A REAL OF STRICTLY POSITIVE EFFECTIVE PACKING DIMENSION THAT DOES NOT COMPUTE A REAL OF EFFECTIVE PACKING DIMENSION ONE

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Abstract. Recently, the Dimension Problem for effective Hausdorff dimension was solved by J. Miller in [Mil], where the author constructs a Turing degree of non-integral Hausdorff dimension. In this article we settle the Dimension Problem for effective packing dimension by constructing a real of strictly positive effective packing dimension that does not compute a real of effective packing dimension one (on the other hand, it is known via [FHP+06, BDS09, DH] that every real of strictly positive effective Hausdorff dimension computes reals whose effective packing dimensions are arbitrarily close to, but not necessarily equal to, one).

1. Introduction and background

1.1. **Our main theorem.** The general subject of this article is that of fractal dimension, a measure-theoretic notion that has its roots in the work of Borel, Lebesgue, Carathéodory, and Hausdorff. In 1919 Hausdorff [Hau19] introduced the notion of Hausdorff dimension, which gives a way of assigning a (nonnegative real-valued) dimension to arbitrary subsets of a given metric space. Quite recently mathematicians also developed the notion of packing dimension [Sul84, Tri82], which is dual to Hausdorff dimension in the following sense. Whereas Hausdorff dimension is computed by considering open covers (from the *exterior*) of a fixed set, packing dimension is computed by considering packings (from the *interior*) of a fixed set. For more information on the classical theory of fractal dimension, including the precise definitions of open covers and packings, consult [Fal]. More recently, the notions of Hausdorff and packing dimension were effectivized by Lutz and others [AHLM07, Lut03], who created computability-theoretic analogs of Hausdorff dimension and packing dimension, called *effective Hausdorff dimension* and *effective packing dimension* (defined below), respectively. Futhermore, Lutz and others [AHLM07, Lut03, May02] were able to relate these new notions to standard computability-theoretic concepts such as Kolmogorov complexity (see Definition 2.7 and Definition 2.8 below). Generally speaking, the relationship between effective fractal dimension and Kolmogorov complexity can be used to characterize "partial randomness."

One of the most interesting and significant computability-theoretic results of the last several years is the solution of the Dimension Problem given by J. Miller in [Mil]. Generally speaking, Miller's result says that one cannot always extract randomness from partial randomness. In particular, Miller constructs a real $f \in 2^\omega$ such that the effective Hausdorff dimension of f is strictly greater than zero, and such that there exists $\alpha \in \mathbb{Q}$, $0 < \alpha < 1$, such that for every $g \in 2^\omega$ that is computable relative to f, the effective Hausdorff dimension of g is at most α. For more information on Cantor space 2^ω and reals $f \in 2^\omega$, see the next section. The definition of effective Hausdorff dimension is given in Definition 2.8 below. Our main theorem (i.e. Theorem 6.1 stated below) is the analog of Miller's solution to the Dimension Problem in the context of effective packing dimension, once the results of

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[FHP⁺06, BDS09, DH] (stated in the following paragraph) have been taken into consideration.

Effective packing dimension (see Definition 2.7 below) is dual to effective Haudorff dimension, and it was first shown by Fortnow, Hitchcock, Pavan, Vinochandran, and Wang in [FHP⁺06] that every real of strictly positive effective packing dimension computes reals whose effective packing dimensions are less than, but arbitrarily close to, one. Later on, different proofs of this fact were discovered by Bienvenu, Doty, and Stephan [BDS09][1] and Bienvenu [DH, Section 12.11]. The original proof of this fact (given in [FHP⁺06]) uses the multisource randomness extractors of Barak, Impagliazzo, and Wigderson [BIW06], which were constructed via a recent result in additive number theory of Bourgain, Katz, and Tao [BKT04] (i.e. an Erdös-Semerédi Theorem for finite fields) which, generally speaking, says that a subset of integers in a finite prime field cannot simultaneously be close to an arithmetic progression and geometric progression.

The main goal of this article is to examine the problem of computing reals of effective packing dimension one from reals of strictly positive effective packing dimension. Intuitively speaking, a real $f \in 2^\omega$ has effective packing dimension at least $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$, if and only if f has infinitely many initial segments whose information density is close to α. One can think of such initial segments as "information packets" of "information density" $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$. The main theorem of this article is similar to Miller's solution to the Dimension Problem, and says that in general Turing reductions cannot always be used to obtain a real of effective packing dimension one from a real of strictly positive effective packing dimension. In other words, a real that possesses infinitely many information packets of strictly positive information density cannot always Turing compute a real with infinitely many information packets of density close to one (though [FHP⁺06], [BDS09], and [DH, Section 12.11] say that Turing redutions can come arbitrarily close to achieving this goal). More precisely, our main theorem is as follows.

Theorem 6.1. *There exists $X \in 2^\omega$ of effective packing dimension at least $\frac{1}{4} > 0$ and such that for every $e \in \omega$ the effective packing dimension of Φ_e^X is strictly less than one whenever $\Phi_e^X \in 2^\omega$ is a total Turing reduction relative to X.*

This entire article is devoted to proving Theorem 6.1.

1.2. **Turing degrees that possess a real of effective packing dimension one.** A major open question in the theory of effective fractal dimension is to classify the Turing degrees that possess a real of effective packing dimension one. In [DG08], Downey and Greenberg show that the class of *computably enumerable* Turing degrees (i.e. the Turing degrees that contain a computably enumerable set) that contain a real of effective packing dimension one coincides with the class of computably enumerable Turing degrees that contain a real of strictly positive effective packing dimension. The authors also showed that this class of Turing degrees is equal to the class of computably enumerable Turing degrees that are *array noncomputable*, as well as the class of computably enumerable Turing degrees that are *c.e. traceable* (for more information on array noncomputability and c.e. traceability, consult [DG08, Nie, DH]).

More recently, however, Downey and Ng [DN] examined the class of Turing degrees that contain a member of effective packing dimension one in general (i.e. not necessarily computably enumerable Turing degrees) and showed that this class does not coincide with the class of c.e. traceable Turing degrees, nor does it coincide with the class of array noncomputable Turing degrees. However, the question of whether or not the class of Turing degrees that contain a member of effective packing dimension one coincides with the class of Turing degrees that contain a member of strictly positive effective packing dimension remained open.

[1]An error was discovered in the construction of the effective packing dimension extractors given in the article [BDS09] that has since been corrected by the authors and posted on the arXiv.

On the other hand, as we mentioned in the previous paragraph, it is known (via [DG08]) that these classes coincide when restricted to the computably enumerable Turing degrees. Our main theorem says that in general the class of Turing degrees of effective packing dimension one is *not* equal to the class of Turing degrees that contain a member of strictly positive effective packing dimension.

2. Preliminaries and notation

2.1. **Basic computability theory.** We now review our notation, and introduce some of the basic concepts from computability theory and effective randomness that play a central role in this article. Most of our basic notation is taken from [Soa, Nie]. We will assume that the reader is familiar with the basic concepts and notation from computability theory such as: convergent and divergent computations (denoted by $\downarrow$ and $\uparrow$, respectively), computable function/set, computably enumerable (c.e.) set, oracle Turing machine, and the *use* of an oracle Turing computation on a given input $x \in \{0, 1, 2, \ldots\}$. For more information on the basics of computability theory, consult [Soa, Nie, DH]. We define the use of a divergent computation to be ∞, where ∞ is a number that satisfies $n < \infty$ for all $n = 0, 1, 2, \ldots, \infty$.

First, we set $\omega = \{0, 1, 2, \ldots\}$; thus ω denotes the set of natural numbers. Let $\{\widehat{\Phi_e}\}_{e\in\omega}$ be an effective listing of all the oracle Turing machines. For any given $e \in \omega$ and $A, B \subseteq \omega$, we write $\widehat{\Phi_e^B} = A$ to mean that A is computable via the oracle Turing machine $\widehat{\Phi_e}$ relative to the oracle B; i.e. for every $x \in \omega$ we have that $\widehat{\Phi_e^B}(x)\downarrow = A(x)$ (recall that we are actually computing the characteristic function of $A \subseteq \omega$ from the characteristic function of $B \subseteq \omega$). We say that A is *Turing reducible* to B if there exists $e \in \omega$ such that $\widehat{\Phi_e^B} = A$.

Remark 2.1. *We leave it to the reader to verify that there is an effective listing of oracle Turing machines $\{\Phi_e\}_{e\in\omega}$ such that for all $A, B \subseteq \omega$ and $e \in \omega$ the following conditions are satisfied:*

1. $(\forall x \in \omega)[\Phi_e^B(x)\downarrow \Rightarrow \Phi_e^B(x) \in \{0, 1\}]$.[2]
2. *For all $x \in \omega$, if $\Phi_e^B(x)\downarrow$ and $\Phi_e^B(x+1)\downarrow$, then the use of the latter computation is strictly larger than the use of the former computation.*

Throughout the rest of this article we will work exclusively with the Turing reductions $\{\Phi_e\}_{e\in\omega}$ of Remark 2.1 above.

Note that since we have defined the use of a divergent computation to be ∞ (above), then condition (2) of Remark 2.1 implies that for all $B \subseteq \omega$, $e, x \in \omega$, if we have that $\Phi_e^B(x)\uparrow$, then for all $y > x$, $y \in \omega$, we also have that $\Phi_e^B(y)\uparrow$. It follows from condition (2) above that the use of the computation of $\Phi_e^B(x)$, for any $B \in 2^\omega$ and $e, x \in \omega$, is always greater than or equal to x.

Finally, recall that there is a computable one-to-one pairing function

$$\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$$

such that the associated projection functions are also computable (see [Soa] for more details). For any given pair of natural numbers, $\langle x, y\rangle$, we write $\pi_1(\langle x, y\rangle) = x$ to denote projection onto the first coordinate, and similarly $\pi_2(\langle x, y\rangle) = y$ denotes projection onto the second coordinate. Moreover, if $A \subseteq \omega \times \omega$ is a set of pairs, then we define

$$\pi_1(A) = \{x \in \omega : (\exists y)[\langle x, y\rangle \in A]\} \text{ and } \pi_2(A) = \{y \in \omega : (\exists x)[\langle x, y\rangle \in A]\}.$$

[2] Since our main theorem deals entirely with effective packing dimension, which is only defined on elements of Cantor space (see the following subsections for more details), it suffices to consider only those Turing reductions whose output lives in $\{0, 1\} \subseteq \omega$, for every input $x \in \omega$.

2.2. **Cantor space and $2^{<\omega}$.** We let $2^{<\omega}$ denote the set of all finite binary sequences (including the empty sequence, which we denote by $\emptyset \in 2^{<\omega}$), and we also let 2^ω denote the set of all infinite binary sequences. Since we can effectively code finite binary strings as natural numbers (and vice versa), we will often simultaneously think of elements of $2^{<\omega}$ as both finite binary strings and natural numbers. For all $\sigma, \tau \in 2^{<\omega}$ and $f \in 2^\omega$, we write $\sigma \subseteq \tau$ to mean that σ is an initial segment of τ and $\sigma \subset \tau$ to mean that σ is a *proper* initial segment of τ. Similarly, we write $\sigma \subset f$ to mean that $\sigma \in 2^{<\omega}$ is an initial segment of $f \in 2^\omega$. There is a topology on 2^ω that is generated by basic clopen sets of the form

$$U_\sigma = \{f \in 2^\omega : \sigma \subset f\},$$

for every $\sigma \in 2^{<\omega}$. The resulting topological space is referred to as *Cantor space*. We often refer to elements of Cantor space $f \in 2^\omega$ as *reals*. For all $f \in 2^\omega$, we will write $f(n) \in \{0,1\}$, $n \in \omega$, to denote the n^{th} bit of f. Furthermore, we will write $\langle 0 \rangle \in 2^{<\omega}$ to refer to the finite binary string consisting of exactly one zero (and no ones), and we will write $0^n \in 2^{<\omega}$, $n \in \omega$, $n \geq 2$, to refer to the finite binary string consisting of exactly n zeros (and no ones); i.e. $0^n = \underbrace{000\cdots 0}_{n} \in 2^{<\omega}$. Henceforth the variable X will always refer to the real $X \in 2^\omega$ of the main theorem (i.e. Theorem 6.1).

Let $\sigma \in 2^{<\omega}$. We write $|\sigma| \in \omega$ to mean the length of sigma – i.e. the number of bits of σ. For every $n \in \omega$, we write $2^{\leq n} \subseteq 2^{<\omega}$ to denote the set of finite binary strings of length at most n (including the empty string $\emptyset \in 2^{<\omega}$ of length 0), $2^{=n} \subseteq 2^{<\omega}$ to denote the set of finite binary strings of length equal to n, and $2^{\geq n} \subseteq 2^{<\omega}$ to denote the set of finite binary strings of length at least n. For every $\sigma \in 2^{<\omega}$ such that $|\sigma| \geq 1$ we write $\sigma^- \in 2^{<\omega}$ to denote the unique initial segment of σ of length $|\sigma| - 1$. This is equivalent to saying that $\sigma^- \in 2^{<\omega}$, $\sigma^- \subset \sigma$, is the finite binary string obtained by deleting the last bit of σ. Also, for all $D \subseteq 2^{<\omega}$ or $D = 2^\omega$, we write $(D)^n$, $n \in \omega$, to denote the n-fold direct product of D with itself, i.e.

$$(D)^n = \underbrace{D \times D \times \cdots \times D}_{n}.$$

If Φ_e, $e \in \omega$, is an oracle Turing machine constructed in Remark 2.1 above, then for all $\sigma \in 2^{<\omega}$ we define $\Phi_e^\sigma \in 2^{<\omega}$ to be the unique finite binary string

$$\Phi_e^\sigma = \langle x_0, x_1, \ldots, x_k \rangle \in 2^{<\omega},$$

where $x_i \in \{0,1\}$, $0 \leq i \leq k$, is the value of the computation $\Phi_e^\sigma(i)\downarrow$, and $k \in \omega$ is the largest number for which the use of the computation $\Phi_e^\sigma(k)\downarrow$ is at most $|\sigma|$. Note that, for a given $e \in \omega$, $\Phi_e^\sigma \in 2^{<\omega}$ is not necessarily uniformly computable in $\sigma \in 2^{<\omega}$. More generally though, for all $D \subseteq 2^{<\omega}$ and $e \in \omega$ we define

$$\Phi_e^D = \{\Phi_e^\sigma \in 2^{<\omega} : \sigma \in D\} \subseteq 2^{<\omega}.$$

Note that, by condition (2) in Remark 2.1 above, it follows that for all $\sigma \in 2^{<\omega}$ and $e \in \omega$, we have that $|\Phi_e^\sigma| \leq |\sigma|$.

For given finite binary strings $\sigma, \tau \in 2^{<\omega}$, let $\sigma\tau$ denote the concatenation of σ and τ (i.e. the string σ followed by the string τ to the right of σ). More generally, for any given sets $D, E \subseteq 2^{<\omega}$, we define the product

$$DE = \{\rho \in 2^{<\omega} : (\exists \sigma \in D, \tau \in E)[\rho = \sigma\tau]\},$$

and for any given $\sigma \in 2^{<\omega}$ and $D \subseteq 2^{<\omega}$, we define $\sigma D \subseteq 2^{<\omega}$ and $D\sigma \subseteq 2^{<\omega}$ to be the same as $\{\sigma\}D \subseteq 2^{<\omega}$ and $D\{\sigma\} \subseteq 2^{<\omega}$, respectively. One can check that this multiplication operation (i.e. concatenation) is associative on individual strings, as well as subsets of $2^{<\omega}$, and so we will never include parentheses when multiplying/concatenating several strings or subsets of $2^{<\omega}$.

We call a subset of strings of the form $\sigma 2^{\leq|\sigma|} \subseteq 2^{<\omega}$, for some $\sigma \in 2^{<\omega}$, a *clump*. For any number $a \in \omega$, we call a set of strings of the form $\sigma 2^{\leq a|\sigma|} \subseteq 2^{<\omega}$ a *generalized clump*.

Clumps, generalized clumps, and other sets of strings called *pruned clumps* (to be defined later on) will play a prominent role in the following sections. Note that a clump is just a generalized clump with $a = 1 \in \omega$, and for a given $\sigma \in 2^{<\omega}$, $a \in \omega$, the generalized clump $\sigma 2^{\leq a|\sigma|} \subseteq 2^{<\omega}$ consists of the extensions of σ (including σ itself) that have length at most $(a+1)|\sigma| \in \omega$.

For any given $f \in 2^\omega$ and $n \in \omega$, we let $f\restriction n = f(0)f(1)\cdots f(n-1) \in 2^{<\omega}$ denote the first n bits of f ($f\restriction 0 = \emptyset \in 2^{<\omega}$, for all $f \in 2^\omega$), and we also define $\hat{f} \subseteq 2^{<\omega}$ via

$$\hat{f} = \{f\restriction 1, f\restriction 2, f\restriction 3, \ldots, f\restriction k, \ldots\} \subseteq 2^{<\omega}.$$

We say that $T \subseteq 2^{<\omega}$ is a *tree* if for every $\sigma \in T$ and $\tau \in 2^{<\omega}$, $\tau \subseteq \sigma$, we have that $\tau \in T$ (i.e. the set $T \subseteq 2^{<\omega}$ is downwards closed with respect to $\subseteq$). We say that $T_0 \subseteq T \subseteq 2^{<\omega}$ is a *subtree* of the tree T whenever T_0 is also a tree. If $T \subseteq 2^{<\omega}$ is a tree, then we define $[T] \subseteq 2^\omega$ via

$$[T] = \{f \in 2^\omega : (\forall n \in \omega)[f\restriction n \in T]\} \subseteq 2^\omega.$$

We say that $\sigma \in T$ is *extendible* whenever there is an infinite path $f \in [T] \subseteq 2^\omega$ such that $\sigma \subset f$. Let $T \subseteq 2^{<\omega}$ be a tree. Then for all $\sigma \in 2^{<\omega}$ and $A \subseteq 2^{<\omega}$, we say that σ is *on* T and A is *on* T to mean that $\sigma \in T$ and $A \subseteq T$, respectively. A set of strings $A \subseteq T \subseteq 2^{<\omega}$ is *dense in* the given tree $T \subseteq 2^{<\omega}$ whenever every $\sigma \in T$ has an extension $\tau \in T$, $\tau \supseteq \sigma$, such that $\tau \in A$. Let $\sigma \in 2^{<\omega}$ and $A \subseteq 2^{<\omega}$. By *downward closure* of σ, we refer to the set of (nonproper) initial segments of σ. Similarly, the *downward closure* of $A \subseteq 2^{<\omega}$ is the union of the downward closures of all $\tau \in A$. In the next section we will define the term "clumpy tree." Clumpy trees are essentially trees that are built via clumps (defined in the second last paragraph above). Let $D \subseteq 2^{<\omega}$, and $\sigma \in 2^{<\omega}$. We say that σ is a *leaf* of D if $\sigma \in D$, and for all $\tau \supset \sigma$, $\tau \in 2^{<\omega}$, we have that $\tau \notin D$. In other words $\sigma \in 2^{<\omega}$ is a leaf of $D \subseteq 2^{<\omega}$ if there is no proper extension of σ in D. Note that every finite nonempty set $D \subseteq 2^{<\omega}$ has at least one leaf. On the other hand, we say that $\sigma \in 2^{<\omega}$ is the *root* of $D \subseteq 2^{<\omega}$ if for all $\tau \in D$ we have that $\sigma \subseteq \tau$. In other words, σ is the (unique) element of D such that every other element of D extends σ. Note that not every set $D \subseteq 2^{<\omega}$ has a root, but it follows by the definition that if the root of D exists then it is unique. Note that for any given $\sigma \in 2^{<\omega}$ and $a \in \omega$, we have that σ is the root of the generalized clump $\sigma 2^{\leq a|\sigma|} \subseteq 2^{<\omega}$.

We conclude this subsection by observing that, by Remark 2.1 above, and our definition of $\Phi_e^\sigma \in 2^{<\omega}$, $\sigma \in 2^{<\omega}$, $e \in \omega$, it follows that if $T \subseteq 2^{<\omega}$ is a tree then so is $\Phi_e^T \subseteq 2^{<\omega}$. Furthermore, for a fixed $e \in \omega$, if T is a computable tree for which $\Phi_e^\sigma \in 2^{<\omega}$ is uniformly computable in $\sigma \in T$, then the tree $\Phi_e^T \subseteq 2^{<\omega}$ is computable. We will use this fact later on in this article.

2.3. **Kolmogorov complexity and effective fractal dimension.** We say that a set $D \subseteq 2^{<\omega}$ is *prefix-free* if for all $\sigma, \tau \in D$ such that $\sigma \neq \tau$ we have that $\sigma \not\subseteq \tau$. In other words $D \subseteq 2^{<\omega}$ is prefix-free if every pair of distinct elements of D are incomparable with respect to $\subseteq$. We say that a Turing machine $M : 2^{<\omega} \to 2^{<\omega}$ is *prefix-free* if the domain of M is prefix-free. It is well-known that there exists a *universal prefix-free* Turing machine, M – i.e. there exists a Turing machine M that is prefix-free and can uniformly and effectively emulate any other prefix-free Turing machine M' (for more information, consult [Nie, Section 2.2]). It is also well-known that there exists a universal Turing machine M_0 – i.e. there exists a Turing machine M_0 that can uniformly and effectively emulate every other Turing machine M_0'. We fix M, M_0 for the rest of this subsection.

We are now ready to state the definitions of plain and prefix-free Kolmogorov complexity. Let $\sigma \in 2^{<\omega}$.

Definition 2.2. *The* plain Kolmogorov complexity *of* $\sigma \in 2^{<\omega}$ *with respect to (the universal Turing machine)* M_0 *is given by*

$$C_{M_0}(\sigma) = \min\{|\tau| \in \omega : \tau \in 2^{<\omega} \text{ and } M_0(\tau)\downarrow = \sigma\}.$$

Definition 2.3. *The* prefix-free Kolmogorov complexity *of* $\sigma \in 2^{<\omega}$ with respect to *(the prefix-free universal Turing machine) M is given by*

$$K_M(\sigma) = \min\{|\tau| \in \omega : \tau \in 2^{<\omega} \text{ and } M(\tau)\downarrow = \sigma\}.$$

It is well-known that if M_0' is any universal Turing machine and M is any prefix-free universal Turing machine, then there exists a constant $C \in \omega$ such that for all $\sigma \in 2^{<\omega}$,

$$|C_{M_0}(\sigma) - C_{M_0'}(\sigma)| \leq C \text{ and } |K_M(\sigma) - K_{M'}(\sigma)| \leq C.$$

Therefore, it follows that the notions of plain and prefix-free Kolmogorov complexity are well-defined up to a constant. With this in mind, for all $\sigma \in 2^{<\omega}$ we write $C(\sigma)$ and $K(\sigma)$ to mean $C_{M_0}(\sigma)$ and $K_M(\sigma)$, respectively, where M_0, M are the (fixed) Turing machines that we defined above, and we will refer to these quantities as the plain and prefix-free Kolmogorov complexity of σ, respectively (with no explicit mention of the machines M_0, M). It follows that all facts and theorems concerning plain and prefix-free Kolmogorov complexity are well-defined only up to a constant, since the notions of plain and prefix-free Kolmogorov complexity are well-defined only up to a constant. Often we will simply refer to the prefix-free Kolmogorov complexity of a given string $\sigma \in 2^{<\omega}$ as *the Kolmogorov complexity of* σ. However, when speaking about plain Kolmogorov complexity we will always use the term *plain* to distinguish this notion from that of prefix-free Kolmogorov complexity. Since M also qualifies as a universal Turing machine, it follows that there is a constant $C_0 \in \omega$ such that for all $\sigma \in 2^{<\omega}$ we have that $K(\sigma) + C_0 \geq C(\sigma)$. Without any loss of generality (i.e. by modifying M slightly), we can assume that $C_0 = 0$ so that we have $K(\sigma) \geq C(\sigma)$, for all $\sigma \in 2^{<\omega}$ (this extra hypothesis simplifies some of our proofs below).

To prove the main theorem of this article we will need a method for showing that every member of various computable sets of strings has low Kolmogorov complexity. There is a standard method that does this that uses the notion of a bounded request set, as well as the Machine Existence Theorem. We now introduce our version of these concepts, some of which differ slightly from the corresponding concepts in [Nie] cited below.

Definition 2.4. [Nie, Definition 2.2.15] *A* bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ *is a computably enumerable set of pairs of the form $\langle r_\sigma, \sigma\rangle$, $r_\sigma \in \omega$, $\sigma \in 2^{<\omega}$, such that*

$$\sum_{r \in \mathcal{R}} 2^{-\pi_1(r)} < 1.$$

We also call a computably enumerable set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ simply a request set.

The first part of the following definition is standard, and can be found in [Nie, Definition 2.2.15]. However, the second part is not standard, but is useful in the context of effective fractal dimension (as we shall see later on). The notion of α-weight (defined below) will play a significant role later on in this article.

Definition 2.5. *Let $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ be a request set. Then the* weight *of $\mathcal{R}$ is given by*

$$\sum_{r \in \mathcal{R}} 2^{-\pi_1(r)},$$

which may be infinite. Note that a request set is a bounded request set if and only if its weight is strictly less than one.

If $D \subseteq 2^{<\omega}$ and $\alpha \in \mathbb{Q}$, $0 \leq \alpha < 1$, we define the α-weight of D *to be*

$$\sum_{\sigma \in D} 2^{-r_\sigma},$$

where $r_\sigma \in \omega$ is the greatest integer less than $\alpha|\sigma| + 1$, and $\sigma \in D$. Here the associated request set consists of those pairs of the form $\langle r_\sigma, \sigma\rangle$, for all $\sigma \in D$. Often times we will have already fixed a value for $\alpha \in \mathbb{Q}$, $0 \leq \alpha < 1$, and we will simply say the weight of $D \subseteq 2^{<\omega}$ *to* *mean the α-weight of $D \subseteq 2^{<\omega}$. Note that, by our definition of $r_\sigma \in \omega$ above, we have that the α-weight of a given set $D \subseteq 2^{<\omega}$ is bounded above by*

$$\sum_{\sigma \in D} 2^{-\alpha|\sigma|}.$$

We will exploit this fact later on.

The following theorem is usually called the *Kraft-Chaitin Theorem.*

Theorem 2.6. [Nie, Theorem 2.2.17] *Let $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ be a bounded request set. Then there exists a prefix-free machine M_0 such that*

$$(\forall r \in \omega)(\forall \sigma \in 2^{<\omega})[\langle r, \sigma\rangle \in \mathcal{R} \Leftrightarrow (\exists \tau \in 2^{<\omega})[|\tau| = r \text{ and } M_0(\tau) = \sigma].$$

Furthermore, it follows (by the previous sentence above) that there exists a constant $C_0 \in \omega$ (depending only on M_0 and not on σ) such that for every $\sigma \in 2^{<\omega}$ and every $r_\sigma \in \omega$ such that $\langle r_\sigma, \sigma\rangle \in \mathcal{R}$, we have that

$$K(\sigma) \leq r_\sigma + C_0.$$

The following definition is not standard, although it is equivalent to the standard definition of effective packing dimension. For the standard definition of effective packing dimension, as well as the proof that it coincides with our definition below, please consult [AHLM07, Lut03, May02]. It is worth noting that effective packing dimension is sometimes referred to as *constructive packing dimension* or *constructive strong dimension.*

Definition 2.7. *Let $f \in 2^\omega$. The* effective packing dimension *of f is given by*

$$\dim_{\mathrm{P}}(f) = \limsup_{n\to\infty} \frac{K(f\restriction n)}{n}.$$

Effective Hausdorff dimension is dual to effective packing dimension. Although the following definition is not the standard one, it is equivalent to the standard definition of effective Hausdorff dimension. Sometimes effective Hausdorff dimension is referred to as *constructive dimension.*

Definition 2.8. *Let $f \in 2^\omega$. The* effective Hausdorff dimension *of f is given by*

$$\dim_{\mathrm{P}}(f) = \liminf_{n\to\infty} \frac{K(f\restriction n)}{n}.$$

It is well-known (see [Nie, Proposition 2.2.8], for example) that there exists a constant $C \in \omega$ such that $0 \leq K(f\restriction n) \leq 2\log(n) + n + C$, for all $f \in 2^\omega$ and $n \in \omega$. From this it follows that for any given $f \in 2^\omega$, the effective packing/Hausdorff dimension of f is a real number between zero and one (inclusive).

3. Clumpy trees, bounded request sets, and Kolmogorov complexity

We begin this section with some important definitions, some of which will be slightly modified in the next section (see Definition 4.1 below for more details).

Definition 3.1. *Let $T \subseteq 2^{<\omega}$ be a computable tree. We say that T is a* clumpy tree[3] *if for every $\sigma \in T$ there is a $\tau \in T \subseteq 2^{<\omega}$ such that $\tau \supset \sigma$ and the clump $\tau 2^{\leq|\tau|} \subseteq 2^{<\omega}$ is on T.*

We say that $T \subseteq 2^{<\omega}$ is a pruned clumpy tree *if T is a computable tree and for every extendible node $\sigma \in T$ there is a string $\tau \in T$, $\tau \supset \sigma$, such that $T \cap \tau 2^{\leq|\tau|}$ contains at least two leaves of the clump $\tau 2^{\leq|\tau|}$. We call $T \cap \tau 2^{\leq|\tau|} \subseteq T \subseteq 2^{<\omega}$ a* pruned clump *of T.*

[3] The terminology "clump" and "clumpy tree" was first introduced by Downey and Greenberg in [DG08], although these concepts were also discovered and used to construct reals of strictly positive effective packing dimension by the author in [Con08].

Note that, by definition, every node $\sigma \in 2^{<\omega}$ on a clumpy tree is extendible. Also note that a clumpy tree is a pruned clumpy tree. In the next section we will define an initial clumpy tree $T_0 \subseteq 2^{<\omega}$ inside of which our entire construction that is the proof of the main theorem (i.e. Theorem 6.1) will take place. Once we have defined T_0, we will use T_0 to modify the definition of pruned clumpy tree slightly, making it a bit more restrictive and easier to talk about for our purposes. All of our pruned clumpy trees will be subtrees of T_0. In Section 5 we will show that whenever pruned clumps live on a tree T_s, $s \in \omega$, that we are constructing, then they have many incomparable nodes in T_s. This fact will make it possible to apply Lemma 3.2 below to find strings of high complexity on T_s, which, in turn will help us to ensure that $X \in [T_s] \subseteq 2^\omega$ of Theorem 6.1 has strictly positive effective packing dimension (via Definition 2.7 above).

Whenever we construct a (pruned) clumpy tree, we will always do so uniformly and computably via finite approximations in stages/substages $k \in \omega$, i.e. we shall have that

$$T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega},$$

where $T^{k+1} \supseteq T^k$ are uniformly computable increasing finite approximations to T. At stage $k = 0$, we will define $T^0 \subseteq 2^{<\omega}$ to be the downward closure of a single string $\sigma \in 2^{<\omega}$. Then, at stage $k+1 \in \omega$, we will properly, uniformly, and computably extend every leaf $\lambda \in T^k$ of the finite tree T^k and add clumps of the form $\tau_\lambda 2^{\leq|\tau_\lambda|} \subseteq 2^{<\omega}$, $\tau_\lambda \supseteq \lambda$, above λ.

We now prove a lemma that essentially gives the main reason for considering pruned clumpy trees in the context of effective packing dimension. Generally speaking, the reason why we consider pruned clumpy trees when examining effective packing dimension is that if $A \subseteq \tau 2^{\leq|\tau|} \subseteq T$ is a pruned clump of nodes on the tree $T \subseteq 2^{<\omega}$ that contains at least $2^{q|\tau|}$-many leaves of $\tau 2^{\leq|\tau|}$, for some $q \in \mathbb{Q}$, then there exists a leaf of $\tau 2^{\leq|\tau|}$, $\sigma \in A \subseteq 2^{<\omega}$, such that

$$K(\sigma) \geq \frac{q}{2}|\sigma| - 1.$$

In other words, inside every pruned clump with sufficiently many leaves of maximal length there are nodes of relatively high Kolmogorov complexity. It follows that in a clumpy tree the set of nodes of relatively high Kolmogorov complexity is a dense set. We will need to find nodes of relatively high Kolmogorov complexity to ensure that the real $X \in 2^\omega$ of Theorem 6.1 is of strictly positive effective packing dimension. Variants of the following lemma and corollary first appeared in [Con08] and [DG08], where (as in this article) they are used to construct reals of strictly positive effective packing dimension that satisfy other properties as well.

Lemma 3.2. *Let $q \in \mathbb{Q}$, $0 \leq q \leq 1$, and $\tau \in 2^{<\omega}$ be given, and let $q_\tau \in \omega$ be the least natural number that is greater than or equal to $q|\tau|$ (note that $q_\tau \in \omega$ can be computed uniformly from τ and q). Then, for any given pruned clump of the form $A \subseteq \tau 2^{\leq|\tau|} \subseteq 2^{<\omega}$ such that A contains at least 2^{q_τ}-many leaves of $\tau 2^{\leq|\tau|}$, there is a leaf $\sigma \in \tau 2^{\leq|\tau|} \subseteq 2^{<\omega}$ in A such that*

$$K(\sigma) > \frac{q}{2}|\sigma| - 1.$$

Proof. There exist 2^{q_τ}-many leaves of $\tau 2^{\leq|\tau|}$, all of length at most $2|\tau| \in \omega$. There are $(2^{q_\tau}-1)$-many nodes of plain complexity at most $q_\tau - 1$. Therefore, by the pigeonhole principle it follows that there exists a node $\sigma \in A$ that is a leaf of the pruned clump $\tau 2^{\leq|\tau|} \subseteq 2^{<\omega}$ and such that σ has plain complexity $C(\sigma) > q_\tau - 1 \geq q|\tau| - 1 \geq \frac{q}{2}|\sigma| - 1$. Now, since we are under the assumption that $K(\rho) \geq C(\rho)$ for all $\rho \in 2^{<\omega}$, the conclusion of the lemma now follows. □

Corollary 3.3. *Let $q \in \mathbb{Q}$, $0 \leq q \leq 1$, and $\tau \in T \subseteq 2^{<\omega}$ be given, and for all $\rho \in 2^{<\omega}$ let $q_\rho \in \omega$ be as in Lemma 3.2 above. Also, suppose that $T \subseteq 2^{<\omega}$ is a pruned clumpy tree such that $\tau \in T$ is extendible and whenever a pruned clump $A \subseteq \rho 2^{\leq|\rho|}$, $\rho \in 2^{<\omega}$, is on T, then* *there are at least 2^{q_ρ}-many leaves of $\rho 2^{\leq|\rho|}$ in A. Then there exists $\sigma \in T$ such that $\sigma \supset \tau$ and*

$$K(\sigma) \geq \frac{q}{2}|\sigma| - 1.$$

Proof. Given $T \subseteq 2^{<\omega}$ and $\tau \in T$ as above, use our hypotheses on τ and T to find $\rho \supset \tau$, $\rho \in 2^{<\omega}$, such that there are at least 2^{q_ρ}-many leaves of $\rho 2^{\leq|\rho|}$ in T. Now, apply the previous lemma to find a leaf $\sigma \in \rho 2^{\leq|\rho|} \subseteq T$, $\sigma \supseteq \rho \supset \tau$, such that $K(\sigma) \geq \frac{q}{2}|\sigma| - 1$. □

Aside from finding nodes of relatively high Kolmogorov complexity to push the effective packing dimension of X (of Theorem 6.1) strictly above zero, we will also need to bound the Kolmogorov complexity of some finite binary strings so that the effective packing dimension of every $Y \in 2^\omega$ such that $Y \leq_T X$ is strictly less than one. We will always use bounded request sets and the Machine Existence Theorem, i.e. Theorem 2.6, to achieve this goal. We now prove an easy lemma that says to bound the effective packing dimension of $Y \in 2^\omega$ by $0 \leq \alpha \leq 1$, $\alpha \in \mathbb{Q}$, it suffices to fix a number $N_0 \in \omega$ and then include every pair of the form $\langle r_n, \sigma_n \rangle$, $n \geq N_0$, in our bounded request set, where $\sigma_n = Y{\upharpoonright}n \in 2^{<\omega}$ is the first n bits of Y, and $r_n \in \omega$ is the greatest integer that is less than or equal to $\alpha|\sigma_n| + 1 \in \mathbb{Q}$ (note that, by our construction of r_n, it follows that $\alpha|\sigma_n| < r_n$).

Lemma 3.4. *Suppose that $N_0 \in \omega$, $\alpha \in \mathbb{Q}$, $\alpha \leq 1$, $Y \in 2^\omega$ are given, and that for all $n \geq N_0$ the pairs $\langle r_n, \sigma_n \rangle_{n \geq N_0}$ are included in a bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$, where $\sigma_n = Y{\upharpoonright}n \in 2^{<\omega}$ and $r_n \in \omega$ is the greatest integer that is less than or equal to $\alpha|\sigma_n|+1 \in \mathbb{Q}$. Then we have that the effective packing dimension of $Y \in 2^\omega$ is at most $\alpha \in \mathbb{Q}$.*

Proof. By the Machine Existence Theorem, i.e. Theorem 2.6 above, we have that there is a constant $C \in \omega$ such that for all $n \geq N_0$

$$K(\sigma_n) \leq \alpha|\sigma_n| + 1 + C.$$

It follows that for all $n \geq N_0$ we have $K(Y{\upharpoonright}n) = K(\sigma_n) \leq \alpha|\sigma_n| + 1 + C = \alpha n + 1 + C$, and therefore the effective packing dimension of Y is given by

$$\limsup_{n\to\infty} \frac{K(Y{\upharpoonright}n)}{n} = \limsup_{n\geq N_0} \frac{K(Y{\upharpoonright}n)}{n} \leq \limsup_{n\geq N_0} \frac{\alpha n + 1 + C}{n} = \alpha.$$

□

Since it is relatively intuitive, and its proof is quite simple, we will henceforth use Lemma 3.4 above without necessarily saying so.

3.1. **A brief overview of the proof of the main theorem.** Now that we have given the reader a glimpse of some of the main ideas used to prove the main theorem (i.e. Theorem 6.1), we will briefly explain how these concepts will be applied in the next two sections which contain most of the proof of the main theorem.

Recall that the main theorem aims to construct a real $X \in 2^\omega$ of strictly positive effective packing dimension and such that for all $Y \in 2^\omega$ such that $Y \leq_T X$, the effective packing dimension of Y is strictly less than one. There are two parts to the proof of the main theorem, i.e. Theorem 6.1. The first part is easier than the second, and aims to construct $X \in 2^\omega$ so that it has strictly positive effective packing dimension. We will essentially achieve this goal via Corollary 3.3 above. The second part of the main theorem aims to bound the effective packing dimension of all total Turing reductions from X to another real $Y \in 2^\omega$. Generally speaking, to achieve this goal we will use the Machine Existence Theorem and bounded request sets. We now explain the overall construction of $X \in 2^\omega$ in stages $s \in \omega$, i.e. $X = \cup_{s\in\omega}\xi_s$, $\xi_s \in 2^{<\omega}$, $\xi_{s+1} \supset \xi_s$.

Section 4 of this article deals with the stage $s = 0$. At stage $s = 0$ our main goal is to construct a clumpy tree $T_0 \subseteq 2^{<\omega}$ inside of which the rest of the construction of $X \in 2^\omega$ will take place. As a warm up to the main theorem, after constructing T_0 we prove that there is a real $X_0 \in [T_0] \subseteq 2^\omega$ such that the effective packing dimension of X_0 is strictly positive, and we also show that every $Y \in [T_0] \subseteq 2^\omega$ has effective packing dimension strictly less than one. Along the way, we develop some important machinery that will be used to handle the most difficult parts of the proof of the main theorem. We also define $\xi_0 = \langle 0 \rangle \in T_0 \subseteq 2^{<\omega}$. At the end of our construction (i.e. at the end of Section 5) we will have that $X \supset \xi_0 = \langle 0 \rangle \in 2^{<\omega}$, where $X \in 2^\omega$ is as in Theorem 6.1. Note that, since $|\xi_0| = 1$, we (trivially) have that $K(\xi_0) \geq \frac{1}{4}|\xi_0| - 1$.

Section 5 of this article handles the construction of $X \in 2^\omega$ at stage $s+1 > 0$. At the beginning of stage $s+1 > 0$, we assume that we are given a string $\xi_s \in 2^{<\omega}$ (ξ_s will be an initial segment of $X \in 2^\omega$) and a pruned clumpy tree $T_s \subseteq T_0 \subseteq 2^{<\omega}$ such that $\xi_s \in T_s$ and such that every pruned clump on T_s has (sufficiently) many incomparable strings in it. Our first goal at stage $s+1$ is to find an extension of ξ_s, $\xi_{s+1} \supset \xi_s$, $\xi_{s+1} \in T_s \subseteq 2^{<\omega}$, such that $K(\xi_{s+1}) \geq \frac{1}{4}|\xi_{s+1}| - 1$. This will be possible since the clumps of T_s have sufficiently many incomparable nodes in them. More specifically, we will show that whenever $A \subseteq \tau 2^{\leq|\tau|} \subset 2^{<\omega}$, $\tau \in A \subseteq 2^{<\omega}$, is a pruned clump on the pruned clumpy tree $T_s \subseteq 2^{<\omega}$, then there are at least $2^{\frac{1}{2}|\tau|}$-many leaves of $\tau 2^{\leq|\tau|}$ on T_s. It will then follow by Corollary 3.3 above that the node $\xi_{s+1} \in T_s$, $\xi_{s+1} \supset \xi_s$, mentioned earlier exists. Our second goal at stage $s+1$ is more difficult; it is to construct a clumpy subtree of T_s, $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$, such that the effective packing dimension of the output of *every* total Turing reduction $Y \in [\Phi_s^{T_{s+1}}] \subseteq 2^\omega$ is strictly less than one. This goal will mostly be achieved via the machinery that we develop in Section 4, when we bound the effective packing dimension of every $X \in [T_0] \subseteq 2^\omega$ by $\frac{4}{5} < 1$.

3.1.1. *Forcing.* Though we will not explicitly do so, we wish to note that one may view our construction of $X \in 2^\omega$, $X = \cup_{s\in\omega}\xi_s$, $\xi_{s+1} \supset \xi_s$, $\xi_s \in 2^{<\omega}$, as a forcing construction. In this case our forcing conditions are pairs of the form $\langle \sigma, T \rangle$, where $\sigma \in 2^{<\omega}$ and $T \subseteq 2^{<\omega}$ is a pruned clumpy tree such that $\sigma \in T$ (σ is an initial segment of X and T is the set of possible extensions of σ). Furthermore, we have that $\langle \sigma_1, T_1 \rangle \geq \langle \sigma_2, T_2 \rangle$ whenever $\sigma_2 \supseteq \sigma_1$ and T_2 is a pruned clumpy subtree of T_1. Our initial forcing condition is given by $\langle \xi_0, T_0 \rangle$, and our generic filter is constructed in stages $s \in \omega$, $s > 0$, so that at stage s we force one more initial segment of X to have relatively high Kolmogorov complexity (so that in the end X will have strictly positive effective packing dimension), and we also force that the effective packing dimension of Φ_s^X is strictly less than one whenever $\Phi_s^X \in 2^\omega$ is a total Turing reduction via a specific linear bound on the use of the computation.

4. Stage $s = 0$

At stage $s = 0$ of our construction we set $\xi_0 = \langle 0 \rangle \in 2^{<\omega}$ and we produce a clumpy tree $T_0 \subseteq 2^{<\omega}$ such that $\xi_0 \in T_0$ and the real $X \in 2^\omega$, $X = \cup_{s\in\omega}\xi_s$, of Theorem 6.1 satisfies $X \in [T_0] \subseteq 2^\omega$. The fact that $T_0 \subseteq 2^{<\omega}$ is a clumpy tree will help us to ensure that the effective packing dimension of $X = \cup_{s\in\omega}\xi_s \in 2^\omega$ is nonzero. We will also construct T_0 sparse enough so that there exists a bounded request set that witnesses that the effective packing dimension of every real $Y \in [T_0] \subseteq 2^\omega$ is strictly less than one. We build $T_0 = \cup_{k\in\omega} T_0^k \subseteq 2^{<\omega}$ in substages $k \in \omega$ as follows.

4.1. **Constructing the clumpy tree $T_0 \subseteq 2^{<\omega}$.** At substage $k = 0$, let $T_0^k = T_0^0 = \{\emptyset, 0\} \subseteq 2^{<\omega}$. Note that we have $\xi_0 = \langle 0 \rangle \in T_0^0 \subseteq T_0 \subseteq 2^{<\omega}$. At subsequent stages $k > 0$, $k \in \omega$, let T_0^{k-1} be given, and define T_0^k to be the downward closure of

$$\bigcup_{\substack{\lambda \in T_0^{k-1} \\ \lambda \text{ a leaf}}} \lambda 2^{=|\lambda|} 0^{2|\lambda|} \subseteq 2^{<\omega}.$$

This ends the construction of $T_0 = \cup_{k\in\omega} T_0^k \subseteq 2^{<\omega}$. It is not difficult to check that T_0 is indeed a clumpy tree.

Note that at substage $k > 0$ we obtain the tree $T_0^k \subseteq 2^{<\omega}$ by extending every leaf $\lambda \in T_0^{k-1}$ via a clump $\lambda 2^{\leq|\lambda|}$, and then extending again via $0^{2|\lambda|}$. Thus, by induction we have that the leaves of T_0^k are all of length 4^k, for every $k \in \omega$, and every node of T_0 is extendible. Fix a leaf $\lambda \in T_0^{k-1} \subseteq 2^{<\omega}$. We say that $\lambda 2^{\leq|\lambda|} \subseteq T_0^k \subseteq T_0 \subseteq 2^{<\omega}$ is a *clump of level k in T_0*. Now, via Lemma 3.3, it follows that there exists a leaf of $\lambda 2^{\leq|\lambda|} \subseteq T_0^k \subseteq 2^{<\omega}$, $\rho \in \lambda 2^{\leq|\lambda|}$, such that $K(\rho) \geq \frac{1}{2}|\rho| - 1$; we will use this fact in the proof of Proposition 4.2 below.

The following definition augments Definition 3.1.

Definition 4.1. *Henceforth, by* clump *we will mean a set of nodes of the form $\sigma 2^{\leq|\sigma|} \subseteq 2^{<\omega}$, $\sigma \in 2^{<\omega}$ such that $\sigma 2^{\leq|\sigma|}$ is on T_0. By* pruned clump *we will mean a set $A \subseteq 2^{<\omega}$ such that $A \subseteq \sigma 2^{\leq|\sigma|}$ for some $\sigma \in 2^{<\omega}$ satisfying $\sigma 2^{\leq|\sigma|} \subseteq T_0 \subseteq 2^{<\omega}$, $\sigma \in A$, A is downwards closed with respect to $\subseteq$ and $\sigma 2^{\leq|\sigma|} \subseteq 2^{<\omega}$, and A contains at least two leaves of $\sigma 2^{\leq|\sigma|}$ (by the way in which we will construct our clumpy trees it will follow that our clumps will contain many more than two leaves). By* pruned clumpy tree *we will mean a tree $T \subseteq 2^{<\omega}$ such that every extendible node on T is a proper initial segment of a pruned clump that is also on T. The definition of clumpy tree remains the same as in Definition 3.1 above.*

Note that, by our construction of T_0 above, it follows that if $\sigma \in 2^{<\omega}$ is the root of a pruned clump (or clump), then $|\sigma| = 4^n$, for some $n \in \omega$. Moreover, if $T \subseteq 2^{<\omega}$ is a pruned clumpy tree and $\tau \in T$ is extendible, then for any given $n \in \omega$ we can find a pruned clump $A \subseteq T \subseteq 2^{<\omega}$ such that if $\sigma \in 2^{<\omega}$ is the root of A then $|\sigma| = 4^m$, for some $m \in \omega$ such that $m \geq n$. We will use these facts in the next section when constructing the pruned clumpy tree $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$.

Throughout the rest of this section we will work exclusively with clumpy trees, except in Proposition 4.10 where we work with "generalized clumpy trees", i.e. trees that are built out of generalized clumps (as defined in Section 2 above) in the same way that clumpy trees are built out of clumps in Definition 3.1 above. We shall utilize pruned clumpy trees in the next section when we construct $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$ at stage $s + 1 > 0$.

4.2. **Finding strings of high Kolmogorov complexity in T_0.**

Proposition 4.2. *There exists a real $X_0 \in [T_0] \subseteq 2^\omega$ such that the effective packing dimension of $X_0 \in 2^\omega$ is greater than or equal to $\frac{1}{2}$.*

Proof. Note that we are *not* constructing the real $X \in 2^\omega$ of Theorem 6.1, but the basic idea behind the construction of X_0 is the one that we shall use to ensure that the effective packing dimension of $X \in 2^\omega$ of Theorem 6.1 is strictly positive. We build $X_0 = \cup_{k\in\omega}\xi_0^k \in 2^\omega$, $\xi_0^{k+1} \supset \xi_0^k$, $\xi_0^k \in 2^{<\omega}$, in stages $k \in \omega$. At stage $k = 0$ let $\xi_0^k = \xi_0^0 = \langle 0 \rangle \in T_0^0 \subseteq T_0 \subseteq 2^{<\omega}$. At stage $k > 0$, suppose that we are given $\xi_0^{k-1} \in T_0 \subseteq 2^{<\omega}$; we shall find a string $\sigma \supset \xi_0^{k-1}$ such that $K(\sigma) \geq \frac{1}{2}|\sigma| - 1$ and set $\xi_0^k = \sigma$. It follows from Lemma 3.3 above that, since $T_0 \subseteq 2^{<\omega}$ is a clumpy tree (as defined in Definition 3.1 above), then such a string $\sigma \in 2^{<\omega}$ exists. Therefore it follows that for every $k \in \omega$ we have

$$K(\xi_0^k) \geq \frac{1}{2}|\xi_0^k| - 1,$$

from which it follows that the effective packing dimension of $X = \cup_{k\in\omega}\xi_0^k \in 2^\omega$ is

$$\limsup_{n\to\infty} \frac{K(X\restriction n)}{n} \geq \limsup_{k\to\infty} \frac{K(\xi_0^k)}{|\xi_0^k|} \geq \frac{|\xi_0^k| - 2}{2|\xi_0^k|} = \frac{1}{2}.$$

□

We will use essentially the same proof in the next section to prove that the real $X \in 2^\omega$ of the main theorem has effective packing dimension at least $\frac{1}{2}$.

4.3. **Bounding the Kolmogorov complexity of strings in T_0.** We now aim to show that the effective packing dimension of *every* $Y_0 \in [T_0] \subseteq 2^\omega$ is bounded above by $\frac{4}{5} < 1$. Along the way we will prove a parallel set of more general lemmas that we will apply in the next section to show that the effective packing dimension of every $Y \leq_T X$, $X, Y \in 2^\omega$, is strictly less than one. The following key lemmas will eventually aid us in showing that a particular bounded request set (to be constructed later on in this section) witnesses the $\frac{4}{5}$ upper bound on the effective packing dimension of *every* $Y_0 \in [T_0] \subseteq 2^\omega$. These lemmas are very important, and will also be used in the next section to prove the most difficult part of the main theorem (i.e. Theorem 6.1) in Section 6.

Lemma 4.3. *Let $l, a \in \omega$ be given, and define*

$$A(l,a) = 2^{=l}0^l 2^{\leq al} \subseteq 2^{<\omega}.$$

If we set $\alpha = \frac{a+2}{a+3} \in \mathbb{Q}$, $\alpha < 1$, then we have that

$$\sum_{\sigma \in A(l,a)} 2^{-\alpha|\sigma|} \leq C_1 2^{-c_1 l},$$

for some constants $C_1 \in \omega$ and $c_1 \in \mathbb{Q}$, $c_1 > 0$, that only depend upon $a \in \omega$ and do not depend upon $l \in \omega$.

Proof. First of all, note that it follows from our construction of $A(l,a) \subseteq 2^{<\omega}$ above that:

(1) Every $\rho \in A(l,a) \subseteq 2^{<\omega}$ has length at least $2l$.
(2) There are exactly 2^l-many strings $\rho_1, \rho_2, \ldots, \rho_{2^l} \in A(l,a)$ such that $|\rho_i| = 2l$ and $\rho_i = (\rho_i \restriction l)\, 0^l \in 2^{<\omega}$.
(3) For every $\tau \in A(l,a)$ there exists $1 \leq i \leq 2^l$ such that $\rho_i \subseteq \tau$.

To prove the lemma, first fix a string $\rho \in A(l,a)$ of length $2l \in \omega$ in $A(l,a) \subseteq 2^{<\omega}$. By our comments in the previous paragraph and our definition of $A(l,a)$ above, we have that $\rho 2^{\leq al} \subseteq A(l,a) \subseteq 2^{<\omega}$ is exactly the set of elements of $A(l,a)$ that extend ρ. Furthermore, we have that

$$\sum_{\sigma \in \rho 2^{\leq al}} 2^{-\alpha|\sigma|} = \sum_{i=2l}^{(a+2)l} 2^{-\alpha i} 2^{i-2l}.$$

The rightmost sum above is a geometric series with initial term $2^{-\alpha 2l}$, and common ratio $2^{1-\alpha}$. Therefore, the expression sums to

$$(2^{-\alpha(2l)}) \frac{2^{(1-\alpha)(al+1)} - 1}{2^{1-\alpha} - 1} \leq (2^{1-\alpha} - 1)^{-1}(2^{1-\alpha})(2^{(-2\alpha + a - \alpha a)l}).$$

Now, $A(l,a) \subseteq 2^{<\omega}$ contains 2^l-many strings of length $2l$, and by our comments in the previous paragraph it follows that to obtain the total weight of the set $A(l,a) \subseteq 2^{<\omega}$, we must multiply the above expression by 2^l, since there are 2^l-many choices for the string $\rho \in A(l,a)$ of length $2l$. Therefore, the total weight of $A(l,a) \subseteq 2^{<\omega}$ is bounded above by

$$(2^{1-\alpha} - 1)^{-1}(2^{1-\alpha})(2^{(-2\alpha + a - \alpha a + 1)l}).$$

Notice that only the exponent of the last factor depends upon $l \in \omega$, and that the exponent is negative if $\alpha > \frac{a+1}{a+2}$. Therefore, by our choice of $\alpha = \frac{a+2}{a+3} > \frac{a+1}{a+2}$, it follows that there exist constants $C_1 \in \omega$, $c_1 \in \mathbb{Q}$, $c_1 > 0$, that depend only on $a \in \omega$ and not on $l \in \omega$ and such that

$$\sum_{\sigma \in A(l,a)} 2^{-\alpha|\sigma|} \leq 2^l \cdot \sum_{\sigma \in \rho 2^{\leq al}} 2^{-\alpha|\sigma|} \leq \frac{2^{1-2\alpha}}{2^{1-\alpha} - 1}(2^{(-2\alpha + a - \alpha a + 1)l}) \leq C_1 2^{-c_1 l}.$$

□

The following modified version of Lemma 4.3 also holds via a similar proof.

Lemma 4.4. *Let $a, l \in \omega$, $a > 0$, be given. Let $\tau_1, \tau_2, \ldots, \tau_{2^l} \in 2^{=l}$ be the (unique) lexicographic listing of* all *finite binary strings of length $l \in \omega$ (any one-to-one listing would suffice) and let $\alpha = \frac{2a+2}{2a+3} \in \mathbb{Q}$, $\alpha < 1$. Then, for any given strings $\rho_1, \rho_2, \ldots, \rho_{2^l} \in 2^{\geq l}$, if we define*

$$A(l, a, \vec{\rho}) = A(l, a, \rho_1, \rho_2, \ldots, \rho_{2^l}) = \bigcup_{1 \leq i \leq 2^l} \tau_i \rho_i 2^{\leq a|\tau_i \rho_i|},$$

then there exist constants $C_1 \in \omega$, $c_1 \in \mathbb{Q}$, $c_1 > 0$, that are independent of $l \in \omega$ and such that

$$\sum_{\sigma \in A(l,a,\vec{\rho})} 2^{-\alpha|\sigma|} \leq C_1 2^{-c_1 l}.$$

Recall that for any given $f \in 2^\omega$, we have that $\hat{f} = \{f\restriction 1, f\restriction 2, f\restriction 3, \ldots\} \subseteq 2^{<\omega}$, where $f\restriction n = f(0)f(1)\cdots f(n-1) \in 2^{<\omega}$, $n \in \omega$, denotes the first n bits of f.

Lemma 4.5. *Let $l, a \in \omega$ be given, and let $\alpha \in \mathbb{Q}$, $A(l, a) \subseteq 2^{<\omega}$, be as in Lemma 4.3 above. Define $\alpha = \frac{a+2}{a+3} \in \mathbb{Q}$, $\alpha < 1$, and*

$$B(l, a) = \bigcup_{\substack{\rho \in A(l,a) \\ \rho \text{ a leaf}}} \rho \hat{0^\infty} \subseteq 2^{<\omega}.$$

Then there exist constants $C_2 \in \omega$, $c_2 \in \mathbb{Q}$, $c_2 > 0$, that are independent of $l \in \omega$ and such that

$$\sum_{\sigma \in B(l,a)} 2^{-\alpha|\sigma|} \leq C_2 2^{-c_2 l}.$$

Proof. Fix $\rho \in A(l, a) \subseteq 2^{<\omega}$ a leaf. By our construction of $A(l, a)$ in lemma 4.3 above, there are $2^{(a+1)l}$-many choices for ρ. Then, by the construction of $B(l, a) \subseteq 2^{<\omega}$, we have that the part of $B(l, a)$ above ρ is exactly $\rho \hat{0^\infty} = \{\rho 0, \rho 00, \ldots, \rho 0^k, \ldots\} \subseteq 2^{<\omega}$. We have that

$$\sum_{\sigma \in \rho \hat{0^\infty}} 2^{-\alpha|\sigma|} = \sum_{i=|\rho|+1}^{\infty} 2^{-\alpha i} = \sum_{i=(a+2)l+1}^{\infty} 2^{-\alpha i} = \frac{2^{-\alpha[(a+2)l+1]}}{1 - 2^{-\alpha}}.$$

Now, recall that there are $2^{(a+1)l}$-many choices for $\rho \in A(l, a) \subseteq 2^{<\omega}$. Therefore, it follows that

$$\sum_{\sigma \in B(l,a)} 2^{-\alpha|\sigma|} = 2^{(a+1)l} \sum_{\sigma \in \rho \hat{0^\infty}} 2^{-\alpha|\sigma|} = 2^{(a+1)l} \frac{2^{-\alpha[(a+2)l+1]}}{1 - 2^{-\alpha}} = \frac{2^{-\alpha}}{1 - 2^{-\alpha}} 2^{[a+1-\alpha(a+2)]l}.$$

Now, since $\alpha = \frac{a+2}{a+3} > \frac{a+1}{a+2}$, then it follows that the exponent of the rightmost expression in the line of equalities above is negative. It now follows that there exist constants $C_2 \in \omega$, $c_2 \in \mathbb{Q}$, $c_2 > 0$ that are independent of $l \in \omega$ and such that

$$\sum_{\sigma \in B(l,a)} 2^{-\alpha|\sigma|} = \frac{2^{-\alpha}}{1 - 2^{-\alpha}} 2^{[a+1-\alpha(a+2)]l} \leq C_2 2^{-c_2 l}.$$

□

Note that the following slightly modified version of Lemma 4.5 above also holds via a very similar proof.

Lemma 4.6. *Let $a, l \in \omega$, $a > 0$, $\alpha = \frac{2a+1}{2a+2} \in \mathbb{Q}$, $\alpha < 1$, $\tau_1, \tau_2, \ldots, \tau_{2^l} \in 2^{=l}$, and $\vec{\rho} \in (2^{\geq l})^{2^l}$ be as in the statement of Lemma 4.4 above, and for every $1 \leq i \leq 2^l$, let $\eta_1^i, \eta_2^i, \ldots, \eta_{2^{a|\tau_i \rho_i|}}^i \in$*

$2^{=a|\tau_i\rho_i|}$ *be the unique lexicographic listing of the elements of* $2^{=a|\tau_i\rho_i|} \subseteq 2^{<\omega}$. *Then, for any given* $\vec{f} \in (2^\omega)^{\sum_{i=1}^{2^l} 2^{a|\tau_i\rho_i|}}$, $\vec{f} = \langle f_{i,j} \rangle_{\substack{1\le i\le 2^l \\ 1\le j\le 2^{a|\tau_i\rho_i|}}}$, $f_{i,j} \in 2^\omega$, *if we define*

$$B(l,a,\vec{\rho},\vec{f}) = B(l,a,\rho_1,\ldots,\rho_{2^l},f_{0,0},\ldots,f_{2^l,2^{a|\tau_{2^l}\rho_{2^l}|}}) = \bigcup_{\substack{1\le i\le 2^l \\ 1\le j\le 2^{a|\tau_i\rho_i|}}} \tau_i\rho_i\eta_j\hat{f}_{i,j},$$

$B(l,a,\vec{\rho},\vec{f}) \subseteq 2^{<\omega}$, *then there exist constants* $C_2 \in \omega$ *and* $c_2 \in \mathbb{Q}$, $c_2 > 0$, *that are independent of* $l \in \omega$ *and such that*

$$\sum_{\sigma\in B(l,a,\vec{\rho},\vec{f})} 2^{-\alpha|\sigma|} \le C_2 2^{-c_2 l}.$$

We now combine Lemmas 4.3 and 4.5 above to obtain the following stronger lemma.

Lemma 4.7. *Let* $a, l \in \omega$ *be given. Set* $\alpha = \frac{a+2}{a+3} \in \mathbb{Q}$, $\alpha < 1$, *and define*

$$D(l,a) = A(l,a) \cup B(l,a) \subseteq 2^{<\omega},$$

then there exist constants $C_0 \in \omega$ *and* $c_0 \in \mathbb{Q}$, $c_0 > 0$ *that are independent of* $l \in \omega$ *(i.e.* C_0, c_0 *depend only on* $a \in \omega$*) and such that*

$$\sum_{\sigma\in D(l,a)} 2^{-\alpha|\sigma|} \le C_0 2^{-c_0 l}.$$

Proof. First, apply Lemmas 4.3 and 4.5 above to obtain constants $C_1, C_2 \in \omega$ and $c_1, c_2 \in \mathbb{Q}$, $c_1, c_2 > 0$, that are independent of $l \in \omega$ and such that

$$\sum_{\sigma\in A(l,a)} 2^{-\alpha|\sigma|} \le C_1 2^{-c_1 l} \quad \text{and} \quad \sum_{\sigma\in B(l,a)} 2^{-\alpha|\sigma|} \le C_2 2^{-c_2 l}.$$

Now, letting $C_0 = 2\max\{C_1, C_2\}$ and $c_0 = \frac{1}{2}\min\{c_1, c_2\}$, $c_0 > 0$, and adding the inequalities above yields

$$\sum_{\sigma\in D(l,a)} 2^{-\alpha|\sigma|} = \sum_{\sigma\in A(l,a)} 2^{-\alpha|\sigma|} + \sum_{\sigma\in B(l,a)} 2^{-\alpha|\sigma|} \le C_1 2^{-c_1 l} + C_2 2^{-c_2 l} \le C_0 2^{-c_0 l}.$$

□

We shall use Lemma 4.7 above to prove Proposition 4.9 below.

We now note that Lemmas 4.4 and 4.6 above can be combined to prove the following lemma.

Lemma 4.8. *Let* $a, l \in \omega$, $a > 0$, $\alpha = \frac{2a+2}{2a+3} \in \mathbb{Q}$, $\alpha < 1$, $\tau_1, \tau_2, \ldots, \tau_{2^l} \in 2^{=l}$, $\vec{\rho} \in (2^{\ge l})^{2^l}$, $\vec{f} \in (2^\omega)^{\sum_{i=1}^{2^l} 2^{a|\tau_i\rho_i|}}$ *be as in the statement of Lemma 4.6 above. If we define*

$$D(l,a,\vec{\rho},\vec{f}) = A(l,a,\vec{\rho}) \cup B(l,a,\vec{\rho},\vec{f}) \subseteq 2^{<\omega},$$

then there exist constants $C_0 \in \omega$ *and* $c_0 \in \mathbb{Q}$, $c_0 > 0$ *that are independent of* $l \in \omega$ *(i.e.* C_0, c_0 *depend only on* $a \in \omega$*) and such that*

$$\sum_{\sigma\in D(l,a)} 2^{-\alpha|\sigma|} \le C_0 2^{-c_0 l}.$$

Proof. The proof is exactly the same as that of Lemma 4.7 above, except that we use Lemmas 4.4 and 4.6 in place of Lemmas 4.3 and 4.5, respectively. □

Lemma 4.8 above will play a crucial role in the proof of Proposition 4.10 below. In turn, Proposition 4.10 will play a crucial role in the proof of the main theorem of this article (i.e. Theorem 6.1). Therefore, it follows that Lemma 4.8 above is very important and will play a key role in the proof of the main theorem.

The key fact about $D(l,a) = A(l,a) \cup B(l,a) \subseteq 2^{<\omega}$ of Lemma 4.7 above that we use to prove Proposition 4.9 below is that at substage $k > 1,\ k \in \omega$, of the construction of the clumpy tree $T_0 \subseteq 2^{<\omega}$ above, we have that

$$T_0^k \setminus T_0^{k-1} \subseteq D(\frac{1}{2}|\lambda|, 2),$$

where $\lambda \in T_0^{k-1} \subseteq 2^{<\omega}$ is any leaf of T_0^{k-1} (by the construction of $T_0^k \subseteq 2^{<\omega}$ it follows that $|\lambda| \in \omega$ is well-defined and always an even number). We will use this fact in the proof of Proposition 4.9 below, and leave its verification, which follows from our constructions of $T_0 = \cup_{k\in\omega} T_0^k \subseteq 2^{<\omega}$ and $D(l,a) \subseteq 2^{<\omega},\ a,l \in \omega$, above, to the reader. Therefore, it follows from what we have just stated that to give short descriptions of every string $\sigma \in T_0^k \setminus T_0^{k-1}$, $k > 1$, it suffices to give short descriptions of every $\sigma \in D(\frac{1}{2}|\lambda|, 2)$, where $\lambda \in T_0^{k-1} \subseteq 2^{<\omega}$ is any leaf of T_0^{k-1}. Lemma 4.7 combined with the Machine Existence Theorem will imply that this is possible for long enough strings (i.e. large enough $k \in \omega$).

Proposition 4.9. *Let $Y_0 \in 2^\omega,\ Y_0 \in [T_0]$. Then we have that the effective packing dimension of Y_0 is less than or equal to $\alpha = \frac{4}{5} < 1$.*

Proof. We shall construct a bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ such that there exists a number $N_0 \in \omega$ so that for every $\sigma \in T_0 \subseteq 2^{<\omega}$, $|\sigma| > N_0$, we have that $\langle r_\sigma, \sigma\rangle \in \mathcal{R}$ and $r_\sigma \in \omega$, $r_\sigma \leq \frac{4}{5}|\sigma| + 1$. It will then follow that the effective packing dimension of every $Y_0 \in [T_0]$ is at most $\frac{4}{5} < 1$.

The construction of $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ is fairly simple. After choosing an appropriate value for $N_0 \in \omega$, we will simply put all pairs of the form $\langle r_\sigma, \sigma\rangle,\ r_\sigma \in \omega,\ \sigma \in T_0,\ |\sigma| > N_0$, into $\mathcal{R}$, where r_σ is the greatest integer less than or equal to $\frac{4}{5}|\sigma| + 1$. Note that for every $\sigma \in T_0$ we have that $r_\sigma \in \omega$ is greater than or equal to $\frac{4}{5}|\sigma| \in \mathbb{Q}$. By choosing $N_0 \in \omega$ large enough, we will ensure that the weight of our bounded request set $\mathcal{R}$ is at most one.

To see that an appropriate value of $N_0 \in \omega$ exists, consider the following line of reasoning. Fix a value of $N_0 \in \omega$, so that we put all pairs $\langle r_\sigma, \sigma\rangle$ such that $|\sigma| > N_0$ into $\mathcal{R}$ (here $r_\sigma \in \omega$ and $\sigma \in T_0$ are as defined in the previous paragraph). Then we have that the weight of $\mathcal{R}$ is:

$$\sum_{\substack{r\in\mathcal{R} \\ |\pi_2(r)|\geq N_0}} 2^{-\pi_1(r)} = \sum_{\substack{\sigma\in T_0 \\ |\sigma|>N_0}} 2^{-r_\sigma} \leq \sum_{\substack{\sigma\in T_0 \\ |\sigma|>N_0}} 2^{-\frac{4}{5}|\sigma|}.$$

Therefore, to find an upper bound for the weight of $\mathcal{R}$, we need to find an upper bound for $\sum_{\substack{\sigma\in T_0 \\ |\sigma|>N_0}} 2^{-\frac{4}{5}|\sigma|}$. But, as we shall see, such an upper bound follows from Lemma 4.7 above.

We now apply Lemma 4.7 with $a = 2$ and thus $\alpha = \frac{a+2}{a+3} = \frac{4}{5} < 1$. Let $C_0 \in \omega$ and $c_0 \in \mathbb{Q},\ c_0 > 0$, be as in the conclusion of Lemma 4.7 above with $a = 2$. Choose $k_0 \in \omega$ large enough so that

$$C_0 \cdot \sum_{k=1}^{\infty} 2^{-\frac{1}{2}c_0 k_0 k} < 1.$$

We define the bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ as follows. For every $\sigma \in T_0 \subseteq 2^{<\omega}$ such that $\sigma \in T_0 \setminus T_0^{k_0}$, put $\langle r_\sigma, \sigma\rangle \in \mathcal{R}$, where $r_\sigma \in \omega$ is the greatest integer that is less than or equal to $\frac{4}{5}|\sigma| + 1 \in \mathbb{Q}$. This is equivalent to setting $N_0 \in \omega$ to be the length of any leaf of $T_0^{k_0} \subseteq 2^{<\omega}$. We claim that the weight of $\mathcal{R}$ is bounded above by $C_0 \cdot \sum_{k=1}^{\infty} 2^{-\frac{1}{2}c_0 k_0 k} < 1$.

To see why this is the case, let $t > k_0$ be a substage of the construction of the clumpy tree $T_0 \subseteq 2^{<\omega}$. Now, by our construction of $T_0^t \subseteq 2^{<\omega}$ and $r_\sigma \in \omega$ above, Lemma 4.7, and the fact that $T_0^t \setminus T_0^{t-1} \subseteq D(\frac{1}{2}|\lambda|, 2)$, it follows that

$$\sum_{\substack{r\in\mathcal{R} \\ \pi_2(r)\in T_0^t\setminus T_0^{t-1}}} 2^{-\pi_1(r)} \leq \sum_{\sigma\in T_0^t\setminus T_0^{t-1}} 2^{-\frac{4}{5}|\sigma|} \leq C_0 2^{-c_0\frac{1}{2}|\lambda|},$$

where $\lambda \in T_0^{t-1}$ is any leaf of $T_0^{t-1} \subseteq 2^{<\omega}$, and hence $|\lambda| = 4^{t-1}$. By induction on $t > k_0$, it is easily verified that for every $t > k_0$ we have that $|\lambda| = 4^{t-1} \geq k_0(t-k_0)$, since it is easily verifiable for $t = k_0+1$, and the height of T_0^t quadruples at each successive stage $t > k_0$, while the quantity $k_0(t-k_0)$ at most doubles at each successive stage $t > k_0$, $t \in \omega$. Therefore, we have that

$$C_0 2^{-c_0\frac{1}{2}|\lambda|} \leq C_0 2^{-\frac{1}{2}c_0 k_0(t-k_0)},$$

which is the $(t-k_0)^{th}$ term in the sum

$$C_0 \cdot \sum_{k=1}^{\infty} 2^{-\frac{1}{2}c_0 k_0 k} < 1.$$

Thus, we count the weight of the node $\sigma \in T_0^t \setminus T_0^{t-1}$, $t > k_0$, in the $(t-k_0)^{th}$ term of the sum above. It follows that the total weight of our bounded request set $\mathcal{R}$ is strictly less than one. □

The proof of Proposition 4.10 (below) is similar to that of Proposition 4.9 (above). The main difference being that we apply Lemma 4.8 in the proof of Proposition 4.10 where we applied Lemma 4.7 in the proof of Proposition 4.9. Fix $a \in \omega$ and recall that a *generalized clump* is a set of strings of the form $\sigma 2^{\leq a|\sigma|} \subseteq 2^{<\omega}$. The next proposition, i.e. Proposition 4.10 below, deals with "generalized clumpy trees" – i.e. trees that are built via generalized clumps, in the same way that clumpy trees are built via clumps.

Proposition 4.10. *Fix $a \in \omega$, $a > 0$, and set $\alpha = \frac{2a+2}{2a+3} \in \mathbb{Q}$, $\alpha < 1$. Let $T \subseteq 2^{<\omega}$ be a tree, constructed as follows. We have that $T = \cup_{k\in\omega} T^k$, where $T^k \subseteq 2^{<\omega}$ is a finite subtree of T for all $k \in \omega$ that is constructed (uniformly and computably) in stages ($k \in \omega$), and with the additional property that $T^0 \subseteq 2^{<\omega}$ is the downward closure of some particular $\sigma \in 2^{<\omega}$. Now, if at every stage of the construction of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$, $k \in \omega$, we define $l_k \in \omega$ to be the length of the longest leaf of the finite subtree $T^k \subseteq 2^{<\omega}$, and at stage $k+1$ we obtain the finite tree $T^{k+1} \supseteq T^k$ by:*

1. *extending each leaf λ of $T^k \subseteq 2^{<\omega}$ to a node $\tau_\lambda \in 2^{<\omega}$ such that $|\tau_\lambda| \geq 2l_k$, and*
2. *further extending these extensions τ_λ by some downwards closed subset of the finite generalized clump $2^{\leq a|\tau_\lambda|} \subseteq 2^{<\omega}$*

(note that we have divided the stage $k+1$ construction up into "two halves," which we will refer to in the proof below). Then the effective packing dimension of every $Z \in [T] \subseteq 2^\omega$ is bounded above by $\alpha < 1$.

Proof. First, we leave it to the reader to check that for every $k \in \omega$, if we denote the set of nodes that we added to $T = \cup_{t\in\omega} T^t$ during the second half of stage $k \in \omega$ and the first half of stage $k+1$ by $D_k \subseteq 2^{<\omega}$, then there exist $\vec{\rho_k} \in (2^{\geq l_k})^{2^{l_k}}$ and $\vec{f_k} \in (2^\omega)^{\sum_{i=1}^{2^{l_k}} 2^{a|\tau_i\rho_i|}}$ as in Lemma 4.8 above such that

$$D_k \subseteq D(l_k, a, \vec{\rho_k}, \vec{f_k}) \subseteq 2^{<\omega}.$$

This is straightforward to verify, and follows from the construction of $T = \cup_{k\in\omega} T^k$ given in the statement of the Lemma, as well as the definition of $D(l, a, \vec{\rho}, \vec{f}) \subseteq 2^{<\omega}$ given in the statement of Lemma 4.8 above.

Now, by Lemma 4.8 above, there exist constants $C_0 \in \omega$ and $c_0 \in \mathbb{Q}$, $c_0 > 0$, that are independent of $l_k \in \omega$ and such that

$$\sum_{\sigma\in D_k} 2^{-\alpha|\sigma|} \leq \sum_{\sigma\in D(l_k,a,\vec{\rho_k},\vec{f_k})} 2^{-\alpha|\sigma|} \leq C_0 2^{-c_0 l_k}.$$

Let $k_0 \in \omega$ be large enough so that for all $k \geq k_0$ we have that $k^2 \leq 2^k$, and

$$C_0 \sum_{k=1}^{\infty} 2^{-c_0 k_0 k} < 1.$$

We now construct a bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ by setting

$$\mathcal{R} = \{\langle r_\sigma, \sigma\rangle : \sigma \in D_k,\ k \geq k_0\},$$

where $r_\sigma \in \omega$ is the greatest integer that is less than or equal to $\alpha|\sigma| + 1 \in \mathbb{Q}$, for each $\sigma \in 2^{<\omega}$. Note that, for all $\sigma \in 2^{<\omega}$, we have that $r_\sigma \geq \alpha|\sigma|$. By our construction of $\mathcal{R}$ it follows that $T \setminus T^{k_0} \subseteq \pi_2(\mathcal{R})$, from which it follows that for every $Z \in [T] \subseteq 2^\omega$, and every initial segment of Z, $\sigma \subset Z$, $\sigma \in 2^{<\omega}$, that was added to $T = \cup_{k\in\omega} T^k$ *after* stage k_0, we have that the pair $\langle r_\sigma, \sigma\rangle$ is in $\mathcal{R}$. Hence, if we can show that $\mathcal{R}$ is indeed a bounded request set, then by our construction of $\mathcal{R}$ and T it will follow that the effective packing dimension of *every* $Z \in [T] \subseteq 2^\omega$ is bounded above by $\alpha \in \mathbb{Q},\ \alpha < 1$.

Note that since our construction of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$ is assumed to be uniformly computable in $k \in \omega$, it follows that T is a computable tree and by our construction of $\mathcal{R} \subseteq \omega \times T$ above it also follows that $\mathcal{R}$ is a computable set. Also note that, by our construction of $\mathcal{R}$ above, we have that the weight of $\mathcal{R}$ is given by

$$\sum_{k=k_0}^{\infty} \sum_{\sigma\in D_k} 2^{-r_\sigma} \leq \sum_{k=k_0}^{\infty} \sum_{\sigma\in D_k} 2^{-\alpha|\sigma|} \leq \sum_{k=k_0}^{\infty} C_0 2^{-c_0 l_k}.$$

Now, note that our hypothesis (1) on the construction of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$ given in the statement of the current proposition above says that, at the very least, we double the length of every leaf at every step of the construction of $T = \cup_{k\in\omega} T^k$, and hence (by induction) it follows that $l_k \geq 2^k$, for all $k \in \omega$. Therefore, since $k_0 \in \omega$ was chosen large enough so that for all $k \geq k_0$ we have $2^k \geq k^2 \geq k_0 k$, it follows that the weight of our request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ is bounded (above) by

$$C_0 \sum_{k=k_0}^{\infty} 2^{-c_0 2^k} \leq C_0 \sum_{k=k_0}^{\infty} 2^{-c_0 k_0 k} \leq C_0 \sum_{k=1}^{\infty} 2^{-c_0 k_0 k} < 1.$$

Hence $\mathcal{R}$ is a bounded request set. □

Proposition 4.10 above is the most impotant result of this section, and will play a major role at the end of the next section, where we use it to bound the effective packing dimension of every $Z \in [\Phi_e^T] \subseteq 2^{<\omega}$, for some given clumpy tree $T \subseteq 2^{<\omega}$ and $e \in \omega$. Although, we should remark that Proposition 4.10 is slightly more general than we actually need in this article.

We have now completely analyzed the stage zero construction of $X \in 2^\omega$. We begin with the clumpy tree $T_0 \subseteq 2^{<\omega}$ that we constructed early in this section. We have shown, via Lemma 3.3 which was key in proving Lemma 4.2 above, that there exists $X_0 \in [T_0] \subseteq 2^\omega$ such that the effective packing dimension of X_0 is at least $\frac{1}{2} > 0$. We will use a similar argument in the proof of Theorem 6.1 to show that the real $X = \cup_{s\in\omega} \xi_s \in 2^\omega$ mentioned in Theorem 6.1, that we construct in the next section, has effective packing dimension at least $\frac{1}{4} > 0$. We then proved a key lemma (i.e. Lemma 4.7), which we later used to prove Proposition 4.9 above. Proposition 4.9 says that the effective packing dimension of *every* $X \in [T_0] \subseteq 2^{<\omega}$ is bounded above by $\frac{4}{5} < 1$. In the following section we will use the most important result of this section, Proposition 4.10, which is similar to Proposition 4.9, to help us show that *every* $Y \in 2^\omega,\ Y \leq_T X$, has effective packing dimension strictly less than one. In fact, the only content of this section that is necessary for a complete understanding of this article is our construction of the initial clumpy tree $T_0 \subseteq 2^{<\omega}$ and Proposition 4.10. Everything else is superfluous and has been included only to help the reader better understand the statements and proofs of these facts, as well as the lemmas and theorems to come in the next section. The next section explains the construction of the string $\xi_{s+1} \supseteq \xi_s$, $\xi_{s+1} \in 2^{<\omega}$, and the tree $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$ at stage $s+1 > 0$. Recall that we will have $X = \cup_{s\in\omega} \xi_s \in \cap_{s\in\omega} T_s \subseteq 2^\omega$, where $X \in 2^\omega$ is the real mentioned in our main theorem, i.e. Theorem 6.1.

5. Stage $s+1>0$

At stage $s+1>0$, we assume that we are given a finite string $\xi_s \in 2^{<\omega}$ and a pruned clumpy tree $T_s \subseteq T_0 \subseteq 2^{<\omega}$ such that $\xi_s \in T_s$ is extendible (recall that we will have $\xi_s \subset X \in [T_s] \subseteq 2^\omega$) and such that whenever $A = A \cap \sigma 2^{\leq|\sigma|} \subseteq \sigma 2^{\leq|\sigma|} \subseteq 2^{<\omega}$, $\sigma 2^{\leq|\sigma|} \subseteq T_0$, $\sigma \in A$, is a pruned clump on T_s, then A contains at least 2^{q_σ}-many leaves of $\sigma 2^{\leq|\sigma|}$, where $q_\sigma \in \omega$ is the smallest number that is greater than or equal to $(1-\sum_{k=1}^{s} 2^{-2k})|\sigma| \geq \frac{1}{2}|\sigma|$. The main goal of our construction at stage $s+1>0$ is to produce a finite string $\xi_{s+1} \in 2^{<\omega}$, $\xi_{s+1} \supset \xi_s$, and a pruned clumpy tree $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$ such that $\xi_{s+1} \in T_{s+1}$ is extendible and $K(\xi_{s+1}) \geq \frac{1}{4}|\xi_{s+1}| - 1$. Furthermore, we will construct the tree $T_{s+1} \subseteq T_s$ so that for all $Z \in [T_{s+1}] \subseteq 2^\omega$ we have that $\xi_{s+1} \subset Z$ and the effective packing dimension of Φ_s^Z is strictly less than one whenever $\Phi_s^Z \in 2^\omega$ is a total Turing reduction relative to the oracle $Z \in 2^\omega$. In the end we will set $X = \cup_{s\in\omega}\xi_s \in 2^\omega$ and we will have that $\{X\} = \cap_{s\in\omega}[T_s] \subseteq 2^\omega$, where $X \in 2^\omega$ is as in Theorem 6.1. Most of this section will be devoted to constructing the clumpy tree $T_{s+1} \subseteq 2^{<\omega}$, and verifying that it satisfies the properties mentioned earlier in this paragraph. In the next section we will put these results together to prove our main theorem (i.e. Theorem 6.1).

We must be somewhat careful when constructing $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$ to ensure that T_{s+1} is still "clumpy enough" to apply Lemma 3.3 so that we may find a string $\xi_{s+1} \in T_{s+1}$, $\xi_{s+1} \supset \xi_s$, of high enough relative complexity and in the end conclude that the effective packing dimension of $X = \cup_{s\in\omega}\xi_s \in 2^\omega$ is bounded below by $\frac{1}{4} > 0$. However, we must also ensure that there are few enough nodes on our trees $\{T_s\}_{s\in\omega}$ to guarantee that for any given $s \in \omega$ and $Z \in [T_s] \subseteq 2^\omega$, whenever $\Phi_s^Z \in 2^\omega$ is a total Turing reduction relative to Z, we have that the effective packing dimension of Φ_e^Z is strictly less than one. Thus, in constructing the tree T_{s+1} from the tree T_s, we shall omit some clumps of T_s, while ensuring that we leave enough clumps on T_{s+1} to guarantee the existence of strings of high enough relative complexity to be able to construct the real $X = \cup_{s\in\omega}\xi_s \in \cap_{t\in\omega}[T_t] \subseteq 2^\omega$ of Theorem 6.1.

5.1. **Constructing $\xi_{s+1} \supset \xi_s$.** Under the assumptions made in the first sentence of this section (above), we may apply Corollary 3.3 to construct $\xi_{s+1} \in 2^{<\omega}$, $\xi_{s+1} \supset \xi_s$, $x_{s+1} \in T_{s+1}$, such that

$$K(\xi_{s+1}) \geq \frac{1}{4}|\xi_{s+1}| - 1.$$

The entire construction of $T_{s+1} \subseteq T_s$ will take place above $\xi_{s+1} \in T_s$. In other words, for all $\sigma \in T_{s+1}$ such that $|\sigma| \leq |\xi_{s+1}|$ we will have that $\sigma \subseteq \xi_{s+1}$.

5.2. **Constructing $T_{s+1} \subseteq T_s$.** The main purpose of constructing T_{s+1}, $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$, $T_{s+1} = \cup_{k\in\omega} T_{s+1}^k$, is to guarantee (i.e. force) that for every $Z \in [T_s] \subseteq 2^\omega$ such that $\Phi_s^Z \in 2^\omega$ is a total Turing reduction, we have that the effective packing dimension of $\Phi_s^Z \leq_T Z$ is strictly less than one. We now proceed with the construction of the pruned clumpy tree $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$.

The entire construction of $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$, $T_{s+1} = \cup_{k\in\omega} T_{s+1}^k$, takes place above the node $\xi_{s+1} \in 2^{<\omega}$ that we specified in the previous subsection. In other words, for all $\rho \in 2^{<\omega}$ such that $|\rho| \leq |\xi_{s+1}|$, we have that $\rho \in T_{s+1}$ if and only if $\rho \subseteq \xi_{s+1}$. Throughout the rest of this section we work above $\xi_{s+1} \in 2^{<\omega}$. In fact, we let $\lambda_{s+1} \in 2^{<\omega}$, $\lambda_{s+1} \supseteq \xi_{s+1}$, be a fixed extendible leaf of $T_s^k \subseteq 2^{<\omega}$ extending $\xi_{s+1} \in 2^{<\omega}$, where $k \in \omega$ is the least number such that $\xi_{s+1} \in T_s^k$, and we work above λ_{s+1}. We will eventually show (by induction) that all nodes in T_s are extendible, and so $\lambda_{s+1} \in T_s$, $\lambda_{s+1} \supseteq \xi_{s+1}$, can be chosen arbitrarily. The construction of $T_{s+1} \subseteq T_s$ is divided up into two cases. Set $T_{s+1}^0 \subseteq 2^{<\omega}$ to be the downward closure of $\lambda_{s+1} \in 2^{<\omega}$.

5.2.1. *Case 1.* The first case says that there exists an (extendible) node $\rho \in T_s$, $\rho \supseteq \lambda_{s+1} \supset \xi_{s+1}$, and $x \in \omega$, such that for all $\tau \in T_s$, $\tau \supseteq \rho$, we have that $\Phi_{s,|\tau|}^{\tau}(x)\uparrow$. Let $k+1>0$

be given, let $l \in \omega$ be the length of the longest leaf of the finite tree $T^k_{s+1} \subseteq 2^{<\omega}$. We now construct the finite tree $T^{k+1}_{s+1} \supseteq T^k_{s+1}$ as follows. Let $T^{k+1}_{s+1} \supseteq T^k_{s+1}$ be the finite tree obtained by first extending every (extendible) leaf $\lambda_0 \in 2^{<\omega}$ of the finite tree $T^k_{s+1} \subseteq 2^{<\omega}$ to a string $\lambda \supseteq \lambda_0$, $\lambda \in 2^{<\omega}$, such that λ is the root of some pruned clump $A \subseteq \lambda 2^{\leq|\lambda|} \subseteq T_s$, $\lambda \in A$, and $2^{-2s-2}|\lambda| \geq 4l \geq 4|\lambda_0|$, and then including the (downward closure of the) pruned clump A in T^{k+1}_{s+1}. Now, by our construction of $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq 2^{<\omega}$ above, it follows that every real $Z \in [T_{s+1}] \subseteq 2^\omega$ extends the finite binary string $\rho \in 2^{<\omega}$. Therefore, by definition of ρ it follows that Φ^Z_s is partial for every $Z \in [T_{s+1}] \subseteq 2^\omega$, and so our construction of $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$, $T_{s+1} = \cup_{k\in\omega} T^k_{s+1}$, has forced Φ^X_s to be a partial reduction (since we will have that $X \in \cap_{t\in\omega}[T_t] \subseteq [T_{s+1}] \subseteq 2^\omega$. Also, by our construction of $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$, it is not difficult to verify that T_{s+1} is indeed a pruned clumpy subtree of T_s.

5.2.2. *Case 2.* The second case says that there does not exist $\rho \in T_s$ as defined in the previous paragraph. In this case it is not difficult to see that for every (extendible) $\rho \in T_s$ such that $\rho \supseteq \lambda_{s+1}$ and every $x \in \omega$, there exists $\tau \in T_s$, $\tau \supseteq \rho$, such that $\Phi^\tau_{s,|\tau|}(x)\downarrow$. The key fact about case two is that it allows us to uniformly and computably calculate $\Phi^\rho_s \in 2^{<\omega}$, for every (extendible) node $\rho \in T_s$. To compute Φ^ρ_s, $\rho \in T_s$, $s \in \omega$, use the hypothesis that we are in case two and ρ is extendible[4] to (uniformly and computably) find a node $\tau \supseteq \rho$, $\tau \in 2^{<\omega}$, such that $\Phi^\tau_{s,|\tau|}(|\rho|+1)\downarrow$. Then search for the *shortest* substring of τ, call it $\sigma \in T_s \subseteq 2^{<\omega}$, $\sigma \subseteq \tau$, such that the use of Φ^σ_s exceeds $|\rho|$ (it follows from our choice of $\tau \in 2^{<\omega}$ and Remark 2.1 (2) that $\sigma = \tau$ is an initial segment of τ that has this property, though it may not be the *shortest* such segment). It follows that $\Phi^\rho_s = (\Phi^\sigma_s)^- \in 2^{<\omega}$. Therefore, we can uniformly compute $\Phi^\rho_s \in 2^{<\omega}$, for any given $\rho \in T_s$.

Next, we specify how to prune the given clumpy tree $T_s \subseteq 2^{<\omega}$ to obtain the clumpy tree $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$. Our construction of $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq T_s$ proceeds in substages $k \in \omega$. Recall that we have already defined T^0_{s+1} to be the downward closure of $\lambda_{s+1} \in T_s$.

At stage $k+1 > 0$ of the construction of $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq T_s \subseteq 2^{<\omega}$, we construct T^{k+1}_{s+1} by extending the leaves of T^k_{s+1}. Let $l \in \omega$ be the length of the longest leaf of the finite tree $T^k_{s+1} \subseteq 2^{<\omega}$, and let λ_0 be a fixed (extendible) leaf of T^k_{s+1}. First, we extend λ_0 to a node $\lambda \supseteq \lambda_0$, $\lambda \in T_s \subseteq 2^{<\omega}$, such that λ is the root of a pruned clump on T_s of the form $C_\lambda = C_\lambda \cap \lambda 2^{\leq|\lambda|} \subseteq 2^{<\omega}$ and we have that $2^{-2s-2}|\lambda| \geq 4l \geq 4|\lambda_0|$ and $|\lambda| = 4^n = 2^{2n} \in \omega$, for some $n \geq s+1$, $n \in \omega$. Let $L_\lambda \subseteq C_\lambda$ denote the leaves of C_λ that are also leaves of $\lambda 2^{\leq|\lambda|}$, and recall that, by our assumptions on T_s (i.e. the first sentence of this section) we have that $|L_\lambda| \geq 2^{(1-\sum_{j=1}^{s} 2^{-2j})|\lambda|}$ (note that, by our assumptions on λ, we have that $(1-\sum_{j=1}^{s} 2^{-2j})|\lambda| \in \omega$). Since we are in case two, for every $\rho \in L_\lambda$ we can uniformly and effectively extend ρ to a node $\rho' \supseteq \rho$, $\rho' \in T_s \subseteq 2^{<\omega}$, such that $|\Phi^{\rho'}_s| \geq 2^{-2s-2}|\lambda| \in \omega$. Now, since $2^{-2s-2}|\lambda| \in \omega$ (again, this follows from our assumptions on $\lambda \in T_s$) there are at most $2^{2^{-2s-2}|\lambda|}$-many nodes in the set $2^{=2^{-2s-2}|\lambda|} \subseteq 2^{<\omega}$. Now, by the pigeonhole principle[5], it follows that there exists a node $\tau \in 2^{=2^{-2s-2}|\lambda|}$ such that there are at least $2^{(1-\sum_{j=1}^{s+1} 2^{-2j})|\lambda|}$-many strings $\rho \in L_\lambda \subseteq 2^{<\omega}$ that have an extension $\rho' \in T_s$ such that $\Phi^{\rho'}_s \supseteq \tau$ (again, note that by our assumptions on λ we have that $(1-\sum_{j=1}^{s+1} 2^{-2j})|\lambda| \in \omega$). Call the set of strings $\rho' \in T_s \subseteq 2^{<\omega}$ of the previous sentence L^0_λ. We define T^{k+1}_{s+1} to be the downward closure of the set of nodes L^0_λ, as λ_0 (and $\lambda \supseteq \lambda_0$) varies over all extendible leaves of T^k_{s+1}. This ends the construction of the finite tree $T^{k+1}_{s+1} \subseteq 2^{<\omega}$, and also ends the construction of $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq 2^{<\omega}$.

Remark 5.1. *Note that the oracle $\emptyset''$ can distinguish between case one and case two, and therefore the sequence of computable trees $\{T_s\}_{s\in\omega}$ can be made computable in $\emptyset''$. Since $\emptyset''$*

[4]We will eventually show that all nodes of the tree $T_s \subseteq 2^{<\omega}$ are extendible.

[5]A similar argument involving the pigeonhole principle is used in [DN] to bound the effective packing dimension of reals.

can also compute $K(\sigma) \in \omega$, uniformly in $\sigma \in 2^{<\omega}$, it follows that the real $X = \cup_{s \in \omega} \xi_s \in 2^\omega$ of Theorem 6.1 can be chosen to be computable in $\emptyset''$. We do not know for sure whether or not this bound on the complexity of X can be improved to $\emptyset'$, although we conjecture that a $\emptyset'$ upper bound for X is attainable.

5.3. **Some basic facts about $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$.** We now verify that $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$ satisfies various properties by mathematical induction. We only consider the base case T^0_{s+1}, and the induction step when T_{s+1} is constructed via case two. The case when T_{s+1} is constructed via case one is always easier than when T_{s+1} was obtained via case two.

The first thing to note about our construction of $T_{s+1} \subseteq T_s \subseteq 2^{<\omega}$ is that T_{s+1} is a pruned clumpy tree. This is because at every substage $k+1 \in \omega$, we add a pruned clump to T^{k+1}_{s+1} above every (extendible) leaf of T^k_{s+1}. Also note that by our construction of $T_{s+1} = \cup_{k \in \omega} T^k_{s+1} \subseteq 2^{<\omega}$, every node $\sigma \in T_{s+1}$ is extendible whenever the same is true of T_s. Since every node of T_0 is extendible (by construction), it follows by induction that every node $\sigma \in T_s$ is extendible, for all $s \in \omega$.

Note that by our construction of $T_{s+1} \subseteq T_s$ above it follows that if for every pruned clump of the form $A = A \cap \sigma 2^{\leq |\sigma|} \subseteq T_s \subseteq 2^{<\omega}$, $\sigma 2^{\leq |\sigma|} \subseteq T_0 \subseteq 2^{<\omega}$, $\sigma \in A \subseteq 2^{<\omega}$, A contains at least $2^{(1-\sum_{j=1}^{s} 2^{-2j})|\sigma|}$-many leaves of $\sigma 2^{\leq |\sigma|}$, then the same is true of every clump in the tree T_{s+1} with the exponent $(1 - \sum_{j=1}^{s} 2^{-2j})|\sigma| \geq \frac{1}{2}|\sigma|$ replaced by $(1 - \sum_{j=1}^{s+1} 2^{-2j})|\sigma| \geq \frac{1}{2}|\sigma|$. In either case, however, we have the uniform lower bound of $\frac{1}{2}|\sigma|$. Since the tree $T_0 \subseteq 2^{<\omega}$ obviously satisfies this condition with $s = 0$, then it follows by induction that every T_s, $s \in \omega$, satisfies this condition also. Therefore, we were justified when applying Corollary 3.3 to the sequence of pruned clumpy trees $T_s \subseteq 2^{<\omega}$, $s \in \omega$, to produce strings ξ_s, $s \in \omega$, such that $K(\xi_s) \geq \frac{1}{4}|\xi_s| - 1$ in Subsection 5.1 above. Generally speaking, this shows that in constructing T_{s+1} from T_s, we did not remove too many nodes in the process and are therefore able to find strings on T_s of relatively high complexity.

Finally, we claim that we can arrange it so that at every substage $k \in \omega$ of the construction of $T_{s+1} = \cup_{k \in \omega} T^k_{s+1}$ we choose the strings $\lambda \supseteq \lambda_0$ (above) so that the length of $\lambda \supseteq \lambda_0$ is independent of $\lambda, \lambda_0 \in 2^{<\omega}$ and depends only on $k \in \omega$. The proof of this claim is by mathematical induction. By our construction of $T_0 = \cup_{k \in \omega} T^k_0 \subseteq 2^{<\omega}$, the claim is true for the tree T_0. Now, assume that the claim holds for the tree T_s; we aim to show that it holds for T_{s+1} as well. Note that the conditions that define $\lambda \supseteq \lambda_0$ only depend on the (largeness of the) length of λ and the fact that λ is the root of some clump in T_0; they do not mention any other properties of λ. Now, since there are finitely many leaves in any T^k_{s+1}, and since every leaf of $T^{k'}_s$ is the initial segment of some pruned clump of $T^{k'+1}_s$, for all $k' \in \omega$, we can wait until some finite tree $T^{k'}_s$ presents itself with pruned clumps above every leaf $\lambda_0 \in T^k_{s+1}$ whose roots are all of sufficiently large and equal length to satisfy the requirements that we imposed on $\lambda \supseteq \lambda_0$. At this point we extend every leaf $\lambda_0 \in T^k_{s+1}$ to a corresponding node $\lambda \supseteq \lambda_0$ such that $\lambda 2^{\leq |\lambda|} \subseteq T_0 \subseteq 2^{<\omega}$, $|\lambda| \in \omega$ is independent of λ_0, and $|\lambda|$ is sufficiently large to satisfy the requirements that we imposed on λ. The claim now follows. Note that the sequence of uniform lengths $|\lambda_k| \in \omega$, $k \in \omega$, that we have just finished choosing for $\lambda_k = \lambda \supset \lambda_0$ during the construction of $T^k_{s+1} \supseteq T^{k-1}_{s+1}$, $k > 0$, is computable, uniformly in k. Also, by our construction of $\lambda_{k+1} \supseteq \lambda_0$ above (i.e. since we required that $2^{-2s-2}|\lambda_{k+1}| \geq 4l \geq 2|\lambda_0|$ at stage $k+1 > 0$ of our construction of $T_{s+1} = \cup_{k \in \omega} T^k_{s+1} \subseteq 2^{<\omega}$ above), it follows that $2^{-2s-2}|\lambda_{k+1}| \geq 4|\lambda_k|$, where $\lambda_k = \lambda \in T^k_{s+1} \subseteq 2^{<\omega}$ is as in the previous sentence.

We have now completed the construction of $T_{s+1} \subseteq T_s \subseteq T_0 \subseteq 2^{<\omega}$ and established some basic facts about the construction, one of which allows us to find the strings ξ_s, $s \in \omega$, in Subsection 5.1 above such that $X = \cup_{s \in \omega} \xi_s \in 2^\omega$ is the real of our main theorem. In the next subsection we will prove a key lemma that will eventually help us to prove the main theorem by verifying that for every $s \in \omega$ and every $Z \in [T_s] \subseteq 2^\omega$, the effective packing dimension of

Φ_s^Z is strictly less than one whenever $\Phi_s^Z \in 2^\omega$ is a total Turing reduction relative to $Z \in 2^\omega$, i.e. $\Phi_s^Z \leq_T Z$.

5.4. **Bounding the effective packing dimension of every** $Z \in [\Phi_s^{T_{s+1}}] \subseteq 2^\omega$**.** The current subsection is entirely devoted to proving the following hard lemma.

Lemma 5.2. *For every $s \in \omega$ such that the pruned clumpy tree $T_{s+1} \subseteq 2^{<\omega}$ was constructed via case two (see above for details), there exists a rational number $0 \leq \alpha_{s+1} < 1$ such that the effective packing dimension of every $Z \in [\Phi_s^{T_{s+1}}] \subseteq 2^\omega$ is at most α_{s+1}.*

Proof. Set $a_{s+1} = 2^{2s+2} \in \omega$, and $\alpha_{s+1} = \frac{2a_{s+1}+2}{2a_{s+1}+3} = \frac{2^{2s+3}+2}{2^{2s+3}+3} \in \mathbb{Q}$, $0 \leq \alpha_{s+1} < 1$. We will use the proof of Proposition 4.10 above to prove the current lemma by weighing the tree of nodes $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$ against a certain generalized clumpy tree (constructed via generalized clumps of the form $\sigma 2^{\leq a|\sigma|} \subseteq 2^{<\omega}$, where $a = a_{s+1} = 2^{2s+2} \in \omega$) that we will construct and that will satisfy the hypotheses of Lemma 4.10. More specifically, we shall construct a generalized clumpy tree $T \subseteq 2^{<\omega}$ that satisfies the hypotheses of Proposition 4.10 and such that for every $l \in \omega$, there are at least as many strings of length l on $T \subseteq 2^{<\omega}$ as there are on $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$. It will then follow that, for every $n \in \omega$, the α_{s+1}-weight of the set $V_n = \{\sigma \in T : |\sigma| \geq n\} \subseteq 2^{<\omega}$ is at least that of $W_n = \{\sigma \in \Phi_s^{T_{s+1}} : |\sigma| \geq n\} \subseteq 2^{<\omega}$, and therefore if we can show that for some $n \in \omega$, the α_{s+1}-weight of V_n is bounded above by one, then the same must be true of W_n. The proof of Proposition 4.10 applied to the generalized clumpy tree T will allow us to conclude that for some $n \in \omega$, the α_{s+1}-weight of V_n is bounded above by one.

We construct the tree $T \subseteq 2^{<\omega}$ in stages $T = \cup_{k\in\omega} T^k$, $T^k \subseteq T^{k+1} \subseteq 2^{<\omega}$, as follows. First, let $l_1 < l_2 < l_3 < \cdots < l_k < \cdots$ be an infinite computable sequence of numbers such that $l_k \in \omega$ is the length of the roots of the pruned clumps that we added to $T_{s+1} = \cup_{k\in\omega} T_{s+1}^k$ at substage $k \in \omega$, $k > 0$, of the construction of T_{s+1} (we showed that $l_k \in \omega$ is well-defined and uniformly computable; see the second last paragraph of Subsection 5.3 above for more details), and let $m_1 < m_2 < \cdots$ be such that $m_i = 2^{-2s-2} l_i \in \omega$, for all $i \in \omega$, $i > 0$. First, we set $T^0 \subseteq 2^{<\omega}$ to be the downward closure of the finite binary string $0^{m_1} \in 2^{<\omega}$. Then, for all $k+1 > 0$, we obtain the finite tree $T^{k+1} \supseteq T^k$ by first including the generalized clump $\lambda 2^{\leq a_{s+1}|\lambda|} \subseteq 2^{<\omega}$ in T^{k+1}, for every leaf λ of T^k, where $a_{s+1} = 2^{2s+2} \in \omega$, and then extending every leaf of $\lambda 2^{\leq a_{s+1}|\lambda|} \subseteq T^{s+1}$, $\lambda \in 2^{<\omega}$ a leaf of T^k, by a string of zeros so that the leaves of $T^{k+1} \subseteq 2^{<\omega}$ are all of length (exactly) $m_{k+2} \in \omega$. This ends the construction of T^{k+1}, and of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$. Note that by our construction of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$ above we have $a_{s+1}|\lambda| = 2^{2s+2} m_{k+1} = l_{k+1}$, for all $k \in \omega$, where $\lambda \in 2^{<\omega}$ is a leaf of T^k (as above).

Now, by our construction of $l_{k+1} \in \omega$ above, it follows that $m_{k+1} = \frac{l_{k+1}}{2^{2s+2}} \geq 4l_k$ (we proved this in the previous subsection). Therefore, by the last sentence of the previous paragraph it follows that after we include the generalized clumps $\lambda 2^{\leq a_{s+1}|\lambda|} = \lambda 2^{\leq a_{s+1} m_{k+1}} = \lambda 2^{\leq l_{k+1}} \subseteq 2^{<\omega}$ in our tree $T^{k+1} \subseteq 2^{<\omega}$, $T^{k+1} \supseteq T^k$, we (at least) double the length of each of the leaves of the resulting intermediate tree to obtain T^{k+1} (note that by construction the leaves of the intermediate tree have length $m_{k+1} + l_{k+1} \leq 2l_{k+1}$). It now follows that $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$ satisfies the hypotheses of Proposition 4.10 with $a = a_{s+1} = 2^{2s+2} \in \omega$. Therefore, the proof of Proposition 4.10 applies to the tree $T \subseteq 2^{<\omega}$, and produces a bounded request set $\mathcal{R} \subseteq \omega \times 2^{<\omega}$ and a number $N \in \omega$ such that for all $\sigma \in T$, $|\sigma| \geq N$, we have that $\langle r_\sigma, \sigma \rangle \in \mathcal{R}$, where $r_\sigma \in \omega$, $r_\sigma \leq \alpha_{s+1}|\sigma| + 1 = \frac{2^{2s+3}+2}{2^{2s+3}+3}|\sigma| + 1 = \frac{2a_{s+1}+2}{2a_{s+1}+3}|\sigma| + 1$. Now, since the tree $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$ is computable, (as we explained in the first paragraph of the proof of the current lemma above) to prove the current lemma it suffices to show that for all $l \in \omega$ there are at least as many nodes of length l in T as there are in $\Phi_s^{T_{s+1}}$. The proof of this fact is by induction on the stages $k \in \omega$ of our construction of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$. First, however, we prove an easier claim which says that for all $k \in \omega$, T^k has at least as many leaves as T_{s+1}^k.

To prove this easier claim, note that the base case is trivial since both T^0 and T_{s+1}^0 are constructed as the downward closures of some finite binary strings and therefore each have a single leaf. Now, at each subsequent stage $k+1$, by our constructions of $T = \cup_{k\in\omega} T^k \subseteq 2^{<\omega}$ and $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq 2^{<\omega}$, respectively, in passing from T^k to T^{k+1} or T^k_{s+1} to T^{k+1}_{s+1}, any single leaf $\lambda \in 2^{<\omega}$ is extended by exactly $2^{l_{k+1}}$-many incomparable nodes in the former case, and at most $2^{l_{k+1}}$-many nodes in the latter case. The claim now follows by induction on $k \in \omega$. In the next two paragraphs we will use what we have shown in the current paragraph to prove that for all $l \in \omega$ there are at least as many nodes of length l in T as there are in $\Phi_s^{T_{s+1}}$.

At stage $k = 0$ we set $T = 0^{m_1} \in 2^{<\omega}$ (above). Now, note that $T^0_{s+1} \subseteq 2^{<\omega}$ is the downward closure of a single $\lambda_{s+1} \in 2^{<\omega}$, $\lambda_{s+1} \supseteq \sigma_{s+1}$, therefore $\Phi^{T^0_{s+1}} \subseteq 2^{<\omega}$ is the downward closure of some $\sigma \in 2^{<\omega}$ as well. Furthermore, note that to construct T^1_{s+1} from T^0_{s+1} we first extend the leaf $\lambda_{s+1} \in T^0_{s+1} \subseteq 2^{<\omega}$ to a node $\lambda \in T_s$, $\lambda \supseteq \lambda_{s+1}$, such that $|\lambda| = l_1 \in \omega$ (where the computable sequence of numbers $l_1 < l_2 < \cdots$ is as in the previous two paragraphs), and then we prune the pruned clump $\lambda 2^{\leq|\lambda|} \cap T_s$ further so as to guarantee the existence of a string $\rho \in 2^{=m_1}$ that every real $f \in [T_{s+1}] \subseteq 2^\omega$ extending λ (which all reals in $[T_{s+1}]$ do, since we are considering the early stage $k = 1$ of the construction of $T_{s+1} = \cup_{k\in\omega} T^k_{s+1} \subseteq 2^{<\omega}$,) must satisfy $\Phi_s^f \supset \rho$. Therefore, it follows that for all $l \in \omega$ such that $l \leq m_1$, we have that $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$ has exactly one node of level l. This proves the base case.

For the induction step, let $k+1 > 0$ be given. We aim to show that for all $l \in \omega$ such that $m_{k+1} < l \leq m_{k+2}$ there are at least as many nodes of length l on $T^{k+1} \subseteq 2^{<\omega}$ as there are on $\Phi_s^{T^{k+2}_{s+1}} \subseteq 2^{<\omega}$. First note that (by what we have shown above) there are at least as many leaves on T^k as there are on T^k_{s+1}. Without any loss of generality we can assume for all $k \in \omega$ that the size of set of nodes $\{\Phi_s^\lambda : \lambda \text{ a leaf of } T^{k+1}_{s+1}\} \subseteq 2^{<\omega}$ is equal to that of $\{\lambda \in 2^{<\omega} : \lambda \text{ a leaf of } T^{k+1}_{s+1}\} \subseteq 2^{<\omega}$, since this assumption could only increase the number of nodes of $\Phi_s^{T_{s+1}}$ of any given level. Now, note that by our construction of $T^{k+1}_{s+1} \supseteq T^k_{s+1}$ above, an argument similar to that of the base case (i.e. by the way we pruned our clumps in passing from T^k_{s+1} to T^{k+1}_{s+1} in the construction of T_{s+1} above), and our assumption in the previous sentence, it follows that for all $k \in \omega$ the number of nodes of level $m_{k+1} \in \omega$ in $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$ is in fact equal to the number of leaves of T^k_{s+1}. Therefore, since there are at least as many leaves on T^k as there are on T^k_{s+1}, and (by construction) the leaves of T^k are all of length $m_{k+1} \in \omega$, then it follows that for any given number of the form $m_i \in \omega$, $i \in \omega$, $i > 0$, the tree T has at least as many nodes as $\Phi_s^{T_{s+1}} \subseteq 2^{<\omega}$ of length m_i. Now, the induction step follows from the fact that at stage $k+1$ of the construction of $T^{k+1} \supseteq T^k$, above every leaf $\lambda \in 2^{<\omega}$ of T^k we *first* include the generalized clump $\lambda 2^{\leq a_{s+1}|\lambda|}$ in T^{k+1}, and *then* we extend the leaves of the clump $\lambda 2^{\leq a|\lambda|}$ by a string of zeros. In other words, in our construction of $T^{k+1} \supseteq T^k$, the leaves of T^k split *as fast as possible.* This completes the proof of Lemma 5.2. □

6. The main theorem

We now state and prove the main theorem of this article, using our results of the previous two sections.

Theorem 6.1. *There exists $X \in 2^\omega$ of effective packing dimension at least $\frac{1}{4} > 0$ and such that for every $e \in \omega$ the effective packing dimension of Φ_e^X is strictly less than one whenever $\Phi_e^X \in 2^\omega$ is a total Turing reduction relative to X.*

Proof. Most of the work has already been done. Let

$$X = \bigcup_{k\in\omega} \xi_k \in 2^\omega,$$

where $\xi_k \in 2^{<\omega}$, $\xi_{k+1} \supset \xi_k$, $k \in \omega$, are as defined in Subsection 5.1 of the previous section. Recall that by our construction of $\xi_{s+1} \in 2^{<\omega}$, $\xi_{s+1} \supset \xi_s$, in the previous section we have that

$$K(\xi_s) \geq \frac{1}{4}|\xi_s| - 1,$$

for all $s \in \omega$, and $X \in \cap_{s\in\omega}[T_s] \subseteq 2^\omega$. Therefore, we have that the effective packing dimension of $X = \cup_{s\in\omega}\xi_s$ is given by

$$\limsup_{n\to\infty} \frac{K(X\restriction n)}{n} \geq \limsup_{s\to\infty} \frac{K(\xi_s)}{|\xi_s|} \geq \limsup_{s\to\infty} \frac{|\xi_s| - 4}{4|\xi_s|} = \frac{1}{4}.$$

Therefore, the effective packing dimension of $X = \cup_{s\in\omega}\xi_s \in 2^\omega$ is at least $\frac{1}{4} > 0$.

Assume now that $\Phi_e^X \in 2^\omega$, $e \in \omega$, $X \in \cap_{s\in\omega}[T_s] \subseteq 2^\omega$, is a total Turing reduction. Then we must have constructed the clumpy tree $T_{e+1} \subseteq 2^{<\omega}$ via case two (otherwise Φ_e^X would have been a partial Turing reduction). By Lemma 5.2 we have that for all $Z \in [\Phi_e^{T_{e+1}}] \subseteq 2^\omega$, the effective packing dimension of Z is at most $\alpha_{e+1} = \frac{2^{2e+3}+2}{2^{2e+3}+3} < 1$. Now, since $X \in \cap_{s\in\omega}[T_s] \subseteq [T_{e+1}] \subseteq 2^\omega$, it follows that $\Phi_e^X \in [\Phi_e^{T_{e+1}}] \subseteq 2^\omega$, and therefore the effective packing dimension of the real Φ_e^X is at most $\alpha_{e+1} < 1$. □

Finally, we wish to point out that our main theorem (i.e. Theorem 6.1 above) can be combined with the results of [FHP+06, BDS09, DH] that we discussed in the first section above in a simple way to yield the following stronger version of our main theorem.

Corollary 6.2. *For every real number $0 < d < 1$, there is a real $Z \in 2^\omega$ of effective packing dimension greater than or equal to d and such that for every $e \in \omega$ the effective packing dimension of Φ_e^Z is strictly less than one whenever $\Phi_e^Z \in 2^\omega$ is a total Turing reduction relative to Z.*

Proof. Let $0 < d < 1$ be any given real number, and let $X \in 2^\omega$ be as in the main theorem (i.e. Theorem 6.1) above. Now, by the results of [FHP+06, BDS09, DH] mentioned in the first section of this article, it follows that there is a real $Z \leq_T X$ such that Z has effective packing dimension greater than or equal to d. Furthermore, by our construction of $X \in 2^\omega$ above and the fact that $Z \leq_T X$, it follows that Z cannot Turing compute a real of effective packing dimension one. □

We end with the following question, which we alluded to in Remark 5.1 above. Recall that Remark 5.1 says that the real $X \in 2^\omega$ can be taken to be computable relative to $\emptyset''$.

Question 6.3. *Is there a real $X \in 2^\omega$, $X \leq_T \emptyset'$, as in Theorem 6.1 above?*

We conjecture that the answer to Question 6.3 above is "yes."

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