily large (finite) size.

4. The key lemma

The main result of this section is a key lemma that we will use in subsequent sections to code *incomputable information* (i.e. the Halting Set $\emptyset'$) into infinite dimensional subspaces of *computable* vector spaces. The main content of this lemma is not new, and can essentially be found in [6, Section 6] among other places. Since we will use this lemma repeatedly in the next few sections we have isolated it in this section. It is provable in $\mathsf{RCA_0}$, as follows.

Lemma 4.1 ($\mathsf{RCA_0}$)**.** *Let $W_0 = \{w_0, w_1, \ldots, w_n\} \subset \mathbb{Q}_\infty = \mathbb{Q}[v_0, v_1, v_2, \ldots]$, $n \in \mathbb{N}$, be a finite set of vectors and suppose that none of the vectors $x_0, x_1, \ldots, x_m$, $m \in \mathbb{N}$, are linear combinations of $\{w_0, w_1, \ldots, w_n\}$. Then, for any given $i, j \in \mathbb{N}$, $i \neq j$, such that no nontrivial linear combination of $\{v_i, v_j\}$ is in the span of W_0 there is a number $n_0 \in \mathbb{N} \subset \mathbb{Q}$ such that x_i is not a linear combination of $\{w_0, w_1, \ldots, w_n, v_i - n_0 v_j\}$, for all $0 \leq i \leq m$.*

Proof. First of all, note that since W_0 is finite it follows that $W = \langle W_0 \rangle$ exists via $\mathsf{RCA_0}$. Let $W_0^k = W_0 \cup \{v_i - k v_j\}$, for $k = 1, \ldots, m+2$. Again, since W_0^k is finite then via $\mathsf{RCA_0}$ we have that the subspaces $W^k = \langle W_0^k \rangle$, $1 \leq k \leq m+2$, exist. Note that for all $1 \leq k < l \leq m+2$ we have that $W^k \cap W^l = W$. For suppose (for a contradiction) that $W^k \cap W^l \supset W$, and choose a vector $x \in (W^k \cap W^l) \setminus W$, and write x as a linear combination of $W_0 \cup \{v_i - k v_j\}$ and a linear combination of $W_0 \cup \{v_i - l v_j\}$ as follows:

$$w + p(v_i - k v_j) = x = w' + p'(v_i - l v_j),\ w, w' \in W,\ p, p' \in \mathbb{Q} \setminus \{0\}.$$

Upon setting these combinations equal and rearranging the terms we get that

$$w - w' = p'(v_i - l v_j) - p(v_i - k v_j),$$

and thus some nontrivial linear combination of $\{v_i, v_j\}$ is in W, a contradiction. Hence, $W^k \cap W^l = W$. Therefore, for each $i = 0, 1, 2, \ldots, m$ there is at most one $1 \leq k_i \leq m+2$ such that $x_i \in W^{k_i}$, and by the (finite) pigeonhole principle (which holds in $\mathsf{RCA_0}$) it follows that there exists some $1 \leq k \leq m+2$ such that $x_i \notin W^k$, for all $0 \leq i \leq n$. Let $n_0 = k$. $\square$

In Section 7 below we will essentially generalize Lemma 4.1 by proving a more complicated version involving linear combinations of the form $v_i - \sum_{j<i} c_j v_j$, $c_j \in \mathbb{Q}$, in place of $v_i - n_0 v_j$. Until then, however, Lemma 4.1 will suffice to do most of our coding.

Before we move on to proving the theorems of part one we make the following important definition that we will revisit in later sections.

Definition 4.2. *Let $V = \mathbb{Q}_\infty / S$, for a subspace $S \subset \mathbb{Q}_\infty$. We say that V is a $2-$based vector space whenever S is generated by elements of the form $v_i - k_{i,j} v_j$, $k_{i,j} \in \mathbb{Q}$, $i, j \in \mathbb{N}$.*

We shall see in Section 8 below that infinite dimensional computable $2-$based vector spaces are different from other infinite dimensional computable vector spaces since they always contain a low infinite dimensional proper subspace. We will also highlight some similarities between constructing infinite dimensional proper subspaces of computable $2-$based vector spaces and some combinatorial constructions such as constructing infinite homogeneous sets in the context of Ramsey's Theorem for pairs (consult [5] for more information on Ramsey's

Theorem for pairs). More specifically, we will examine the reverse mathematical strength of the following principle, which we take to imply $\mathsf{RCA_0}$ and the following axiom:

$\mathsf{INF}_0^{\mathsf{2B}}$: Every infinite dimensional 2−based vector space contains a proper infinite dimensional subspace.

Meanwhile, in Section 5 we will examine the reverse mathematical strength of

$\mathsf{COINF}_0^{\mathsf{2B}}$: Every infinite dimensional 2−based vector space contains an infinite/coinfinite dimensional subspace.

and in Section 6 we will examine the reverse mathematical strength of

$\mathsf{COFIN}_0^{\mathsf{2B}}$: Every infinite dimensional 2−based vector space contains a cofinite dimensional proper subspace.

Note that we have

$$\mathsf{INF}_0^{\mathsf{2B}} = \mathsf{COINF}_0^{\mathsf{2B}} \vee \mathsf{COFIN}_0^{\mathsf{2B}}.$$

Three of the four vector spaces that we construct in the next two sections below will be infinite dimensional computable 2−based vector spaces - i.e. we will demonstrate that we can require the vector spaces V in the statements of Theorems 1.1, 1.2, and 1.3 above to be 2−based. More precisely, we will show that $\mathsf{COINF}_0^{\mathsf{2B}}$ and $\mathsf{COFIN}_0^{\mathsf{2B}}$ each imply $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$. Since it is obvious that $\mathsf{COINF_0}$ and $\mathsf{COFIN_0}$ imply $\mathsf{COINF}_0^{\mathsf{2B}}$ and $\mathsf{COFIN}_0^{\mathsf{2B}}$, respectively, it will follow that $\mathsf{COINF_0}$ and $\mathsf{COFIN_0}$ each imply $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$ as well. Moreover, in Section 8 below we will show that $\mathsf{INF}_0^{\mathsf{2B}}$ is a consequence of $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. Taken together these results say, generally speaking, that constructing an infinite dimensional proper subspace of a 2−based vector space is easier/simpler than constructing an infinite/coinfinite dimensional subspace or an infinite/cofinite dimensional proper subspace. Furthermore, the fact that $\mathsf{INF}_0^{\mathsf{2B}}$ is provable in $\mathsf{WKL_0}$ is interesting in the context of the main theorem of this article (i.e. Theorem 1.4 above) which says that $\mathsf{INF_0}$ is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$, because it implies that the vector space V that we will construct in the main theorem cannot be a 2−based vector space, and also gives further evidence that $\mathsf{INF_0}$ is a nonrobust algebraic theorem.

5. Coding into infinite dimensional subspaces of coinfinite dimension

The main purpose of this section is to prove Theorem 1.1, which was also proven independently by Downey and Turetsky. As a consequence of our proof of Theorem 1.1 we will deduce Theorems 1.3, 1.5, and 1.7. We reason in $\mathsf{RCA_0}$.

Theorem 1.1 ($\mathsf{RCA_0}$). *There exists a computable infinite dimensional* $(2-$ *based*$)$ *vector space V such that every infinite/coinfinite dimensional subspace of V computes the Halting Set $\emptyset'$.*

Proof. We will construct V as a quotient of $\mathbb{Q}_\infty$, i.e. $V = \mathbb{Q}_\infty/S$ for some subspace $S \subset \mathbb{Q}_\infty$, and we will use Lemma 4.1 above to ensure that V, S are computable (recall that V is computable iff S is computable). We construct S by first enumerating a generating set S_0 such that $S = \langle S_0 \rangle$, in stages, $S_0 = \cup_{s\in\mathbb{N}} S_0^s$.

The main idea behind the construction of S_0 is as follows. We enumerate $v_i - kv_{i+1}$ into S_0^{s+1} for some $k \in \mathbb{Q}$, $i, j \in \mathbb{N}$, via Lemma 4.1 above whenever i enters A_0 at stage $s \in \mathbb{N}$. This has the effect of collapsing $\overline{v}_i$ and $\overline{v}_{i+1}$ in the quotient space $V = \mathbb{Q}_\infty/S$. The end result is that, for all $i, j \in \mathbb{N}$, $\overline{v}_i$ and $\overline{v}_j$ are scalar multiples in V iff i, j belong to the same component of A_0. Now, suppose that we are given an infinite/coinfinite dimensional subspace W of V such that infinitely many $\overline{v}_i$ are in W and infinitely many $\overline{v}_i$ are not in W. Then if we let $\{k_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence of indices such that for all i, $\overline{v}_{k_i} \notin W$ and there is a $k_i < j < k_{i+1}$ such that $\overline{v}_j \in W$, then it follows that $\{k_i\}_{i\in\mathbb{N}}$ computes the Halting Set $\emptyset'$ since $k_i \geq a_i^c$ for all $i \in \mathbb{N}$. Hence W computes $\emptyset'$. The case in which W contains only finitely many $\overline{v}_i$ is a bit more complicated and described precisely below[6]. We now explicitly construct $S_0 \subset \mathbb{Q}_\infty$, $\langle S_0 \rangle = S$, as follows.

At stage 0 set $S_0^0 = \emptyset$. At stage $s+1 > 0$ we are given a finite set of generators $S_0^s \subset \mathbb{Q}_\infty$, and via $\mathsf{RCA_0}$ we can determine which of the vectors $u_0, u_1, \ldots, u_s \in \mathbb{Q}_\infty$ (recall that $\{u_i\}_{i\in\mathbb{N}}$ is an effective listing of the elements of $\mathbb{Q}_\infty$) are in $\langle S_0^s \rangle$ – i.e. via $\mathsf{RCA_0}$ the set $Z^{s+1} = \{0 \leq i \leq s : u_i \notin \langle S_0^s \rangle\}$ exists. Now, find the unique $x \in \mathbb{N}$ for which $x \in A_0^s \setminus A_0^{s-1}$ and enumerate $v_x - k_{s+1}v_{x+1}$ into $S_0^{s+1} \supset S_0^s$ for some $k_{s+1} \in \mathbb{Q}$ such that $u_z \notin \langle S_0^{s+1} \rangle$ for all $z \in Z^{s+1}$. This ends the construction of $S_0 = \cup_{s\in\mathbb{N}} S_0^s$. We claim that the number $k_{s+1} \in \mathbb{Q}$ above exists and can be obtained uniformly and effectively via Lemma 4.1. The only difficulty in applying Lemma 4.1 in this situation is showing that the span of $\{v_x, v_{x+1}\}$ has trivial intersection with the span of S_0^s. To see why this is the case first of all note that for all $i \in \mathbb{N}$ we enumerate at most one element of the form $v_i - kv_{i+1}$, $k \in \mathbb{Q}$, into S_0 (and we do this precisely when i is enumerated into A_0). Now, upon examining the minimal and maximal index of a (nontrivial) linear combination of elements in S_0^s, l, it follows that neither index is canceled (in l), from which it follows that l cannot be in the span of $\{v_x, v_{x+1}\}$ and so Lemma 4.1 applies as we previously claimed. Furthermore, we have that $S = \langle S_0 \rangle$ is computable since by our construction of S for all $s \in \mathbb{N}$ we have that $u_s \in S$ iff $u_s \in \langle S_0^{s+1} \rangle$. By our construction of S_0 it is also clear that $V = \mathbb{Q}_\infty / \langle S_0 \rangle$ is a 2−based vector space.

Now, we claim that

(1) For all $n \in \mathbb{N}$ the set $\{\overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \ldots, \overline{v}_{a_n^c}\} \subset V$ is linearly independent in V.
(2) For any given vector $\overline{x} \in V$ there exists $n \in \mathbb{N}$ such that $\overline{x} = \sum_{j<n} c_j \overline{v}_{a_j^c}$, for some $c_j \in \mathbb{Q}$, $0 \leq j < n$.

(In other words, $\overline{v}_0 = \overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \overline{v}_{a_2^c}, \ldots$ form a basis for V.)

To see why (1) holds, first define the n^{th} *component of* $\mathbb{Q}_\infty$, denoted $\mathbb{Q}_{\infty,n} \subset \mathbb{Q}_\infty$, to be the subspace of $\mathbb{Q}_\infty$ generated by all v_i such that $i \in \mathbb{N}$ in the n^{th} component of A_0. Now, given any linear combination L of elements in $S_0 \cap \mathbb{Q}_{\infty,n}$, note that (by our construction of S_0) the minimal and maximal indices of v_i, $i \in \mathbb{N}$, appearing in this linear combination cannot be canceled and therefore must appear in any representation of L with respect to the standard basis $\{v_0, v_1, v_2 \ldots\} \subset \mathbb{Q}_\infty$. Therefore, no nontrivial linear combination of $\{v_{a_0^c}, v_{a_1^c}, \ldots, v_{a_n^c}\}$ is in S, and thus no nontrivial linear combination of $\{\overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \ldots, \overline{v}_{a_n^c}\}$ is zero (in V).

To see why (2) holds, let $x \in \mathbb{Q}_\infty$, $\overline{x} \in V$, be given and let $s_0 \in \mathbb{N}$ be large enough so that A_0 has settled on $\{0, \ldots, a_n^c\} \subset \mathbb{N}$, where $n \in \mathbb{N}$ is large enough so that a_n^c is larger than the index of any v_i appearing in the standard representation of $x \in \mathbb{Q}_\infty$. Now, by our construction of S_0 every $\overline{v}_i$, $i \in A_0$, appearing in the standard representation of $\overline{x} \in V$ is a scalar multiple of $\overline{v}_{i+1}$. It follows that every $\overline{v}_i$, $i \in A_0$, is a scalar multiple of some $\overline{v}_{a_j^c}$, $j, a_j^c \in \mathbb{N}$, where i and a_j^c belong to the same component of A_0. (2) now follows.

As a consequence of (1) we have that V is infinite dimensional (i.e. V is of arbitrarily large finite dimension). Let $W \subset V$ be an infinite/coinfinite dimensional subspace of V. We will define a function $f : \mathbb{N} \to \mathbb{N}$ recursively in W such that for all $x \in \mathbb{N}$ we have that $f(x) \geq a_x^c$. Since W is coinfinite dimensional it follows that $\overline{v}_j \notin W$ for infinitely many $j \in \mathbb{N}$. First, suppose that $\overline{v}_i \in W$ for infinitely many $i \in \mathbb{N}$. By our construction of S and $V = \mathbb{Q}_\infty / S$ above, we know that $\overline{v}_j \in W$ iff $v_k \in W$, $j, k \in \mathbb{N}$, for all k in the same component of A_0 as j. Define a function $f : \mathbb{N} \to \mathbb{N}$ recursively as follows. First, let $i_0 \in \mathbb{N}$ be the least number such that $\overline{v}_{i_0} \in W$, and let $j_0 > i_0$ be the least number greater than i_0 such that $\overline{v}_{j_0} \notin W$. Set $f(0) = j_0$. For $x > 0$, $x \in \mathbb{N}$, assume that $f(x-1)$ is defined and let $i_x > f(x-1)$ be the least number such that $\overline{v}_{i_x} \in W$ and let $j_x > i_x$ be the least number greater than i_x such that $\overline{v}_{j_x} \notin W$ and set $f(x) = j_x$. This ends the construction of f. By Σ_1-induction and our

[6]In this case the proof is also not that complicated, and we encourage the motivated reader to work it our for themselves.

previous remarks it follows that for all $x \in \mathbb{N}$ we have that $f(x) \geq a_x^c$, and hence $f \leq_T W$ computes the Halting Set $\emptyset'$.

Now suppose that for cofinitely many $i \in \mathbb{N}$ we have that $\overline{v}_i \notin W$. Moreover assume that $n_0 \in \mathbb{N}$ is such that for all $n \geq n_0$ we have that $\overline{v}_n \notin W$ and (via $\mathsf{RCA_0}$) take a sequence of nonzero elements in $W \subset V$, $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots$, such that when represented as linear combinations of $\{\overline{v}_0, \overline{v}_1, \overline{v}_2, \ldots\}$ (via some representative of the corresponding equivalence class in $V = \mathbb{Q}_\infty/S$) we have that the maximal index of all $\overline{v}_j$ occurring in some (i.e. our) representation of $\overline{w}_i$ is strictly less than the minimal index of all $\overline{v}_j$ occurring in some (i.e. our) representation of $\overline{w}_{i+1}$, and that the minimal index of any $\overline{v}_j$ occurring in some (i.e. our) representation of $\overline{w}_0$ is strictly greater than n_0. This is possible via finite-dimensional linear algebra (i.e. Gaussian elimination) and the fact that W is infinite dimensional. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(x)$ is the maximal index of any $\overline{v}_i$ occurring in our representation of $\overline{w}_x$. We claim that $f(x) \geq a_x^c$ for all $x \in \mathbb{N}$. For suppose not. Then by the finite pigeonhole principle it follows that there must be $x, x+1 \in \mathbb{N}$ such that $f(x), f(x+1)$ belong to the same component of A_0. But by our construction of $S = \langle S_0 \rangle$ above this means that $\overline{w}_x$ is a nonzero scalar multiple of some standard basis vector $\overline{v}_k$ (in V), a contradiction. So $f(x) \geq a_x^c$ for all $x \in \mathbb{N}$ and hence $f \leq_T W$ computes the Halting Set $\emptyset'$. □

Interpreting the relativized version of Theorem 1.1 in the context of reverse mathematics yields the following result (recall that $\mathsf{ACA_0}$ is equivalent to saying that "for any set A, the Halting Set relative to A, A', exists"). We briefly sketch this argument now.

Theorem 1.5. $\mathsf{COINF_0}$ *and* $\mathsf{COINF_0^{2B}}$ *are equivalent to* $\mathsf{ACA_0}$ *over* $\mathsf{RCA_0}$*.*

Proof. Without loss of generality we prove the theorem for $\mathsf{COINF_0}$ (the proof for $\mathsf{COINF_0^{2B}}$ is identical). Let $X \subseteq \mathbb{N}$ be given. We will use the principle $\mathsf{COINF_0}$ to show that X' exists. Using $\mathsf{RCA_0}$ construct the infinite dimensional vector space V_X of Theorem 1.1 above relative to X by using A_X in place of A_0 where $A_X \subset \mathbb{N}$ is defined analogously to A_0, except that we replace the Halting Set $\emptyset'$ with the Halting Set relative to A, A'. The rest of the argument remains the same and in the end we conclude that $X \oplus W_X$ computes X', for any infinite/coinfinite dimensional subspace $W_X \subset V_X$, and hence X' exists since $\mathsf{COINF_0}$ says that some W_X exists. □

We now sketch the proof of Theorem 1.3 as a modification of the proof of Theorem 1.1 above.

Theorem 1.3 ($\mathsf{RCA_0}$)**.** *There exists a computable infinite dimensional* $(2-$ *based*$)$ *vector space* V*, and* $0 \neq x \in V$*, such that every infinite dimensional subspace of* V *not containing* x *computes the Halting Set* $\emptyset'$*.*

Proof. We reason in $\mathsf{RCA_0}$, following the proof of Theorem 1.1 above. In the construction of S_0 in proof of Theorem 1.1 above at stage $s+1$ we enumerated a vector of the form $v_i - k_{s+1}v_{i+1}$ into S_0^{s+1} (via Lemma 4.1). Now, instead of doing this we enumerate a vector of the form $v_0 - k_{s+1}v_i$ into S_0^{s+1} (via Lemma 4.1) whenever i is enumerated into A_0. Then, arguing along the same lines as in the proof of Theorem 1.1, we have that V is a $2-$based computable vector space and that $\overline{v}_i$ is a scalar multiple of $\overline{v}_0$ for every $i \in A_0$. We also have that $0 \neq \overline{x}_0$ since every linear combination L of vectors in $S_0 \subset \mathbb{Q}_\infty$ must have some v_i occurring for some $i > 0$ (here i is the maximal index of any v_j appearing in L). So, if we let W be an infinite dimensional subspace of V not containing $x = \overline{v}_0 \neq 0$, then every representative (in terms of the standard spanning set $\{\overline{v}_j\}_{j\in\mathbb{N}}$) of every nonzero element of W must have some $\overline{v}_i$ appearing such that $i \in A_0^c$. Now, if $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots \in W$ is a sequence of nonzero vectors in V such that the minimal index of some (i.e. our) representation of $\overline{w}_{n+1}$ (expressed as a linear combination of $\{\overline{v}_j\}_{j\in\mathbb{N}}$) is strictly greater than the maximal index of some (i.e. our) representation of $\overline{w}_n$, for all $n \in \mathbb{N}$ (as before this sequence may be obtained via Gaussian elimination), then it follows that the function $f : \mathbb{N} \to \mathbb{N}$ defined by setting

$f(x)$ equal to the maximal index of our representation of $\overline{w}_x$ satisfies $f(x) \geq a_x^c$ for all $x \in \mathbb{N}$. Therefore $f \leq_T W$ computes $\emptyset'$. $\square$

As above, upon interpreting the relativized version of Theorem 1.3 in the context of reverse mathematics we obtain the following result. The proof is similar to that of Theorem 1.5 above.

Theorem 1.7. $\mathsf{x}-\mathsf{INF}_0$ *is equivalent to* ACA_0 *over* RCA_0.

6. Coding into infinite dimensional subspaces of cofinite dimension

The main goal of this section is to prove Theorem 1.2. As a consequence we will derive Theorem 1.6. We reason in RCA_0. As we stated earlier, we will present two proofs of Theorem 1.2. First, we will present our more complicated proof that constructs a $2-$based vector space V as in the statement of Theorem 1.2 above. Afterwards we will give a simpler proof of Theorem 1.2 due to Downey, Greenberg, Kach, Lempp, Miller, Ng, and Turetsky that is based on [14] and in which the vector space V is not necessarily $2-$based. Our theorem that V can be $2-$based is interesting in the context of Section 8 below.

Theorem 1.2 (RCA_0)**.** *There exists a computable infinite dimensional* $(2-$ *based) vector space* V *such that every infinite/cofinite dimensional proper subspace of* V *computes the Halting Set* $\emptyset'$*.*

First proof of Theorem 1.2. We will construct V as a quotient of $\mathbb{Q}_\infty$, i.e. $V = \mathbb{Q}_\infty/S$, for some computable subspace $S \subset \mathbb{Q}_\infty$. Hence V will be a computable vector space. We will construct the subspace S by enumerating a set of generators $S_0 \subset S$ for S. From our construction of S_0 it will be clear that V is in fact a $2-$based vector space. Let $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ denote a computable pairing function that is a bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$ such that $x, y \leq \langle x, y\rangle$ for all $x, y \in \mathbb{N}$ (see [39, Theorem II.2.2][40, 41] for more details). Recall that the ordered triple $\langle a, b, c\rangle \in \mathbb{N}$, $a, b, c \in \mathbb{N}$ is actually a shorthand for the nested ordered pairs $\langle\langle a, b\rangle, c\rangle \in \mathbb{N}$ and also exists in RCA_0.

The main idea behind our construction of $S, S_0 \subset \mathbb{Q}_\infty$ is as follows. For every standard basis vector of $\mathbb{Q}_\infty$, v_n, $n \in \mathbb{N}$, we will enumerate generators of the form $v_n - k_{n,x,i}v_{\langle n+1,x,i\rangle}$, $k_{n,x,i} \in \mathbb{Q}$, for all $x, i \in \mathbb{N}$ such that $0 \leq i \leq a_x^c$ into S_0. This has the effect of collapsing v_n and $v_{\langle n+1,x,i\rangle}$, $0 \leq i \leq a_x^c$ in the quotient space $V = \mathbb{Q}_\infty/S$. The end result is that v_n is a scalar multiple of $v_{\langle n+1,x,i\rangle}$ whenever $x, i \in \mathbb{N}$ are such that $1 \leq i \leq a_x^c$. Now, suppose that we are given an infinite/cofinite proper subspace of V, called $W \subset V$. Then, since W is a proper subspace there is some $n_0 \in \mathbb{N}$ such that $\overline{v}_{n_0} \notin W$, and so for every $x \in \mathbb{N}$ the vectors $\overline{v}_{\langle n_0+1,x,i\rangle} \notin W$, for all $0 \leq i \leq a_x^c$. Recall that for every $x \in \mathbb{N}$ the vectors $\overline{v}_{\langle n_0+1,x,i\rangle}$, $0 \leq i \leq a_x^c$, are all scalar multiples of $\overline{v}_{n_0}$ and therefore they are also scalar multiples of each other. Therefore, any linear combination of these vectors is not in W. Later on in this proof we will use the fact that W has cofinite dimension in V to show that for every $x \in \mathbb{N}$ there is a linear combination $l_x \in V$ of elements of the form $\overline{v}_{\langle n_0+1,x,j\rangle}$, $j \in \mathbb{N}$, in W. It follows from our previous remarks and our definition of $\overline{v}_{n_0}$ above that the maximum $j \in \mathbb{N}$ for which $\overline{v}_{\langle n_0+1,x,j\rangle}$ occurs in the linear combination l_x must satisfy $j > a_x^c$ (otherwise $\overline{v}_{n_0} \in W$, a contradiction). Hence, by effectively searching for and finding the linear combinations l_x, $x \in \mathbb{N}$, we can compute a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) \geq a_x^c$ for all $x \in \mathbb{N}$, and so W computes $\emptyset'$. We are now ready to give a detailed proof of Theorem 1.2.

We construct the generating set $S_0 \subset \mathbb{Q}_\infty$ in stages, $S_0 = \cup_{s\in\mathbb{N}} S_0^s$, $S_0^{s+1} \supseteq S_0^s$. At stage $s = 0$ set $S_0^0 = \emptyset$. At stage $s+1 > 0$ we define $S_0^{s+1} \supset S_0^s$ as follows. Without any loss of generality we assume that for all $x \in \mathbb{N}$, $\phi_x(x)$ does not halt before stage x. At stage $s+1 > 0$ we enumerate vectors of the form $v_s - k_{s+1,x,i}v_{\langle s+1,x,i\rangle}$, $k_{s,x,i} \in \mathbb{Q}$, into S_0^{s+1} for all $0 \leq x \leq s$ and $0 \leq i \leq a_x^{c,s}$. As in the previous section we choose the (finitely many) numbers $k_{s,x,i} \in \mathbb{Q}$ in the previous sentence one-at-a-time via Lemma 4.1 above[7] so as to guarantee that for all $0 \leq k \leq s$ we have that $u_k \in \langle S_0^{s+1} \rangle$ iff $u_k \in \langle S_0^s \rangle$. This ends our construction of S_0 and $S = \langle S_0 \rangle$. Note that S is computable since for all $k \in \mathbb{N}$ we have that $u_k \in S$ iff $u_k \in S_0^k$. Hence $V = \mathbb{Q}_\infty/S$ is computable. It is also clear by our construction of S_0 that V is a 2−based vector space.

To see that V is infinite dimensional we will show that for all $n, x \in \mathbb{N}$ the vectors $\{\overline{v}_{\langle n,x,i \rangle}\}_{i > a_x^c}$ are linearly independent in V. We prove this via proof by contradiction. Fix $n, x \in \mathbb{N}$ and suppose (for a contradiction) that some linear combination of $\{v_{\langle n,x,i \rangle}\}_{i > a_x^c}$, which we will call L, is in S. Then L is also a linear combination of elements of S_0, which we will call L_0. By our construction of S_0 we know that every generator of S_0 is of the form $v_p - kv_{\langle p+1,x,i \rangle}$, for some $p, x, i \in \mathbb{N}$, $k \in \mathbb{Q}$, and $n < n+1 \leq \langle n+1, x, i \rangle$. It follows that every $m \in \mathbb{N}$ can appear as the maximal index of an element of S_0 *at most once.* Let m_0 be the maximal index of any generator appearing in L_0. Then it follows that m_0 cannot be canceled by any other generator in S_0 appearing in L_0. Hence, $m_0 = \langle n, x, i \rangle$, for some $i > a_x^c$. But this is a contradiction since by our construction of S_0 no element of S_0 has m_0 as its maximal index (because $i > a_x^c$ is too large to be in S_0). This proves that for all $n, x \in \mathbb{N}$ the vectors $\{\overline{v}_{\langle n,x,i \rangle}\}_{i > a_x^c}$ are linearly independent in V. Hence V is infinite dimensional.

Now, let $\overline{W} \subset V$ be an infinite/cofinite dimensional proper subspace. We claim that $\overline{W}$ computes the Halting Set $\emptyset'$. To see why, first of all let $W \subset \mathbb{Q}_\infty$, $W \leq_T \overline{W}$, be the pullback of $\overline{W}$ and using the fact that W is a proper subspace of $\mathbb{Q}_\infty$ (since $\overline{W}$ is a proper subspace of V) let $n \in \mathbb{N}$ be such that $v_n \notin W$. We claim that for every $x \in \mathbb{N}$ there exists a linear combination of the vectors $\{v_{\langle n+1,x,i \rangle}\}_{i \in \mathbb{N}}$ in W. Using the fact that W has cofinite dimension in $\mathbb{Q}_\infty$ (since $\overline{W}$ has cofinite dimension in V) let $m \in \mathbb{N}$ be such that $\langle v_0, \ldots, v_m, W \rangle = \mathbb{Q}_\infty$. Then for all $i > m$ either $v_{\langle n,x,i \rangle} \in W$, in which case we are done, or else we can write $v_{\langle n,x,i \rangle} = w_i + z_i$, for some nonzero vectors $w_i \in W$ and $z_i \in \langle v_0, \ldots, v_m \rangle$. Now, since there are infinitely many vectors of the form $v_{\langle n+1,x,i \rangle}$, $i > m$, and only finitely many $v_0, \ldots, v_m$, we can use Gaussian elimination on the equations $v_{\langle n,x,i \rangle} = w_i + z_i$, $m < i \leq 2m+1$, to eliminate any occurrence of the vectors $v_0, \ldots, v_m$ in the "z_i part" of the equations, thus constructing a linear combination of $v_{\langle n,x,i \rangle}$, $m < i \leq 2m+1$, that lies in W. This proves that for every $x \in \mathbb{N}$ there is a linear combination of the vectors $\{v_{\langle n,x,i \rangle}\}_{i \in \mathbb{N}}$ in W. The rest of the argument that $W \leq_T \overline{W}$ computes the Halting Set $\emptyset'$ was given in the last few sentences of the second paragraph of this proof. This completes our proof of Theorem 1.2. □

We now present the second (simpler) proof of Theorem 1.2 above due to Downey, Greenberg, Kach, Lempp, Miller, Ng, and Turetsky that is based on a construction in [14]. Recall that the vector space constructed in the second proof below is *not necessarily* a 2−based vector space.

Second proof of Theorem 1.2. Let $V = \mathbb{Q}_\infty/S$ be the infinite dimensional computable vector space constructed in [14, Theorem 1.5], $S \subset \mathbb{Q}_\infty$ a subspace, and let $\overline{W} \subset V$ be an infinite/cofinite dimensional proper subspace of V. Let W denote the pullback of $\overline{W}$ in $\mathbb{Q}_\infty$ (recall that W is computable in $\overline{W}$), let $\{x_0, \ldots, x_n\}$, $n \in \mathbb{N}$, be finitely many vectors such that $X = \langle x_0, \ldots, x_n \rangle \subset \mathbb{Q}_\infty$ is a complement for W in $\mathbb{Q}_\infty$. It follows that every vector $z \in \mathbb{Q}_\infty$ can be written *uniquely* as a sum $z = x + w$ with $x \in X$ and $w \in W$. In [14] the authors explain that every finite dimensional subspace of V computes the Halting Set $\emptyset'$ (because it is computably enumerable and of PA degree; for more information see [14]), and so X computes $\emptyset'$. We will show how to compute X from W. For every $z \in \mathbb{Q}_\infty$ (computably)

[7]As in the previous section, to argue that no linear combination l of v_s and $v_{\langle s+1,x,i \rangle}$ is currently in the span of S_0^{s+1} assume the opposite and derive a contradiction by examining the minimal and maximal indices of the elements currently in S_0^{s+1} whose span includes l. We leave the details to the reader but remark that the argument is similar to one that we give in the next paragraph when we show that V is infinite dimensional.

find vectors $x_z \in X$ and $w_z \in W$ such that $z = x_z + w_z$. We can find x_z, w_z computably and uniformly in z since we know that they exist ahead of time (by our hypothesis on W and X) and they satisfy a computably enumerable (i.e. Σ_1) relation. It follows that $z \in X$ iff $w_z = 0$ (since X is a complement for W in $\mathbb{Q}_\infty$). Hence, we have a computable procedure, uniform in $z \in \mathbb{Q}_\infty$, for deciding whether or not $z \in X$. So W computes X, and hence W computes the Halting Set $\emptyset'$. □

As in the previous section, interpreting the relativized version of Theorem 1.2 above in the context of reverse mathematics yields the following reverse mathematical theorem (we omit its proof).

Theorem 1.6. COFIN_0 *and* $\mathsf{COFIN}_0^{2\mathsf{B}}$ *are equivalent to* ACA_0 *over* RCA_0.

7. Our Main Theorem: Coding the Halting Set into arbitrary infinite dimensional subspaces

Theorem 1.4 (RCA_0)**.** *There exists a computable infinite dimensional vector space V such that every infinite dimensional proper subspace of V computes the Halting Set $\emptyset'$.*

Proof. We will construct $V = \mathbb{Q}_\infty/S$ for some computable (infinite dimensional) subspace $S \subset \mathbb{Q}_\infty$ generated by vectors of the form

$$(*) \qquad v_n - \sum_{i<n} c_i v_i,\ n \in \mathbb{N},\ c_i \in \mathbb{Q},$$

with at most one such generator for each $n > 0$.

Generally speaking, the main idea behind our construction of S_0 is as follows. At stage $s+1 > 0$, $s \in \mathbb{N}$, of the construction we enumerate some $v = v_x - \sum_{i<x} c_i v_i$, $c_i \in \mathbb{Q}$, into $S_0^{s+1} = S_0^s \cup \{v\}$ for the unique $x \in \mathbb{N}$ that enters A_0 at stage s. Furthermore, generally speaking we do this in a way that guarantees that no vector of the form

$$\sum_{i\le x} c_i \overline{v}_i,\ c_i \in \{q_0, \dots, q_s\},$$

i.e. no vector with "small coefficients," is contained in a proper subspace of $\langle \overline{v}_0, \overline{v}_1, \dots, \overline{v}_{x-1}\rangle \subset \mathbb{Q}_\infty/S_0^{s+1}$ spanned by vectors of the form

$$\sum_{i<x} c_i \overline{v}_i,\ c_i \in \{q_0, \dots, q_s\},$$

i.e. vectors with "small coefficients." In other words, whenever we create a new linear dependence relation $\overline{v}_x = \sum_{i<x} c_i \overline{v}_i$, $c_i \in \mathbb{Q}$, in $V/\langle S_0^{s+1}\rangle$, we have that some $c_i = q_j$, where $j > s$. Hence the linear dependence relation is computable in every proper subspace of V since the coefficients of the vectors bound the stage at which the linear dependence was created in V. Using this fact, and the fact that the subspace $W \subset V$ of Theorem 1.4 above is infinite dimensional, we will be able to construct an infinite set of linearly independent vectors $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots \in W \subset V$, and since $\overline{v}_i \in \langle \overline{v}_j : j < i\rangle \subset V = \mathbb{Q}_\infty/\langle S_0\rangle$ for every $i \in A_0$ (by our construction of S_0 outlined above), it will follow that the maximal index of any $\overline{v}_i$ appearing in $\overline{w}_0, \dots, \overline{w}_i$ is at least a_i^c, from which it will follow that W computes the Halting Set $\emptyset'$. More details are given below.

We now explain precisely how to enumerate the computable set of generators $S_0 \subset \mathbb{Q}_\infty$ for S. Recall that the generators of S_0 will all take the form of $(*)$ above, and we say that v_n, $n \in \mathbb{N}$, *has been substituted* whenever there exists s such that S_0^s contains a generator of the form $(*)$. We construct $S_0 = \cup_s S_0^s$, $|S_0^s| = s$, $S_0^{s+1} \supset S_0^s$, in stages s as follows. At stage $s = 0$ set $S_0^0 = \emptyset$.

At stage $s+1 > 0$ assume that we are given S_0^s and search for the unique $x_s > 0$, $x_s \in \mathbb{N}$, such that $x_s \in A_0^s \setminus A_0^{s-1}$. Let $z_s \in \mathbb{N}$ be the largest number enumerated into A_0 by stage $s+1$ and let $A_0^{c,s}$ denote the complement of A_0^c at stage s (of the construction of A_0, that we assume is running simultaneously with our construction of S_0). Let $n_s \in \mathbb{N}$ be the largest number such that $a^{c,s}_{n_s} < x_s$ (note that n_s always exists since we are assuming that $a^c_0 = 0$). By Σ_1-induction and our construction of S_0 (still in progress) we will have that at all stages $0 < t \leq s$ every generator enumerated into S^t_0 is of the form $(*)$ with $n \in A^t_0$, and moreover for every $n \in A^t_0$ there is such a generator in S^s_0. We proceed by substituting v_{x_s} at stage $s+1$, but first we require the following definition/notation.

Definition 7.1. *For all $n \in \mathbb{N}$ and $v = \sum_{j<n} c_j v_j \in \mathbb{Q}_\infty$ let $\hat{v} \in \mathbb{Q}_\infty$ denote the vector obtained by repeatedly substituting, in order from highest index to lowest index, for every index $j \in \mathbb{N}$ in the expression for v (in terms of the standard basis $\{v_k\}_{k\in\mathbb{N}}$) such that v_j has been substituted by the end of stage s. Each substitution made is given by the rule*

$$v_j = \sum_{i<j} c_i v_i,$$

where $v_j - \sum_{i<j} c_i v_i \in S^s_0$. For every $X \subset \mathbb{Q}_\infty$ we define $\hat{X} = \{\hat{v} : v \in X\} \subset \mathbb{Q}_\infty$.

Note that $\hat{v}$ is equal to v modulo $\langle S^s_0 \rangle$, and therefore $\hat{v}$ is equal to v modulo $\langle S_0 \rangle = S$. Moreover, it follows that all indices appearing in $\hat{v}$ belong to the complement of $A^s_0 \subset \mathbb{N}$. Finally, note that one may pass uniformly (in $v \in \mathbb{Q}_\infty, s \in \mathbb{N},$) and computably from v to $\hat{v}$ at stage $s+1$.

Let $Y_s \subset \mathbb{Q}_\infty$ be the (finite) set of vectors $v \in \mathbb{Q}_\infty$ such that the unique representation of v with respect to the standard basis vectors $v_0, v_1, \ldots, v_{z_s}$ has all of its coefficients in $\{q_0, \ldots, q_s\} \subset \mathbb{Q}$ and such that $\hat{v} \in \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$. Let Z_s be defined similarly to Y_0, except that the resulting vector $\hat{v}$ must lie in $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}}, v_{x_s} \rangle$. Let $0 \in \hat{P}_s \subset \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ be the union of all proper (or, equivalently, n_s-dimensional) subspaces spanned by vectors in $\hat{Y}^s_0$. Note that $Y_s, \hat{Y}^s_0, Z_s, \hat{Z}^s_0, \hat{P}_s$ are all uniformly computable in s. $\hat{Y}^s_0$ and $\hat{Z}^s_0$ are finite sets of vectors in $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ and $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}}, v_{x_s} \rangle$, respectively, while $\hat{P}_s$ is the union of finitely-many subspaces of codimension one in $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$. For all $z \in Z_s$, let $\tilde{z}$ be the projection of $\hat{z} \in \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}}, v_{x_s} \rangle$ onto $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$. Now, for all $z \in Z_s$ let $d_{\hat{z}} \in \mathbb{Q}$ be the v_{x_s}-coefficient of $\hat{z}$. Let $D \in \mathbb{Q}$, $D > 0$, be strictly larger than the maximum of the absolute values of the (finitely-many) $d_{\hat{z}}$, $z \in Z_s$. For every $z \in Z_s$, either $\tilde{z} \in \hat{P}_s$, or else there is closed ball[8] around $\tilde{z}$ in $\langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ that does not intersect $\hat{P}_s$. Since Z_s is a finite set of vectors and since the intersection of finitely many open sets is open it follows that there is a small but strictly positive uniform radius $r_s \in \mathbb{Q}$ such that for all $\tilde{z}$ not in $\hat{P}_s$, $z \in Z_s$, the closed ball $\hat{B}_{s,\tilde{z}} \subset \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ with center $\tilde{z}$ and radius $Dr_s \in \mathbb{Q}$, $Dr_s > 0$, does not meet $\hat{P}_s$. Let $B_{s,0} \subset \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ denote the closed ball with center 0 and radius Dr_s. Now, for every $z \in Z_s \cup \{0\}$ the complement of $\hat{P}_s$ in $\hat{B}_{s,\tilde{z}}$ is an open and dense (in $\hat{B}_{s,\tilde{z}}$) by the Baire Category Theorem. It follows that there exist nonzero vectors $y^0_s, y^1_s, \ldots, y^{s+1}_s \in B_{s,0} \in \langle v_{a^{c,s}_0}, v_{a^{c,s}_1}, \ldots, v_{a^{c,s}_{n_s}} \rangle$ (of the form $\sum_{i<x_s} c_i v_i$) such that for all $z \in Z_s$ and $0 \leq i \leq s+1$ we have that $\tilde{z} + y^i_s \in B_{s,\tilde{z}} \setminus P_s$ and $\langle S^s_0, y^i_s \rangle \cap \langle S^s_0, y^j_s \rangle = \langle S^s_0 \rangle \subset \mathbb{Q}_\infty$ for $0 \leq i < j \leq s+1$.[9] By an argument similar to the one we gave in the proof of the key lemma (see Section 4 above for more information) involving the finitary pigeonhole principle, there exists $0 \leq j_0 \leq s+1$ such that for all $u_0, u_1, \ldots, u_s$

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[8]Here we are viewing $\mathbb{Q}_\infty$ as a metric space where the metric is induced via the standard basis vectors $v_0, v_1, v_2, \ldots$.

[9]Here we are proving a stronger version of the key lemma (i.e. Lemma 4.1) above. The proofs of the key lemma above and the crucial lemmas of [14] can also be phrased in terms of the Baire Category Theorem. In our opinion the Baire Category Theorem is essentially the key to proving most of the results in part one. More generally, in our opinion the Baire Category Theorem is the key to coding *incomputable information* into nontrivial subspaces of *computable* vector spaces.

we have that $u_i \in \langle S_0^s, y_s^{j_0} \rangle$ iff $u_i \in \langle S_0^s \rangle$. Finally, we enumerate

$$v_{x_s} - \frac{1}{D} y_s^{j_0} = v_{x_s} - \sum_{i<x_s} c_i v_i$$

into S_0^{s+1} and proceed to the next stage of the construction of S_0. This completes our construction of $S_0 = \cup_{s \in \mathbb{N}} S_0^s$ and $S = \langle S_0 \rangle$.

Remark 7.2. *Note that, by our construction of $y_s^{j_0}$ in the previous paragraph, upon substituting $y_s^{j_0}$ for v_{x_s} in every $z \in \hat{Z}_s \backslash \hat{Y}_s = \widehat{Z_s \setminus Y_s}$ it follows that none of the resulting vectors lie in $\hat{P}_s$. In other words, none of the resulting vectors live in a codimension one subspace of $\langle v_{a_0^{c,s}}, v_{a_1^{c,s}}, \ldots, v_{a_{n_s}^{c,s}} \rangle$ spanned by linear combinations of $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{z_s}$ with coefficients in $\{q_0, \ldots, q_s\} \subset \mathbb{Q}$. This is the key property of $S_0 = \cup_s S_0^s$ that we will use in the final paragraph of the current proof below to prove that every infinite dimensional proper subspace of $V = \mathbb{Q}_\infty / S$ computes the Halting Set $\emptyset'$.*

We now verify that $V = \mathbb{Q}_\infty / S$ has the properties listed in the statement Theorem 1.4 above. By our construction of $S = \langle S_0 \rangle \subset \mathbb{Q}_\infty$, we have that $u_n \in S$ iff $u_n \in \langle S_0^{n+1} \rangle$, for all $n \in \mathbb{N}$, and so S and $V = \mathbb{Q}_\infty / S$ are computable. Furthermore, since we substituted v_n exactly once for every $n \in A_0$ (as in previous sections) it follows that:

(1) For all $n \in \mathbb{N}$, $\overline{v}_0 = \overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \ldots, \overline{v}_{a_n^c}$ are linearly independent.
(2) For every nonzero $\overline{v} \in V$ there exists $n \in \mathbb{N}$ such that $\overline{v}$ can be expressed as a linear combination of $\overline{v}_0 = \overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \ldots, \overline{v}_{a_n^c}$.

(In other words, $\overline{v}_0 = \overline{v}_{a_0^c}, \overline{v}_{a_1^c}, \overline{v}_{a_2^c}, \ldots$ form a basis for V.) It follows that V is infinite dimensional.

Now, suppose that $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots \in V$ is an infinite sequence of linearly independent vectors in V. It follows from (1) and (2) above, along with some elementary finite-dimensional linear algebra, that if $f : \mathbb{N} \to \mathbb{N}$ is defined by setting $f(n)$, $n \in \mathbb{N}$, to be the maximal index appearing in any expression of $\overline{w}_0, \ldots, \overline{w}_n$ with respect to the standard spanning set $\overline{v}_0, \ldots, \overline{v}_n$, then we must have that $f(n) \geq a_n^c$, for all $n \in \mathbb{N}$ (otherwise there exists $n_0 \in \mathbb{N}$ and $(n_0+2)-$many vectors $\overline{w}_0, \ldots, \overline{w}_{n_0+1}$ such that $\overline{w}_0, \ldots, \overline{w}_{n_0+1} \in \langle \overline{v}_0 = \overline{v}_{a_0^c}, \ldots, \overline{v}_{a_{n_0}^c} \rangle$ are linearly independent, a contradiction since RCA_0 proves that dimension is well-defined for finite dimensional vector spaces). In the following paragraph we prove that we can always compute an infinite linearly independent set of vectors in V when given oracle access to an infinite dimensional proper subspace $W \subset V$. It will then follow that W computes the Halting Set $\emptyset'$.

Suppose that we are given an infinite dimensional proper subspace W of $V = \mathbb{Q}_\infty / S$, and let $n_0 \in \mathbb{N}$ be such that $\overline{v}_{n_0} \notin W$. Then for all $n \geq n_0$, $W \cap \langle \overline{v}_0, \ldots, \overline{v}_n \rangle \subset V$ is contained in a codimension one subspace of $\langle \overline{v}_0, \ldots, \overline{v}_n \rangle \subset V$. Let $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots$ be an effective enumeration of the nonzero elements of $W \subset V$, written as $\mathbb{Q}$-linear combinations of the standard spanning set $\overline{v}_0, \overline{v}_1, \overline{v}_2, \ldots$ such that some $\overline{v}_n$, $n > n_0$, occurs in the expression of $\overline{w}_i$ for all $i \in \mathbb{N}$. Let $w_0, w_1, w_2, \ldots \subseteq \mathbb{Q}_\infty$ be the corresponding sequence of linear combinations in $\mathbb{Q}_\infty$. The sequence $\overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots$ exists because W is infinite dimensional. Let $g : \omega \to \omega$ be computed from W such that for every $x \in \mathbb{N}$, $\overline{w}_{g(x)}$ is not contained in the span of $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{g(x)-1}$ in the vector space $\mathbb{Q}_\infty / \langle S_0^{t_x} \rangle$, where $t_x \in \mathbb{N}$ is least such that the coefficients of $\overline{w}_0, \ldots, \overline{w}_{g(x)}$ (with respect to $\overline{v}_0, \overline{v}_1, \overline{v}_2, \ldots$) all lie in $\{q_0, \ldots, q_{t_x-1}\}$. Note that g is a total computable function relative to W since W is infinite dimensional and $S_0^{t_x}$ is finite. We claim that $\overline{w}_{g(0)}, \overline{w}_{g(1)}, \overline{w}_{g(2)}, \ldots$ is an infinite linearly independent subset of V, and hence computes $\emptyset'$. Suppose for a contradiction and via Σ_1-induction that $n \in \mathbb{N}$ is least such that $\overline{w}_{g(n)}$ is a linear combination of $\overline{w}_{g(0)}, \ldots, \overline{w}_{g(n-1)}$. Then we can computably find the least $s_0 \in \mathbb{N}$ such that this is the case in the quotient $\mathbb{Q}_\infty / \langle S_0^{s_0} \rangle$. We claim that $s_0 < t_n$, contradicting our definition of g. Suppose for a contradiction that the coefficients of $\overline{w}_{g(0)}, \ldots, \overline{w}_{g(n)}$ all live in $\{q_0, \ldots, q_{s_0-1}\}$ (i.e. suppose that $s_0 \geq t_n$), then in our construction of $S_0^{s_0}$ at stage

s_0 it follows that $w_{g(0)}, \ldots, w_{g(n)-1} \in Y_{s_0} \subset \mathbb{Q}_\infty$, $\langle \hat{w}_{g(0)}, \ldots, \hat{w}_{g(n)-1} \rangle \subseteq \hat{P}_{s_0} \subset \mathbb{Q}_\infty$, and $w_{g(n)} \in Z_{s_0} \backslash Y_{s_0} \subset \mathbb{Q}_\infty$. Now, by our construction of $S_0^{s_0}$ at stage s_0 and Remark 7.2 above, we cannot have that $\overline{w}_{g(n)} \in \mathbb{Q}_\infty / \langle S_0^{s_0} \rangle$ is a linear combination of $\overline{w}_{g(0)}, \ldots, \overline{w}_{g(n-1)} \in \mathbb{Q}_\infty / \langle S_0^{s_0} \rangle$, a contradiction. $\square$

As above, interpreting the relativized version of Theorem 1.4 above in the context of reverse mathematics yields the following theorem.

Theorem 1.8. INF_0 *is equivalent to* ACA_0 *over* RCA_0.

This concludes the first part of this article. In the next part (i.e. part two) we will come at the problem of coding into infinite dimensional proper subspaces from the opposite perspective of showing that there always exists a (computability theoretically) "simple" such subspace. More precisely, we will prove that every infinite dimensional computable $2-$based vector space V contains a *low* infinite dimensional proper subspace $W \subset V$. Therefore, it is impossible to code the Halting Set $\emptyset'$ into every infinite dimensional proper subspace of a $2-$based vector space. This is interesting given what we have seen in part one (i.e. Theorems 1.1, 1.2, 1.3, and 1.4 above). We will also draw some general parallels between INF_0^{2B} and combinatorial principles such as Ramsey's Theorem for pairs. More information and discussion follows.

8. Constructing infinite dimensional proper subspaces of $2-$based vector spaces

The main purpose of this section is to classify the reverse mathematical strength of INF_0^{2B}, which we introduced in Section 4. We restate INF_0^{2B} for the reader's convenience.

INF_0^{2B}: Every infinite dimensional $2-$based vector space contains a proper infinite dimensional subspace.

Recall that a vector space V is a $2-$based *vector space* if $V = \mathbb{Q}_\infty / S$ for a subspace $S \subseteq \mathbb{Q}_\infty$ generated by vectors of the form $v_i - k_{i,j} v_j$, $k_{i,j} \in \mathbb{Q}$, $i, j \in \mathbb{N}$.

More specifically, we will show that INF_0^{2B} is implied by WKL_0 over RCA_0. This is the main theorem of part two. It follows that the vector space V in Theorem 1.4 *cannot* be a $2-$based vector space. This is interesting in the context of Theorems 1.1, 1.2, and 1.3 above, because these theorems show that $2-$based vector spaces are quite useful for coding into various subclasses of infinite dimensional proper subspaces of infinite dimensional computable vector spaces. More precisely, we have that the statements COINF_0^{2B} and COFIN_0^{2B} each imply ACA_0 over RCA_0, but the seemingly weaker statement INF_0^{2B} is actually provable in the strictly weaker system WKL_0 (see Section 1 for more details). Along the way to proving this fact we will highlight some general similarities between INF_0^{2B} and various combinatorial theorems such as Ramsey's Theorem for pairs.

We will also show that, in the context of $\omega-$models, any infinite dimensional computable vector space V that does not contain a low infinite dimensional proper subspace $W \subset V$ *must* contain many different (i.e. "densely many") *computable* finite dimensional subspaces. This is a stronger version of the theorem that follows from [14] that says, in the context of $\omega-$models, every infinite dimensional vector space V that does not contain a low infinite dimensional proper subspace $W \subset V$ must contain many different (i.e. "densely many") *low* finite dimensional subspaces. Our theorem is interesting because it says that any vector space in which all nontrivial subspaces are incomputable, like the one constructed in [14] in which all nontrivial subspaces are of (incomputable) PA degree, *must* contain a low infinite dimensional proper subspace. It also implies that in the context of $\omega-$models our construction of an infinite dimensional computable vector space V in Theorem 1.4 above *must* contain many different finite dimensional computable subspaces. More details and discussions follow.

8.1. INF_0^{2B} and weak König's lemma.

Theorem 8.1 (RCA_0). WKL_0 *implies* INF_0^{2B}.

Proof. We reason in WKL_0. Let $V = \mathbb{Q}_\infty/S$ be an infinite dimensional 2−based vector space and let $\{\overline{v}_k\}_{k\in\mathbb{N}} \subset V = \mathbb{Q}_\infty/S$ be the images of $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_\infty$, respectively, under the canonical quotient map $\mathbb{Q}_\infty \to V$.

First we claim that if $\overline{W} \subseteq V$ is a subspace, then the span of $V_0 = \{\overline{v}_k : \overline{v}_k \notin \overline{W},\ k \in \mathbb{N}\}$ does not contain any nonzero $\overline{v}_k \in V$ such that $\overline{v}_k \in \overline{W}$. To see why this is the case, suppose otherwise (for a contradiction). Let $\overline{W}_0 = \langle V_0 \rangle \subseteq V$, and W, W_0 be the preimages of $\overline{W}, \overline{W}_0$ in $\mathbb{Q}_\infty$, respectively. Note that $\overline{W}_0, W_0$ may not exist since we are reasoning in WKL_0; we only introduced these subspaces so that we can write statements like "$\overline{x} \in \overline{W}_0$" as a shorthand for saying that "$\overline{x}$ is a linear combination of elements of V_0." The entire proof of the claim in the rest of this paragraph can be done without mentioning $\overline{W}_0, W_0$ via the substitution described in the previous sentence. By hypothesis we have that some $0 \neq \overline{v}_{k_0} \in \overline{W}$, $k_0 \in \mathbb{N}$, is a linear combination of $\{\overline{v}_j\}_{j\in\mathbb{N}}$ in V_0. Lifting this relation to $\mathbb{Q}_\infty$ implies that there exists $z \in S$ and $w_0 \in W_0$ such that

$$v_{k_0} = w_0 + z,$$

where $v_{k_0} \in \mathbb{Q}_\infty \setminus S$.

Now, since V is a 2−based vector space we have that $z \in S$ is a linear combination of vectors of the form $v_i - k_{i,j}v_j$, $k_{i,j} \in \mathbb{Q}$, $i, j \in \mathbb{N}$. Furthermore, note that for all $v = v_i - k_{i,j}v_j \in \mathbb{Q}_\infty$, $k_{i,j} \in \mathbb{Q}$, $i, j \in \mathbb{N}$, (since $S \subseteq W \subseteq \mathbb{Q}_\infty$ are subspaces) we have that $v_i \in W$ iff $v_j \in W$. Hence $v_i \in W_0$ iff $v_j \in W_0$. It now follows that it is impossible to write $v_{k_0} \in W$ as a linear combination of elements of standard basis vectors $v_k \notin W$ and generators of S, a contradiction. Intuitively speaking this is the case because our previous remarks imply that one cannot use Gaussian elimination via the generators of S of the form $v_i - k_{i,j}v_j$ as above to go from linear combinations of standard basis vectors of the form $v_k \notin W$ to the standard basis vector $v_{k_0} \in W$, which is a contradiction. A more formal proof would proceed as follows. Note that every standard basis vector appearing in w_0 must be canceled by another occurrence in z (since $v_{k_0} = w_0 + z$ and $v_{k_0} \in W$). Hence, by our previous remarks in this paragraph it follows that $-w_0$ is a sum of generators of S, hence $w_0 \in S$, and so $v_{k_0} \in S$, a contradiction.

Now, use WKL_0 to construct a nontrivial subspace $\overline{Z} \subset V$ containing $\overline{v}_0 \in V = \mathbb{Q}_\infty/S$ and not containing $\overline{v}_1 \in V$ (see [14] for more details on how to do this). If $\overline{Z}$ is infinite dimensional then we are done. Assume that $\overline{Z}$ is finite dimensional. Then, using WKL_0 (again) along with our remarks in the previous paragraph, construct a subspace $\overline{Z}_0 \subset V$ containing all $\overline{v}_k \notin \overline{W}$, $k \in \mathbb{N}$, and not containing $\overline{v}_0$ (see [14] for more details on how to do this). This is possible since (by the previous paragraph) we know that the intersection of $\overline{Z}$ and the span of all $\overline{v}_k \notin \overline{Z}$ is trivial (i.e. zero). Note that $\overline{v}_k \notin \overline{Z}$ for cofinitely many $k \in \mathbb{N}$ since $\overline{Z}$ is finite dimensional. It follows that $\overline{Z}_0$ is an infinite/cofinite dimensional proper subspace of V. □

Remark 8.2. *Intuitively speaking, the reason why* INF_0^{2B} *is provable in* WKL_0 *but* INF_0 *is not is because constructing "complements" is easier in 2−based vector spaces than in general vector spaces (here we are thinking of* $\overline{W}_0$ *in the proof above as a "complement" for* $\overline{W}$*). In other words, our construction of* $\overline{Z}_0$ *in the proof of Theorem 8.1 above is valid in* WKL_0 *because of our remarks in the second paragraph of the proof, which only apply to 2−based vector spaces. In this sense 2−based vector spaces are "simpler" than other vector spaces such as the vector space that we built in Theorem 1.4 above.*

Remark 8.3. *Note that the proof of Theorem 8.1 above is divided into two cases, one in which we produce an infinite/coinfinite dimensional subspace, and the other in which we produce an infinite/cofinite dimensional proper subspace (see the last two paragraphs of the* *proof of Theorem 8.1 for more details). Our constructions in Theorems 1.1 and 1.2 above imply that any proof of Theorem 8.1 must be divided into these two cases because if V is the vector space of Theorem 1.1 then the proof of Theorem 8.1 applied to V and the Jockusch-Soare Low Basis Theorem* [23] *produces an infinite/cofinite dimensional low proper subspace. Meanwhile, if V is the vector space of Theorem 1.2 then the proof of Theorem 8.1 applied to V and the Jockusch-Soare Low Basis Theorem constructs an infinite/coinfinite dimensional low subspace. In other words, Theorems 1.1 and 1.2 above and the Low Basis Theorem imply that any proof of Theorem 8.1 must be nonuniform (i.e. divided into cases).*

8.1.1. *Some similarities between our proof of INF_0^{2B} via WKL_0 and proofs of various combinatorial principles such as Ramsey's Theorem for pairs.* We now comment that our proof of INF_0^{2B} via WKL_0 bears some resemblance to well-known proofs of various combinatorial principles such as Ramsey's Theorem for pairs and the Chain/Antichain Principle for infinite partial orders. For more information on these theorems, consult [5, 21]. We will assume that the reader is generally familiar with these theorems and their proofs, as well as the standard proof of the Jockusch-Soare Low Basis Theorem. For more information on the Low Basis Theorem, consult [23, 40, 41].

In the proof of Ramsey's Theorem for pairs one is given a 2-coloring c of pairs of natural numbers via the colors RED and BLUE and one must construct an infinite homogeneous set of natural numbers for the coloring c. Generally speaking, the proof is as follows. Begin by trying to construct an infinite RED homogeneous set in stages by carefully adding one new number to the homogeneous set at every stage (for more information on the proof and what we mean by carefully, see [5]). Then one argues that if at any stage one cannot extend the current RED homogeneous set by one element, the obstruction is caused by an infinite BLUE homogeneous set that one can find via a different construction. One key property of this proof is that it is nonuniform. In other words, we cannot tell at the beginning of the proof whether we will construct an infinite RED homogeneous set or an infinite BLUE homogeneous set, and at no finite stage of the proof is this ever decided (unless we have already constructed an infinite BLUE homogeneous set at some previous finite stage in which case the proof is finished). One can make similar observations about the proof of the Chain/Antichain Principle (CAC), as well as other combinatorial principles related to Ramsey's Theorem for pairs.

The proof of Theorem 8.1 above via WKL_0 is similar to that of Ramsey's Theorem for pairs. To see how, first of all note that our proof of INF_0^{2B} (via WKL_0) is nonuniform, since it is divided into two cases: one in which $\overline{W}$ is infinite dimensional and one in which $\overline{W}$ is finite dimensional. Furthermore, the similarities between the proofs becomes much more apparent when we think about proving INF_0^{2B} via WKL_0 in the context of $\omega-$models and via the (proof of the) Low Basis Theorem. To compute a low infinite dimensional proper subspace of V in this context one applies the proof of the Jockusch-Soare Low Basis Theorem [23]. The proof of the Low Basis Theorem proceeds in stages, and at every stage we may keep on increasing the dimension of our current subspace by one, unless at some stage we force divergence and land ourselves in a finite dimensional subspace. This is analogous to getting stuck building an infinite RED homogeneous set in the paragraph above. In this case, however, via a different construction we can actually construct a finite dimensional subspace (see the proof of Theorem 8.1 above for more details) and use this subspace, along with the fact that V is a $2-$based vector space (see the proof of Theorem 8.1 above for more details), to help us construct an infinite/cofinite dimensional proper subspace of V. This is analogous to building an infinite BLUE homogeneous set in the previous paragraph. A more precise description of the proof of INF_0^{2B} outlined in paragraph is given in the proof of Theorem 8.6 in the next subsection below.

This general similarity between INF_0^{2B} and Ramsey's Theorem for pairs is interesting in light of the fact that the reverse mathematical strength of INF_0^{2B} is still open. Like many combinatorial theorems related to Ramsey's Theorem for pairs it may be the case that INF_0^{2B} is not equivalent to any of the "big five" subsystems of second order arithmetic in the context of ω-models, which would be interesting. In other words, the reverse mathematical strength of INF_0^{2B} could lie strictly between RCA_0 and WKL_0.

Question 8.4. *Determine the reverse mathematical strength of* INF_0^{2B}

(i) *over* RCA_0*, and*
(ii) *in the context of* ω*-models that satisfy* RCA_0*.*

8.2. **Computable finite dimensional subspaces of infinite dimensional computable vector spaces.** Throughout this subsection we will work exclusively in the context of $\omega-$models and classical effective algebra (i.e. we abandon the lens of reverse mathematics). We begin with a definition.

Definition 8.5. *Let V be an infinite dimensional computable vector space, and let $\mathcal{P}(F)$ be a property of the finite dimensional subspaces F of V. For example, $\mathcal{P}(F)$ could say that F is computable. We say that $\mathcal{P}$ is* dense *in V if for any given finite set of vectors $x_0, \ldots, x_n \in V$, $n \in \mathbb{N}$, there exists $m \geq n$, $m \in \mathbb{N}$, and a finite set of vectors $x_0, \ldots, x_m$ extending $x_0, \ldots, x_n$ such that $\mathcal{P}(\langle x_0, \ldots, x_m \rangle)$ holds. Our terminology comes from the topology on Cantor space.*

We are now ready to state the main theorem of this subsection. The main theorem is interesting because it says that in order for a computable infinite dimensional vector space (over a computable field) to have all of its infinite dimensional proper subspaces be complicated (i.e. nonlow) it must have lots of simple (i.e. computable) finite dimensional subspaces. Conversely, it also says that in order for a computable infinite dimensional vector space to have all of its finite dimensional subspaces be complicated (i.e. incomputable) it must have at least one simple (i.e. low) infinite dimensional proper subspace. This is a stronger version of a consequence of [14] which says that in any computable infinite dimensional vector space V there is either a low infinite dimensional proper subspace, or else the class of low finite dimensional nonzero subspaces is dense in V.

Theorem 8.6. *Let V be an infinite dimensional computable vector space (over a computable field). Then either:*

(1) *V contains a low infinite dimensional proper subspace $W \subsetneq V$, or else*
(2) *The class of computable finite dimensional subspaces is dense in V.*

We assume that the reader is familiar with the basics of Π^0_1-classes (i.e. effectively closed sets), $2^{<\omega}$ (the set of all finite binary sequences), Cantor space 2^ω, and the standard proof of the Low Basis Theorem relative to the oracle $\emptyset'$. These topics live at the core of computability theory, and extensive information and background on these topics can be found in [3, 4, 11, 12, 23, 30, 34, 35, 40, 41]. Recall that a tree is simply a subset of $2^{<\omega}$ that is closed downward under initial segments $\subseteq$ and that a Π^0_1-class is a subset of Cantor space that can be represented as the set of infinite binary paths through an infinite binary computable tree in $2^{<\omega}$. Also recall that the class of subspaces of a computable vector space V that do not contain a given set of finitely many vectors is a Π^0_1-class. Now, let $V = \{u_0, u_1, u_2, \ldots\}$ be a computable presentation of an infinite dimensional computable vector space (over some computable field), and for all $n \in \omega$ let $P_n \subseteq 2^{<\omega}$ be a computable tree in $2^{<\omega}$ such that the set of infinite paths through P_n code the subspaces of V that do not contain $u_n \in V$. In other words, for all $n \in \omega$, $f \in 2^\omega$ is the characteristic function of a subspace of V not containing u_n if and only if f is an infinite path through P_n. Here we are identifying each finite binary string $\sigma \in 2^{<\omega}$ with the finite set of vectors $F_\sigma = \{u_i : \sigma(i) = 1\} \subset V$, where $\sigma(i)$ denotes the i^{th} bit of σ, and we think of σ as the characteristic function of F_σ. Note that this identification is computable. Also, we say that a string $\sigma \in 2^{<\omega}$ is *extendible* on a tree $T \subseteq 2^{<\omega}$ whenever $\sigma \in T$ and there is an infinite path $f_\sigma \supset \sigma$, $f_\sigma \in 2^\omega$, on T extending σ. Since V is infinite dimensional it follows that for any finite set of vectors $F \subset V$ whose span does not contain $u_n \in V$, $n \in \omega$, there exists $\sigma \in P_n$ such that $F \subseteq \sigma$ and σ is extendible on P_n. We are now ready to prove the current theorem.

Proof of Theorem 8.6. Let a finite set of vectors $F_0 = \{w_0, w_1, \dots, w_k\} \subset V$, $k \in \omega$, be given, and let $n_0 \in \omega$ be such that $u_{n_0} \in V$ is not in the span of F_0. It follows that there exists $\sigma_0 \in P_{n_0}$ such that $F_0 \subseteq \sigma_0$ and σ_0 is extendible on P_{n_0}. We will show that either V contains a computable finite dimensional subspace containing F_0 or else V contains an infinite dimensional proper subspace not containing u_{n_0}. First, however, recall that the standard proof of the Low Basis Theorem for the computable tree P_{n_0} is carried out relative to the oracle $\emptyset'$ and produces, uniformly in $\emptyset'$, a sequence of finite binary strings $\{\sigma_s\}_{s\in\omega}$, $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_s \subset \cdots$, such that σ_0 is as defined above, $\sigma_n \in P_{n_0}$, and σ_n is a proper initial segment of σ_{n+1} for all $n \in \omega$. The proof also simultaneously produces a sequence of computable trees $\{T_s\}_{s\in\omega}$, uniformly in $\emptyset'$, $T_0 = P_{n_0} \supseteq T_1 \supseteq T_2 \supseteq \cdots T_s \supseteq \cdots$, such that $\sigma_n \in T_n$ and σ_n is extendible (via an infinite path $f_n \supset \sigma_n$) in T_n, for all $n \in \omega$.

Now, assume that for every $n \in \omega$ and extendible $\tau_n^0 \in T_n$ extending $\sigma_n \in T_n$, there exists an extendible string $\tau_n^1 \in T_n$ extending τ_n^0 and such that the span of τ_n^1 is strictly larger than the span of τ_n^0 (i.e. in the proof of the Low Basis Theorem we never land in a finite dimensional subspace). In other words, τ_n^1 contains a vector that is not in the span of τ_n^0. In this case one can give a slightly modified version of the proof of the Low Basis Theorem that constructs a low infinite path $f_{n_0} \in 2^\omega$ through P_{n_0} such that f_{n_0} is/codes an infinite dimensional subspace of V not containing u_{n_0}. Generally speaking, one does this by modifying the proof of the Low Basis Theorem so that one always extends the strings σ_n (in the paragraph above) to longer strings σ_{n+1} with a strictly larger span. This is possible using a $\emptyset'$ oracle (which we have in the proof of the Low Basis Theorem) and our assumption in the first sentence of this paragraph.

Suppose now, on the other hand, that there exists $n \in \omega$ and extendible $\tau \in T_n$ extending $\sigma_n \in T_n$ such that every extendible node $\rho \in T_n$ extending τ has the same span as $\tau \supseteq \sigma_0$ (i.e. somewhere during the proof of the Low Basis Theorem we forced divergence and landed ourselves in the finite dimensional subspace spanned by τ). Recall that τ is an extension of σ_n which is an extension of σ_0 and so the span of τ contains F_0. We claim that the span of τ is computable. This follows from the fact that T_n has a *unique* infinite path extending τ, and the well-known fact that an isolated point in a Π_1^0-class is computable. To see why T_n has a unique path extending τ, note that τ is extendible in T_n by hypothesis, and by hypothesis we also have that every vector on T_n extending τ is in the span of τ. Now, since every infinite path of T_n codes a subspace of u_{n_0}, it must be the case that the unique path on T_n extending τ is given by (i.e. codes) the span of τ. $\square$

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