ON THE COMPLEXITY OF RADICALS IN NONCOMMUTATIVE RINGS

CHRIS J. CONIDIS

Abstract. This article expands upon the recent work by Downey, Lempp, and Mileti [3], who classified the complexity of the nilradical and Jacobson radical of commutative rings in terms of the arithmetical hierarchy.

Let R be a computable (not necessarily commutative) ring with identity. Then it follows from the definitions that the prime radical of R is Π^1_1, and the Levitzki radical of R is Π^0_2. We show that these upper bounds for the complexity of the prime and Levitzki radicals are optimal by constructing two noncommutative computable rings with identity, such that the prime radical of one is Π^1_1-complete, while the Levitzki radical of the other is Π^0_2-complete.

1. Introduction

One of the first and most important questions to be studied in computable ring theory is the ideal membership problem. The analysis of this problem dates back to the work of Kronecker [8], who showed that every ideal in a computable presentation of $\mathbb{Z}[X_1, X_2, \ldots, X_N]$ is decidable. These results were later expanded by Van der Waerden [14], who showed that there does not exist a single universal splitting algorithm for factoring polynomials over all computable fields, and others. Frölich and Shepherdson [7] were first to give formal definitions in terms of recursive functions and Turing machines. They also showed, among other things, that there exists a single computable field with no splitting algorithm. By computable ring, we mean the following.

Definition 1.1. A *computable ring* (with identity) is a computable subset R of natural numbers, together with computable binary operations $+$ and $\cdot$ on R, and elements $0, 1 \in R$, such that $(R, 0, 1, +, \cdot)$ is a ring (with identity $1 \in R$). Throughout this article we use R to denote both the domain of the ring, as well as the ordered 5-tuple $(R, 0, 1, +, \cdot)$.

More recently, there has been an interest in the complexity of radicals in rings in terms of the arithmetical hierarchy. In particular, Downey, Lempp, and Mileti [3] have completely classified the complexity of the nilradical and Jacobson radical in commutative computable rings, showing that the former is Σ^0_1-complete, while the latter is Π^0_2-complete (the arithmetical and analytical hierarchies are formally introduced in the next section).

We now define two radicals, which differ from the nilradical and Jacobson radical in noncommutative rings. The first is called the *prime radical*, while the second is known as the *Levitzki radical*. These radicals can be thought of as generalizations of the Jacobson radical, and some of the theorems related to the Jacobson radical can be generalized to these radicals as well. The main purpose of this article is to determine the complexity of the prime radical and Jacobson radical in a general noncommutative ring R.

Let R be a (possibly noncommutative) ring with identity. By *ideal* we mean two-sided ideal.

Definition 1.2. An ideal $P \subset R$ is *prime* if whenever $AB \subseteq P$, for ideals $A, B \subseteq R$ then either $A \subseteq P$, or else $B \subseteq P$. This is equivalent to saying that for any two elements $a, b \in R$, we have that either $a \in P$ or $b \in P$ whenever $aRb \subseteq P$.

Definition 1.3. An ideal $P \subseteq R$ is *semiprime* if $A \subseteq P$ whenever A is an ideal such that $A^2 \subseteq P$.

It can be shown that an ideal $P \subseteq R$ is semiprime if and only if it is an intersection of prime ideals.

Definition 1.4. The intersection of all prime ideals in R is called the *prime radical* of R (it is also known as the *lower nilradical of* R, or the *Baer-McCoy radical of* R). This is the smallest semiprime ideal of R.

We now define the Levitzki radical of R.

The author was partially supported by NSERC grant PGS D2-344244-2007. Furthermore, he would like to acknowledge the helpful input he received from his thesis advisors, R.I. Soare, D.R. Hirschfeldt, and A. Montalbán. He would also like to especially thank J.R. Mileti for suggesting this topic, A. Montalbán for his help in editing the preliminary drafts, and the anonymous referee for finding an error in the initial draft.

Definition 1.5. A subset S of R is *locally nilpotent* if every subring of R (without identity) generated by a finite number of elements of S is nilpotent.

It can be proved that if A and B are locally nilpotent subsets of R, then so are RAR, RBR, and $A+B$. From these facts it can be shown that there exists a largest locally nilpotent subset of R, and that this subset is an ideal (see Section 4).

Definition 1.6. The *Levitzki radical* of R is the largest locally nilpotent subset of R.

Most of the typical problems that one encounters in algebra have *arithmetical* solutions. This means that their solutions can be expressed in relatively simple terms. For example, if R is a computable commutative ring, then by definition it follows that the nilradical of R is Σ^0_1, and a well-known result from classical commutative ring theory says that for every $r \in R$, r is in the Jacobson radical of R if and only if

$$(\forall x \in R)(\exists a \in R)[(1-rx)a = 1]. \tag{1}$$

From this result it follows that the Jacobson radical of R is Π^0_2 (the Π comes from the $\forall$ to the far left, and the number 2 comes from the number of alternations of quantifiers in the expression). On the other hand, Downey, Lempp, and Mileti [3] have constructed computable commutative rings R_0 and R_1 such that the nilradical of R_0 is Σ^0_1-complete, and the Jacobson radical of R_1 is Π^0_2-complete, thus showing that the simplest characterization of the nilradical is the standard definition, while the simplest characterization of the Jacobson radical is (1) above. Many more examples of arithmetical ring-theoretic constructions exist, see for example [2, 3, 5, 6].

Above the arithmetical hierarchy lies the analytic hierarchy. Analytical sets are more complex than arithmetical sets, because to define an analytic set one is allowed to quantify over both number variables (as in the arithmetical case), as well as function (or set) variables. The reader should note that every arithmetical set is analytical, but not vice versa. For example, the standard definition of the Jacobson radical of a commutative ring R is the intersection of all maximal ideals in R. Since this definition quantifies over all the maximal ideals of R, it follows from the definition that the Jacobson radical of a computable ring is analytic. However, (1) above gives a different (arithmetical) characterization of the Jacobson radical, from which it follows that the Jacobson radical of a computable ring is always in fact arithmetical. In the next section we define a well-known set called WF (the set of computable indices for well-founded trees) that is analytic but not arithmetic.

When a set $X \subseteq \omega$ is shown to be analytical but not arithmetical, it implies that function or set quantifiers are necessary to define X via a computable predicate. For example, in Section 3, we construct a computable ring R whose prime radical is Π^1_1-complete. It follows that the prime radical of R is analytical but not arithmetical. One consequence of this construction is that any effective definition of the prime radical *must* involve quantifying over sets of natural numbers. In other words, one must say something like "the prime radical of a ring R is the intersection of all the prime ideals in R" (here we are quantifying over all prime ideals of R). The superscript 1 in Π^1_1 says that we are allowed to quantify over sets, while the subscript 1 says that only one set quantifier is necessary in the definition of the prime radical.

By definition, it follows that if R is a computable ring, then the prime radical of R is a Π^1_1 set, and the Levitzki radical of R is a Π^0_2 set. The main purpose of this article is to show that these upper bounds on the complexity of the prime radical and Levitzki radical are sharp, by constructing computable rings R_0 and R_1 such that the prime radical of R_0 is Π^1_1-complete, and the Levitzki radical of R_1 is Π^0_2-complete. More formally, the main goal of this article is to prove Theorems 1.7 and 1.8 below. The proof of Theorem 1.7 is given in Section 3, while the proof of Theorem 1.8 is given in Section 4. The formal definition of completeness is given in the next section, but, intuitively, to say that a set X is Γ-complete means that

(1) X belongs to the complexity class Γ.
(2) The complexity of X is maximal among Γ-sets, in the sense that every Γ-set can be (computably) reduced to X.

Our main goal in this article is to prove the following theorems.

Theorem 1.7. *There exists a noncommutative computable ring R such that the prime radical of R is Π^1_1-complete.*

Theorem 1.8. *There exists a noncommutative computable ring R such that the Levitzki radical of R is Π^0_2-complete.*

2. Preliminaries

2.1. **Background.** Let ω denote the set of natural numbers, i.e. $\omega = \{0, 1, 2, 3, \dots\}$. By *ring* we mean a (possibly noncommutative) ring with identity. We assume that the reader is familiar with the basic definitions of ring theory, as well as those of (oracle) Turing machines and (relative) computation. Standard texts in commutative ring theory include [1, 4, 10, 11]. A standard text on noncommutative rings is [9]. Two standard references in computability theory are [12, 13].

Fix a computable bijection $p_2 : \omega \times \omega \to \omega$, and numbers $x, y \in \omega$. We will denote $p_2(x, y)$ by $\langle x, y\rangle$. Furthermore, for every $n \in \omega,\ n \geq 3$, define a function $p_n : \omega^n \to \omega$ by

$$p_n(x_0, x_1, x_2, \dots, x_{n-1}) = \langle x_0, p_{n-1}(x_1, x_2, \dots, x_{n-1})\rangle.$$

It follows (by induction) that p_n is a computable bijection, and that

$$p_n(x_0, x_1, x_2, \dots, x_{n-1}) = \langle x_0, \langle x_1, \langle x_2, \langle \cdots \langle x_{n-2}, x_{n-1}\rangle\rangle \cdots\rangle.$$

For every $n, x_0, x_1, \dots, x_{n-1} \in \omega$, we let

$$\langle x_0, x_1, x_2, \dots, x_{n-1}\rangle = p_n(x_0, x_1, x_2, \dots, x_{n-1}).$$

We now review the construction of the arithmetical hierarchy. Fix natural numbers $m, n \geq 1$.

(1) We say that a set $X \subseteq \omega^m$ is Σ^0_n, and write $X \in \Sigma^0_n$, if there exists a computable set $A \subseteq \omega^{n+m}$ such that for every $x_1, x_2, \dots, x_m \in \omega$ we have that
$$(x_1, x_2, \dots, x_m) \in X \Leftrightarrow \exists a_1 \forall a_2 \exists \cdots Q a_n[(x_1, x_2, \dots, x_m, a_1, a_2, \dots, a_n) \in A],$$
where Q is $\exists$ if n is odd, and $\forall$ if n is even.

(2) A set $X \subseteq \omega^m$ is Π^0_n, and write $X \in \Pi^0_n$, if there exists a computable set $A \subseteq \omega^{n+m}$ such that for every $x_1, x_2, \dots, x_m \in \omega$ we have that
$$(x_1, x_2, \dots, x_m) \in X \Leftrightarrow \forall a_1 \exists a_2 \forall \cdots Q a_n[(x_1, x_2, \dots, x_m, a_1, a_2, \dots, a_n) \in A],$$
where Q is $\exists$ if n is even, and $\forall$ if n is odd.

Definition 2.1. A Σ^0_n (resp. Π^0_n) set $X \subseteq \omega$ is called Σ^0_n (Π^0_n)-*complete* if for every set $Y \in \Sigma^0_n$ (Π^0_n) there is a computable function $h_Y : \omega \to \omega$ such that for every $n \in \omega$, $n \in Y$ if and only if $h_Y(n) \in X$.

For our purposes, we are most interested in Π^0_2 sets, since the proof of Theorem 1.8 involves reducing every Π^0_2 set to the Levitzki radical of a noncommutative computable ring. With this in mind, we state the following standard computability-theoretic result. Recall that if $\{\varphi_e\}_{e\in\omega}$ is an effective listing of the partial computable functions, then, for every $e \in \omega$, the e^{th} computably enumerable (c.e.) set is defined to be

$$W_e = \{x \in \omega : \varphi_e(x)\downarrow\}.$$

Proposition 2.2. *The set*

$$\mathrm{Inf} = \{e \in \omega :\ |W_e| = \infty\}$$

is Π^0_2-complete.

Therefore, to show that a given set X is Π^0_2-complete, it suffices to find a computable function h such that for all $n \in \omega$, $n \in \mathrm{Inf}$ if and only if $h(n) \in X$.

We now define what it means for a set $X \subset \omega$ to be Π^1_1. Recall that ω^ω denotes the set of functions $f : \omega \to \omega$.

Definition 2.3. We say that a set $X \subset \omega^m$ is Π^1_1, and write $X \in \Pi^1_1$, if there exists a number $n \in \omega$, and a computable set $A \subseteq \omega^\omega \times \omega^{m+n}$, such that for all $x_1, x_2, \dots, x_m \in \omega$ we have that

$$(x_1, x_2, \dots, x_m) \in X \Leftrightarrow \forall f \exists a_1 \forall a_2 \cdots Q a_n[(f, a_1, a_2, \dots, a_n, x_1, x_2, \dots, x_m) \in A],$$

where Q is $\forall$ if n is even, and $\exists$ if n is odd.

A well-known result says that, without loss of generality, we can always assume that $n = 1$ in Definition 2.3.

Definition 2.4. A Π^1_1 set $X \subseteq \omega$ is called Π^1_1-*complete* if for every set $Y \in \Pi^1_1$, there is a computable function $h_Y : \omega \to \omega$ such that for every $n \in \omega$, $n \in Y$ if and only if $h_Y(n) \in X$.

We now construct an example of a Π^1_1-complete set called WF (the set of computable indices for well founded trees).

Let $\omega^{<\omega}$ denote the set of finite strings of natural numbers. For any $\sigma, \tau \in \omega^{<\omega}$ write $\sigma \subseteq \tau$ to mean that σ is an initial segment of τ. A nonempty subset T of $\omega^{<\omega}$ is *closed downwards* if for every $\sigma \in T$ and every $\tau \in \omega^{<\omega}$ such that $\tau \subseteq \sigma$, we have that $\tau \in T$. We call subsets of $\omega^{<\omega}$ that are closed downwards *trees*.

Let $T \subseteq \omega^{<\omega}$ be a tree, and $\sigma \in T$. We say that σ is an *extendible node* if there exists an infinite path through T extending σ – i.e. if there exists $f \in \omega^\omega$ such that for every $n \in \omega$, $f \upharpoonright n \in T$. Here $f \upharpoonright n = \langle f(0), f(1), \ldots, f(n-1) \rangle \in \omega^{<\omega}$ denotes the first n bits of f. We also say that T is *well-founded* if no $\sigma \in T$ is an extendible node. By definition it follows that if T is a computable tree, then the property of T being well-founded is Π^1_1. It turns out that this property is also Π^1_1-complete.

Proposition 2.5. *Let $\{T_e\}_{e \in \omega}$ be an effective listing of all computable trees. Then the set*

$$\mathrm{WF} = \{e \in \omega : T_e \text{ is a well–founded tree}\}$$

is Π^1_1-complete.

Hence, to show that a given set X is Π^1_1-complete, it suffices to find a computable function h such that for all $n \in \omega$, $n \in \mathrm{WF}$ if and only if $h(n) \in X$.

Now that we have given the reader the necessary preliminaries, we are ready to prove Theorems 1.7 and 1.8. Throughout this article, R will always denote a (possibly) noncommutative ring with identity. In Section 3 we prove Theorem 1.7, and in Section 4 we prove Theorem 1.8. As an aside, it may interest the reader to know that in a general noncommutative ring R, if B denotes the prime radical of R, L denotes the Levitzki radical of R, N denotes the nilradical of R, and J denotes the Jacobson radical of R, then we have that

$$B \subseteq L \subseteq N \subseteq J,$$

and the inclusions are strict in general.

3. Prime Radical

Recall that the prime radical of a (possibly) noncommutative ring R is defined to be the intersection of all the prime ideals of R. From this it follows that the prime radical of a computable (possibly) noncommutative ring R is a Π^1_1 set. Hence, the most that one could hope for is to construct a computable (noncommutative) ring R whose prime radical is Π^1_1-complete. With this observation in mind, we prove the following theorem.

Theorem 1.7. *There exists a noncommutative computable ring R such that the prime radical of R is Π^1_1-complete.*

First, however, we require some definitions. Let R be a ring.

Definition 3.1. For any elements $a, b \in R$, we say that *a divides b* if b is contained in the (two-sided) ideal generated by a, i.e. $b \in \langle a \rangle$.

Definition 3.2. A nonempty set $S \subseteq R$ is called an *m-system* if, for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$.

Definition 3.3. Let R be a ring with identity. For any two-sided ideal $I \subseteq R$, define

$$\sqrt{I} = \{s \in R : \text{ every } m\text{−system containing } s \text{ meets } I\}.$$

Theorem 3.4. *The prime radical of R is equal to $\sqrt{\langle 0 \rangle}$.*

Let $\mathbb{Q}[\overrightarrow{X}] = \mathbb{Q}[X_0, X_1, X_2, \ldots]$ be the noncommutative polynomial ring in countably many indeterminates over the field of rational numbers $\mathbb{Q}$. Throughout the remainder of this section we will only consider rings R of the form $R = \mathbb{Q}[\overrightarrow{X}]/I$, for some two-sided ideal $I \subseteq \mathbb{Q}[\overrightarrow{X}]$. In this case we use the notation $\overline{X} \in R$ to denote the image of $X \in \mathbb{Q}[\overrightarrow{X}]$ under the canonical map $\varphi : \mathbb{Q}[\overrightarrow{X}] \to R$. By *monomial*, we mean nonconstant monomial. An element $r \in R$ is said to be a *monomial* if it is equivalent to the image of a monomial under φ.

Definition 3.5. A nonempty set $S \subseteq R$ is a *monomial m-system* if, for any $a, b \in S$, there is a monomial $r \in R$ such that $arb \in S$.

We now prove a simple proposition that allows us to construct monomial m-systems in R.

Proposition 3.6. *Let $x_0 = \overline{X_n} \in R$, for some $n \in \omega$, and for every $i > 0$, let $x_i = x_{i-1} m_{i-1} x_{i-1}$ for some monomial $m_{i-1} \in R$. Then, if $i, j \in \omega$ are given, with $i \leq j$, there exist monomials $m_0, m_1 \in R$ such that $x_{j+1} = x_i m_0 x_j$ and $x_{j+1} = x_j m_1 x_i$. It follows that the set $X = \{x_0, x_1, x_2, \ldots\} \subset R$ is a monomial m-system.*

Proof. We prove the existence of m_0. The proof of the existence of m_1 is similar.

The proof is by induction on $j = \max\{i, j\}$. If $j = 0$, then since $i \leq j$, we have that $i = j = 0$ and by definition of $x_1 = x_0 m_0 x_0$, the proposition holds. A similar argument shows that the proposition holds if $i = j$, so assume that $i < j$. Before we prove the induction step, we make the obvious observation that, by construction, for every $n \in \omega$, $x_n \in \mathbb{Q}[\overrightarrow{X}]$ is a monomial.

If $j > 0$, assume that the proposition holds for $j - 1$; we shall show that the proposition also holds for j. By the induction hypothesis and the fact that $i < j$, there is a monomial m' such that $x_j = x_i m' x_{j-1}$. Now, we have that $x_{j+1} = x_j m_j x_j$, and so $x_{j+1} = x_i(m' x_{j-1} m_j)x_j$. Hence, the desired monomial m_0 is equal to $m' x_{j-1} m_j$. This proves the induction step, and thus completes the proof of the proposition. □

Having given the necessary background, we are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Let $\mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be the polynomial ring over the field of rational numbers $\mathbb{Q}$, with indeterminates X_σ, for every $\sigma \in \omega^{<\omega}$. Let $T \subset \omega^{<\omega}$ be a computable tree containing every node of length 1, and such that the set of extendible nodes in T of length 1 is Π^1_1-complete. Such a tree $T \subset \omega^{<\omega}$ may be constructed as follows. First, put all nodes of length 1 in T. Then, if $\{T_e\}_{e\in\omega}$ is an effective listing of the computable trees in $\omega^{<\omega}$, for every $e \in \omega$ put the tree T_e above the node $\langle e \rangle$ (of length 1) into T. By the construction of T, it follows that T is a computable tree in $\omega^{<\omega}$.

We shall construct a computable ring R of the form $R = \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]/I$, for some (computable) ideal $I \subset \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ such that I is generated by a computable set of monomials. Furthermore, the prime radical of R shall be Π^1_1-complete.

Let a computable function $F : \omega^{<\omega} \to \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be defined as follows. $F(\emptyset) = 1$, and if $\sigma \in \omega^{<\omega}$ is such that $|\sigma| > 0$, then define $F(\sigma) = F(\sigma^-)X_\sigma F(\sigma^-)$, where σ^- is the unique initial segment of σ such that $|\sigma^-| = |\sigma| - 1$. Note that (by induction we have that) for every node $\rho \in \omega^{<\omega}$, $F(\rho)$ is a monomial of degree $2^{|\rho|} - 1$, unless $\rho = \emptyset$ in which case $F(\rho) = 1$. Using the function F, we now construct the computable ideal I such that $R = \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]/I$.

Let $I \subseteq \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be the ideal generated by the monomials $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ such that m does not divide any monomial of the form $F(\sigma)$, $\sigma \in T$. Note that if a monomial $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ contains an occurrence of some indeterminate X_σ, where $\sigma \notin T$, then it follows that m cannot divide any element of the form $F(\tau)$, $\tau \in T$, and thus by definition of I we have that $m \in I$. We also have the following proposition.

Proposition 3.7. *Let $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be a monomial, and let $\sigma \in \omega^{<\omega}$ be maximal such that X_σ appears in m. Then $m \notin I$ if and only if m divides $F(\sigma)$.*

Proof. If m divides $F(\sigma)$, then by definition of I it follows that $m \notin I$.

Now, suppose that $m \notin I$. Then there is some $\tau \in \omega^{<\omega}$ such that m divides $F(\tau)$. Note that we must have $\sigma \subseteq \tau$ since otherwise, by the construction of F, we know that X_σ does not appear in $F(\tau)$, and so m cannot divide $F(\tau)$. It suffices to show that if $\tau \supsetneq \sigma$, then m divides $F(\tau^-)$, where τ^- is the unique initial segment of τ of length $|\tau| - 1$. Suppose that $\tau \supsetneq \sigma$. By definition of F, we have that $F(\tau) = F(\tau^-)X_\tau F(\tau^-)$. Now, by definition of σ and the fact that $\tau \supsetneq \sigma$, we know that the indeterminate X_τ does not appear in m. Therefore, since m divides $F(\tau) = F(\tau^-)X_\tau F(\tau^-)$, it follows that m must also divide $F(\tau^-)$. □

Corollary 3.8. *The ideal $I \subset R$ is computable.*

Proof. Since the ideal I is generated by monomials, it follows that a polynomial $p \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ is in the ideal I if and only if every monomial summand m of p is in I. Proposition 3.7 gives a method for deciding whether or not a given monomial is in I, and so it also gives a method for deciding whether or not $p \in I$. □

The following corollary is a consequence of the proof of Proposition 3.7.

Corollary 3.9. *If $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ is a monomial such that $m \notin I$, and if $\sigma \in \omega^{<\omega}$ is maximal such that X_σ appears in m, then X_σ is unique. In other words, if $\sigma, \tau \in \omega^{<\omega}$ and X_σ and X_τ appear in m, then σ and τ are comparable.*

Now that we have constructed the computable ring $R = \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]/I$, it remains to show that $\sqrt{\langle 0 \rangle} \subseteq R$ is Π^1_1-complete. With this in mind, we prove the following proposition.

Proposition 3.10. *For every $\sigma \in \omega^{<\omega}$, $X_\sigma \notin \sqrt{\langle 0 \rangle} \subseteq R$ if and only if there is an infinite path through T extending $\sigma \in T$.*

Proof. First, we claim that if $\sigma \in \omega^{<\omega}$ is an extendible node of T, then there is a monomial m-system containing X_σ but not containing 0. The proof is as follows. Let $f \in \omega^\omega$ be an infinite path through T extending σ. Then, by Proposition 3.6, and the constructions of F and I, it follows that the image of F restricted to f (in the quotient R) is a monomial m-system containing X_σ but not containing 0.

Now, let $\sigma \in T \subset \omega^{<\omega}$, and suppose that there is an m-system S in R containing X_σ, but not containing 0. In this case we claim that there is an infinite path in T extending σ. To construct such a path, first set $y_0 = X_\sigma \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$, and for every number $n > 0$, let $y_n \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$, $y_n \notin I$, be of the form $y_n = y_{n-1} r_{n-1} y_{n-1}$, for some $r_{n-1} \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$.

To prove that there is an infinite path in T extending $\sigma \in T$, we first prove the following lemma which says that there is an infinite, finitely branching tree $T_0 \subseteq T$ above σ. Then we apply König's Lemma to the tree $T_0 \subseteq \omega^{<\omega}$ to get an infinite path in $T_0 \subseteq T$ extending σ.

Lemma 3.11. *There is an infinite, finitely branching tree $T_0 \subseteq T$ such that for every $\tau \subseteq \sigma$, $\tau \in T_0$, and for all $\tau \in T_0$, if $\tau \nsubseteq \sigma$, then $\tau \supset \sigma$.*

Proof. We begin by giving several definitions and constructions which shall aid us in the proof of Lemma 3.11. Let $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be a monomial and $p \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ be a polynomial.

Definition 3.12. We say that m is an *essential monomial summand* of p if m is a summand of p such that $m \notin I$ (i.e. $\overline{m} \neq 0 \in R$).

For every $n \in \omega$, define

$$Y_n = \{\tau \in \omega^{<\omega} : X_\tau \text{ appears in an essential monomial summand of } y_n\}.$$

Now, define $T_0 \subseteq \omega^{<\omega}$ to be the downward closure of the set

$$\{\rho \in \omega^{<\omega} : (\exists n \in \omega)[\rho \in Y_n]\},$$

and for every $s \in \omega$, let T_0^s be the downward closure of the set

$$\{\rho \in \omega^{<\omega} : (\exists n \leq s)[\rho \in Y_n]\}.$$

By definition, it follows that $T_0 = \cup_{s\in\omega} T_0^s$ and T_0 is a tree. Also, since for every $n \in \omega$, the set of $\sigma \in \omega^{<\omega}$ such that X_σ appears in $y_n \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ is finite, it follows that for every $s \in \omega$, T_0^s is a finite (and hence finitely branching) tree. Moreover, recall that if $\tau \notin T$ then (by definition of I) it follows that $X_\tau \in I$. Therefore, if $\tau \in \omega^{<\omega}$ is such that $\tau \notin T$ and X_τ appears in some monomial summand m of y_n for some $n \in \omega$, then m is not an essential monomial summand of y_n. Hence, by definition of Y_n, $n \in \omega$, and T_0, it follows that T_0 is a subtree of T. It remains to be shown that every initial segment of σ is in T_0, every node $\tau \in T_0$ is comparable to σ, and that T_0 is an infinite, finitely branching tree.

By assumption, we know that $\sigma \in T$. It follows that $y_0 = X_\sigma \notin I$, and thus X_σ is an essential summand of y_0. Therefore, by the construction of T_0, it follows that every initial segment of σ belongs to T_0. Furthermore, by induction on $n \in \omega$, it follows that for every $n \in \omega$ and every monomial summand m of y_n, X_σ appears in m. Now, by Corollary 3.9 and the definition of T_0, it follows that if $\tau \in T_0$ then τ is comparable to σ.

We now show that T_0 is infinite by showing that T_0 contains nodes of arbitrarily large length. First note that (by induction on $n \in \omega$ it follows that) for all $n \in \omega$, every monomial summand of y_n has degree at least 2^n. Furthermore, by definition of I, it follows that if $m \in \mathbb{Q}[\overrightarrow{X_{\omega^{<\omega}}}]$ is a monomial of degree 2^n, then m cannot divide $F(\rho)$ for any $\rho \in T$ of length less than n (since in this case $F(\rho)$ has degree $2^{|\rho|} - 1 < 2^{|\rho|}$). Hence, by definition of I, if m is an essential summand of y_n, then m must divide some $F(\rho)$, where $|\rho| \geq n$. Now, by Proposition 3.7, it follows that if m is an essential summand of y_n, then m contains an occurrence of some indeterminate X_ρ, $\rho \in T$, $|\rho| \geq n$. We have now shown that *every* essential monomial summand m of y_n contains an occurrence of some indeterminate X_ρ, where $\rho \in T$ and $|\rho| \geq n$. By assumption, we have that $y_n \notin I$, for every $n \in \omega$. Hence, for every $n \in \omega$ there exists an essential monomial summand m of y_n. Therefore, by definition of T_0, it follows that T_0 contains nodes of arbitrarily large length. Next, we complete the proof of Lemma 3.11 by showing that T_0 is a finitely branching tree.

To show that T_0 is finitely branching, fix a node $\tau \in T_0$, and let $n \in \omega$ be large enough so that every essential monomial summand m of y_n contains an occurrence of an indeterminate of the form X_ρ, for some node $\rho \in \omega^{<\omega}$ such that $|\rho| > |\tau|$ (the previous paragraph explains why such an n exists). We claim that the sets

$$S_0 = \{\rho \in T_0 : |\rho| = |\tau| + 1 \text{ and } \rho \supset \tau\}$$

and

$$S_1 = \{\rho \in T_0^n : |\rho| = |\tau| + 1 \text{ and } \rho \supset \tau\}$$

are equal. Since $T_0 = \cup_{s\in\omega} T_0^s$, it follows that $S_0 \supseteq S_1$. We need to show that $S_0 \subseteq S_1$. Suppose, for a contradiction, that there exists a node $\rho \in S_0 \setminus S_1$. Then, by definition of S_0, S_1, T_0, T_0^n, it follows that there exists a number $m > n$ and a node $\rho_0 \supseteq \rho$ such that X_{ρ_0} appears in an essential monomial summand of y_m. However, by definition of n, and the fact that $m > n$, it follows that every monomial summand of y_m contains an occurrence of some indeterminate X_λ, where $|\lambda| > |\tau|$ and $\lambda\restriction(|\tau| + 1) \in S_1$. Since $\rho \notin S_1$, it follows that λ and ρ_0 are incomparable nodes for all such λ. Now, by Corollary 3.9 it follows that no monomial summand of y_m in which X_{ρ_0} appears is an essential monomial summand of y_m, a contradiction. Thus, we have shown that $S_0 = S_1$, and therefore T_0 is finitely branching. □

Applying König's Lemma to the infinite, finitely branching tree $T_0 \subseteq T \subset \omega^{<\omega}$ yields an infinite path $f \in \omega^\omega$ through T extending $\sigma \in T$. This completes the proof of Proposition 3.10. □

We now construct a computable function $h : \omega \to R$ by setting, for every number $n \in \omega$ corresponding to the node $\langle n\rangle \in \omega^{<\omega}$, $h(n) = \overline{X_{\langle n\rangle}} \in R$. Proposition 3.10 says that $n \in \mathrm{WF}$ if and only if $h(n) \in \sqrt{\langle 0\rangle} \subset R$. Therefore, $\sqrt{\langle 0\rangle} \subset R$ is Π^1_1-complete. This completes the proof of Theorem 1.7. □

4. Levitzki radical

In this section we prove the following theorem.

Theorem 1.8. *There exists a noncommutative computable ring R such that the Levitzki radical of R is Π^0_2-complete.*

First, however, we require two definitions and a proposition. Let R be a ring.

Definition 4.1. A set $S \subset R$ is *locally nilpotent* if, for any finite subset $S_0 = \{s_0, s_1, \dots, s_n\} \subseteq S$, there is a number $N = N(S_0) \in \omega$ such that any product of N elements from $\{s_0, s_1, \dots, s_n\}$ is zero. This is equivalent to Definition 1.5.

The following proposition is standard. We omit its proof.

Proposition 4.2. *Let I, J be locally nilpotent one-sided ideals in R. Then RIR, and $I + J$ are locally nilpotent.*

We now define the Levitzki radical of a ring R.

Definition 4.3. The *Levitzki radical* of R, $L \subset R$, is the largest locally nilpotent ideal in R. By Proposition 4.2, we have that

$$L = \{x \in R : xR \text{ is locally nilpotent}\},$$

and that $L \subset R$ is an ideal.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. We have already remarked that, for all $x \in R$, $x \in L$ if and only if xR is locally nilpotent. In other words, $x \in L$ if and only if

$$(\forall\langle x_0, x_1, \dots, x_n\rangle \in xR)(\exists N)(\forall\sigma \in n^N)[\prod_{i<|\sigma|} x_{\sigma(i)} = 0].$$

Hence, it follows that if R is a computable ring, then $L \in \Pi^0_2$ (the last quantifier in the expression above is bounded[1]). We now show that this (upper) bound on the complexity of the Levitzki radical is sharp by constructing a computable ring R whose Levitzki radical is Π^0_2-complete. The rest of this section is dedicated to the construction of R.

R shall be a quotient of the form $\mathbb{Q}[\overrightarrow{X}]/I$, where $\mathbb{Q}[\overrightarrow{X}] = \mathbb{Q}[X_0, X_1, X_2, \dots]$ is the noncommutative polynomial ring in countably many variables over the rational numbers $\mathbb{Q}$, and $I \subset \mathbb{Q}[\overrightarrow{X}]$ is a (two-sided) ideal. We construct $I = \cup_{s\in\omega} I_s$ in stages such that for all $s \in \omega$, $I_s \subseteq I_{s+1}$.

At stage 0 define I_0 to be the computable ideal in $\mathbb{Q}[\overrightarrow{X}]$ generated by the monomials $m \in \mathbb{Q}[\overrightarrow{X}]$ such that there are numbers $e_0, e_1, i, j \in \omega$ with $e_0 \neq e_1$, and indeterminates $X_{\langle e_0,i\rangle}, X_{\langle e_1,j\rangle}$ both occurring in m.

[1]Note that if $p(y, x)$ is a computable formula with free variables $y, x \in \omega$, then for every $n \in \omega$ the formula $q(x) = (\forall y < n)p(y, x)$ is also computable. It follows that the bounded quantifier above does not contribute to the arithmetical complexity of the formula which defines the Levitzki radical.

At stage $s+1$, we are given the computable ideal I_s, and add to it all monomials m of degree greater than $s+1$, such that the only indeterminates appearing in m are in the set $\{X_{\langle e,0\rangle}, X_{\langle e,1\rangle}, \ldots, X_{\langle e,s\rangle}\}$, where $e \in \omega$ is such that $W_{e,s+1} \neq W_{e,s}$ (without loss of generality we assume that at every stage s there exists a unique $e \in \omega$ such that $W_{e,s+1} \neq W_{e,s}$).

We now verify that $I = \cup_{s\in\omega} I_s$ is computable. To see why this is the case, first note that I is generated by monomials. Thus, it suffices to show that the set of monomials that generate I, $M \subset I$, is a computable set. To see why this is the case, note that, by the construction of $I = \cup_{s\in\omega} I_s$, we have that for any monomial $m \in \mathbb{Q}[\overrightarrow{X}]$ of degree d, $m \in I$ if and only if $m \in I_d$. For every $X \in \mathbb{Q}[\overrightarrow{X}]$, let $\overline{X}$ denote the image of X under the canonical quotient map $\varphi : \mathbb{Q}[\overrightarrow{X}] \to R$.

Recall that the set Inf=$\{e \in \omega : W_e$ is infinite$\}$ is Π^0_2-complete. Therefore, to show that $L \subset R$ is Π^0_2-complete, it suffices to exhibit a computable function $h : \omega \to \mathbb{Q}[\overrightarrow{X}]$ such that for every $e \in \omega$, $e \in$ Inf if and only if $\overline{h(e)} \in L$. We claim that the computable map $h : \omega \to R$ such that $h(e) = \overline{X_{\langle e,0\rangle}}$ satisfies this condition.

To verify that the function h above has the desired property, we shall prove that for every $e \in \omega$, $\overline{X_{\langle e,0\rangle}} \in L$ if and only if W_e is infinite. It suffices to prove the following proposition.

Proposition 4.4. *For every $e \in \omega$, the right ideal $\overline{X_{\langle e,0\rangle}} \cdot R$, is locally nilpotent if and only if W_e is infinite.*

Proof. First, suppose that W_e is finite. Then there is a stage $s_e \in \omega$ such that for all $s \geq s_e$ we have $W_{e,s+1} = W_e$. Fix a number $n \in \omega$. By the construction of $R = \mathbb{Q}[\overrightarrow{X}]/I$, we have that $X = (X_{\langle e,0\rangle} \cdot X_{\langle e,s_e+1\rangle})^n \notin I_{s_e}$, since, by the construction of I_{s_e}, we have that $X_{\langle e,0\rangle} \notin I_{s_e}$, and no (monomial) generator of I_{s_e} contains an appearance of the indeterminate $X_{\langle e,s_e+1\rangle}$ (and $X_{\langle e,s_e+1\rangle}$ appears in X). Furthermore, since at all stages $t \geq s_e$ we do not enumerate any new elements into W_e, then by the construction of $I = \cup_{s\in\mathbb{N}} I_s$, it follows that we do not enumerate X into I at any stage $t \geq s_e$. Therefore, $X \notin I$, and so $\overline{X} \neq 0 \in R$. It follows that $\overline{X_{\langle e,0\rangle}} \cdot R$ is not locally nilpotent, and hence $\overline{X_{\langle e,0\rangle}} \notin L$.

Now suppose that W_e is infinite. Let $\overline{m_0}, \overline{m_1}, \ldots, \overline{m_n} \in R$ be nonzero, and let $M \in \omega$ be large so that, for all $0 \leq i \leq n$ and $X_{\langle e,j\rangle}$ occurring in m_i, we have that $M > \max\{e, j\}$. We shall show that there exists a number $N \in \omega$ such that

$$\prod_{k=0}^{N} (\overline{X_{\langle e,0\rangle}}) \cdot \overline{m_{i_k}} = 0, \tag{2}$$

where $i_k \in \{0, 1, \ldots, n\}$ for all $0 \leq k \leq N$. Without loss of generality, we can assume that for all $0 \leq i \leq n$, we have that $m_i \in \mathbb{Q}[\overrightarrow{X}]$ is a monomial. Therefore, it follows that the product in (2) above is a monomial of degree greater than or equal to N. Now, by the construction of R, if we let $N = s$, where $s \in \omega$ is the least stage greater than M such that $W_{e,s+1} \neq W_{e,s}$ (note that s exists, since W_e is infinite), then we have that (2) holds, as required. Hence, $\overline{X_{\langle e,0\rangle}} \cdot R$ is locally nilpotent. □

This completes the proof of Theorem 1.8. □

References

1. M.F. Atiyah and I.G. MacDonald, *Introduction to commutative algebra*, Perseus, 1969.
2. C.J. Conidis, *Chain conditions in computable rings*, to appear in the Transactions of the American Mathematical Society.
3. R.G. Downey, S. Lempp, and J.R. Mileti, *Ideals in computable rings*, J. Algebra **314** (2007), 872–887.
4. D.S. Dummit and R.M. Foote, *Abstract algebra*, John Wiley & Sons, 1999.
5. H.M. Friedman, S.G. Simpson, and R.L. Smith, *Countable algebra and set existence axioms*, Ann. Pure and Appl. Logic **25** (1983), 141–181.
6. ———, *Addendum to: "countable algebra and set existence axioms"*, Ann. Pure and Appl. Logic **28** (1985), 319–320.
7. A. Frölich and J.C. Shepherdson, *Effective procedures in field theory*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **248** (1956), 407–432.
8. L. Kronecker, *Grundzüge einer arithmetischen theorie der algebraischen groössen*, J. Reine Angew. Math. **92** (1882), 1–122.
9. T.Y. Lam, *A first course in noncommutative rings, second edition*, Springer-Verlag, 2001.
10. S. Lang, *Algebra*, Springer-Verlag, 1993.
11. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 2004.
12. H. Rogers, *Theory of recursive functions and effective computability*, MIT Press, 1987.
13. R.I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, 1987.
14. B.L. van der Waerden, *Algebra, vol. 1*, Springer-Verlag, 2003.

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA