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EFFECTIVE PACKING DIMENSION OF Π^0_1-CLASSES

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ABSTRACT. We construct a Π^0_1-class X that has classical packing dimension 0 and effective packing dimension 1. This implies that, unlike in the case of effective Hausdorff dimension, there is no natural correspondence principle (as defined by Lutz) for effective packing dimension. We also examine the relationship between upper box dimension and packing dimension for Π^0_1-classes.

1. INTRODUCTION

A major theme of computability theory is the effectivization of classical mathematics. To do this one takes an existing (i.e. classical) mathematical notion and develops a new computability-theoretic analogue of that notion. Afterwards, one tries to determine the similarities and differences between the old classical notion and its new effective counterpart. This article examines the classical notion of packing dimension, as well as its effective counterpart which is called either *effective packing dimension* or *effective strong dimension*.

In [7] Lutz effectivized the notion of Hausdorff dimension to obtain the notion of effective Hausdorff dimension. Furthermore, he conjectured that for Hausdorff dimension there is a *correspondence principle*. By correspondence principle we mean a theorem which says that there is a certain (natural) class of sets whose classical and effective Hausdorff dimensions are equal. Hitchcock [5] found such a class by showing that if X is a union of Π^0_1-classes, then the classical and effective Hausdorff dimensions of X are the same (for more information on Π^0_1-classes see [10, 11]). This is a beautiful and useful result, because it allows one to compute the classical Hausdorff dimension of a set by determining its effective Hausdorff dimension, which, as is shown in [7], is the supremum of the effective Hausdorff dimensions of its individual points.

Later, Athreya, Hitchcock, Lutz, and Mayordomo [1] effectivized the classical notion of packing dimension to obtain the notion of effective packing dimension. They also wondered whether or not there existed a correspondence principle for this new notion of dimension. The main theorem of this article shows that there

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is a Π^0_1-class X that has classical packing dimension 0 (in fact X is countable) and effective packing dimension 1. Hence, there is no possibility for a reasonable correspondence principle of the same sort as the one for Hausdorff dimension.

The plan of the rest of the paper is as follows. Section 2 contains the necessary definitions and notational conventions. Section 3 consists of the proof of the main theorem which says that there is a Π^0_1-class X with classical packing dimension 0 and effective packing dimension 1. Finally, section 4 contains two theorems. The first proves that the effective packing dimension of a Π^0_1-class is always less than or equal to its upper box dimension. The second theorem shows that there is a Π^0_1-class X that has effective packing dimension 0 and upper box dimension 1.

For further information on computability theory, effective randomness, and dimension theory, consult [2, 3, 8, 9].

2. Definitions and notation

2.1. **Cantor space and Π^0_1-classes.**

In this article ω denotes the set of natural numbers, $2^{<\omega}$ denotes the set of finite binary sequences, and 2^ω denotes the set of infinite binary sequences (i.e. Cantor space). For any $\sigma \in 2^{<\omega}$, let $|\sigma|$ denote the length of σ. For any $\tau \in 2^{<\omega}$ and $n \in \omega$ we write $\mathbf{C}^\tau_n$ for the set of nodes $\{\tau\sigma \in 2^{<\omega} : 1 \leq |\sigma| \leq n\}$ — i.e. the "cone" above τ of length n. Also, let $\mathbf{C}^\tau$ denote the set $\{\sigma \in 2^{<\omega}\colon \sigma \text{ extends } \tau\}$. 2^n denotes the set of strings of length $n \in \omega$.

For all $f \in 2^\omega$ and $n \in \omega$, $f\restriction n$ denotes the first n bits of f. We write $\sigma \subseteq \tau$ to mean that $\sigma \in 2^{<\omega}$ is an initial segment of $\tau \in 2^{<\omega}$; in other words τ extends σ. Also, if $f \in 2^\omega$ and $\sigma \in 2^{<\omega}$ then $\sigma \subseteq f$ means that σ is an initial segment of f. A set $A \subseteq 2^{<\omega}$ is prefix-free if for any $\sigma, \tau \in A$ such that $\sigma \neq \tau$, we have $\sigma \nsubseteq \tau$. If $A \subseteq 2^{<\omega}$ and $k \in \omega$ then $A_{<k} = \{\sigma \in A : |\sigma| < k\}$. We denote the plain and prefix-free Kolmogorov complexity of a string $\sigma \in 2^{<\omega}$ by $C(\sigma)$ and $K(\sigma)$, respectively. For more information on plain and prefix-free Kolmogorov complexity see [6].

A set $X \subseteq 2^\omega$ is a Π^0_1-class if there is a computable tree $T \subseteq 2^{<\omega}$ such that X is the set of paths through T.

2.2. **Packing dimension.**

In this section we define the notion of classical packing dimension. For more information on classical packing dimension see [4].

For every $k \in \omega$ let $\mathcal{A}_k$ be the collection of prefix-free sets $A \subseteq 2^{<\omega}$ such that $A_{<k} = \emptyset$. For every $X \subseteq 2^\omega$ we now define

$$\mathcal{A}_k(X) = \left\{ A \in \mathcal{A}_k : X \subseteq \bigcup_{\alpha \in A} \mathbf{C}^\alpha \right\},$$

$$\mathcal{B}_k(X) = \{A \in \mathcal{A}_k : (\forall \alpha \in A)[\mathbf{C}^\alpha \cap X \neq \emptyset]\}.$$

$\mathcal{A}_k(X)$ is the set of all *covers* of X, while $\mathcal{B}_k$ denotes the set of all *packings* of X. For more information on covers and packings, see [4].

If $X \subseteq 2^\omega$, $s \in [0, \infty)$, and $k \in \omega$, then we define the quantity

$$P^s_k(X) = \sup_{B \in \mathcal{B}_k(X)} \sum_{\beta \in B} 2^{-s|\beta|}$$

which is decreasing in k, and so the limit

$$P^s_\infty(X) = \lim_{k\to\infty} P^s_k(X)$$

exists, though it may be infinite. We now define the *s-dimensional packing (outer) cylinder measure* of X:

$$P^s(X) = \inf\left\{\sum_{i=0}^{\infty} P^s_\infty(X_i) \ : \ X \subseteq \bigcup_{i=0}^{\infty} X_i\right\}.$$

Definition 2.1. The *packing dimension* of $X \subseteq 2^\omega$ is $\dim_{\mathrm{P}}(X) = \inf\{s \in [0,\infty) : P^s(X) = 0\}$.

Throughout this article we use a well-known characterization of packing dimension as a modified box dimension, which we define next.

2.3. **Modified box dimension.** For every $X \subseteq 2^\omega$, and $n \in \omega$, let

$$N_n(X) = |\{\sigma \in 2^n \ : \ (\exists f \in X)\sigma \subseteq f\}|.$$

Now, the *upper box dimension* of X is given by

$$\overline{\dim}_{\mathrm{B}}(X) = \limsup_{n\to\infty} \frac{\log(N_n(X))}{n}.$$

Though we will not mention it again, it is worth noting that X also has a lower box dimension $\underline{\dim}_{\mathrm{B}}(X)$, which is obtained by replacing lim sup by lim inf in the definition of upper box dimension. If $\underline{\dim}_{\mathrm{B}}(X) = \overline{\dim}_{\mathrm{B}}(X)$, then the box dimension of X $\dim_{\mathrm{B}}(X)$ is defined and equal to this number. As we will see in section 4, upper box dimensions are easy to compute, but poorly behaved. A more well-behaved notion is the *modified upper box dimension* of X

$$\overline{\dim}_{\mathrm{MB}}(X) = \inf\left\{\sup_i \overline{\dim}_{\mathrm{B}}(X_i) \ : \ X \subseteq \bigcup_{i=0}^{\infty} X_i\right\};$$

moreover, it is equal to the packing dimension of X. In fact, the following theorem is well-known (for a proof see [4]).

Theorem 2.2. *For all $X \subseteq 2^\omega$, $0 \le \dim_{\mathrm{H}}(X) \le \overline{\dim}_{\mathrm{MB}}(X) = \dim_{\mathrm{P}}(X) \le \overline{\dim}_{\mathrm{B}}(X) \le 1$.*

From now on we will make no distinction between the modified upper box dimension of X and the (classical) packing dimension of X.

2.4. **Packing dimension and s-gales.** In this section we define the effective packing dimension of a set $X \subseteq 2^\omega$. For a more complete guide to effective dimension theory which includes the definition of effective Hausdorff dimension, consult [1, 7].

Definition 2.3. Fix a number $s \in [0,\infty)$. An *s-supergale* is a function $d : 2^{<\omega} \to \mathbb{R}^{\ge 0}$ that satisfies, for all $\sigma \in 2^{<\omega}$, the following condition

$$d(\sigma) \ge 2^{-s}[d(\sigma 0) + d(\sigma 1)].$$

Replacing $\ge$ with $=$ in the definition above gives the definition for an *s-gale*. 1-gales are called martingales, and 1-supergales are called supermartingales. An s-gale or s-supergale is Σ^0_1 if it may be computably approximated from below by a uniform sequence of rational numbers.

Intuitively, s-gales are thought of as strategies for betting on the bits of some binary sequence $f \in 2^\omega$ (in order). In particular, if $\sigma \in 2^{<\omega}$ is an initial segment of f, then $d(\sigma)$ is the capital that one would have after placing $|\sigma|$-many bets. The parameter s is thought of as "fairness factor" because as s decreases it becomes more difficult to increase one's capital. Next we define what it means for s-gales and s-supergales to succeed strongly on a sequence $f \in 2^\omega$.

Definition 2.4. Let d be an s-gale or an s-supergale, for some $s \in [0, \infty)$. We say that d *succeeds strongly* on $f \in 2^\omega$ if

$$\liminf_{n\to\infty} d(f\restriction n) = \infty.$$

The strong success set of d is the set

$$S^\infty_{\mathrm{str}}[d] = \{f \in 2^\omega \ : \ d \text{ succeeds strongly on } f\}.$$

The following surprising result of Lutz gives a characterization of classical packing dimension in terms of s-gales.

Theorem 2.5 (Lutz)**.** *For any* $X \subseteq 2^\omega$*,*

$$\dim_{\mathrm{P}}(X) = \inf\left\{ s : \begin{array}{c} \text{there is an s} - \text{gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}.$$

Now, by effectivizing the notions of s-gales and s-supergales, we obtain the following definition of effective packing dimension.

Definition 2.6. The *effective packing dimension* of $X \subseteq 2^\omega$ is

$$\mathrm{cDim}(X) = \inf\left\{ s : \begin{array}{c} \text{there is a } \Sigma^0_1 \ s\text{−gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}.$$

For all $f \in 2^\omega$ define $\mathrm{Dim}(f) = \mathrm{cDim}(\{f\})$.

The following are two well-known and useful theorems about effective packing dimension. The first characterizes effective packing dimension of points in Cantor space in terms of the prefix-free Kolmogorov complexity of their initial segments. The second says that the effective packing dimension of a set $X \subseteq 2^\omega$ is the supremum of the dimensions of its individual points. In other words, effective packing dimension is absolutely stable.

Theorem 2.7. *For all* $f \in 2^\omega$*,*

$$\mathrm{cDim}(f) = \limsup_{n\to\infty} \frac{K(f\restriction n)}{n}.$$

Theorem 2.8. *For all* $X \subseteq 2^\omega$*,*

$$\mathrm{cDim}(X) = \sup_{f\in X} \mathrm{Dim}(f).$$

The next section is devoted to proving the main theorem of this article, which says that there is a Π^0_1-class X such that $\dim_{\mathrm{P}}(X) = 0$ and $\mathrm{cDim}(X) = 1$.

3. No correspondence principle for effective packing dimension of Π^0_1-classes

Theorem 3.1. *There exists a (countable)* Π^0_1*-class* X *such that* $\dim_{\mathrm{P}}(X) = \overline{\dim}_{\mathrm{MB}}(X)=0$ *and* $\mathrm{cDim}(X) = 1$.

Proof. We construct a computable tree $T = \cup_s T_s$, $T_s \subset T_{s+1}$, in stages such that X is the set of paths through T. Furthermore, T_{s+1} is obtained from T_s by extending the leaves of T_s. Also, every path of T will be computable, except for one distinguished path $t = \cup_i \tau_i \in X$ ($t \in 2^\omega$, $\tau_i \in 2^{<\omega}, \tau_i \subset \tau_{i+1}$), such that

$K(\tau_i) \geq (1-2^{-i})|\tau_i|$. From these facts it follows that X is countable, and therefore has a classical packing dimension of 0. However, since $t \in X$ has effective packing dimension 1, it follows from Theorems 2.7 and 2.8 that the same must hold of X. All that is left to do is build T and show that it has these properties. Though it does not necessarily follow from the construction below, we wish to note that the following construction can be modified so that $t \in 2^\omega$ is in fact a left-c.e. real.

The construction of T

Stage 0: Put the nodes $\emptyset, 0, 1,$ into T_0, and set $\tau_0^0 = 0$.

Stage $s+1$: Let $\tau = \tau_r^s$ for the largest $r \leq s$ for which τ_r^s is currently defined.

We begin by enumerating all $\sigma \in T_s$ into T_{s+1}. If $\lambda \in T_s$ is the length-lexicographically least leaf of T_s extending τ, we computably determine a number n such that there exists a node $\rho \in \mathbf{C}_n^\lambda$ such that $K(\rho) \geq (1-2^{-r-1})|\rho|$. Note that n can be determined effectively since, by the definition of plain Kolmogorov complexity C, and a simple counting argument it follows that for any $\lambda \in 2^{<\omega}$, $n \in \mathbb{N}$, if $M = \max\{C(\sigma) : \sigma \in \mathbf{C}_n^\lambda\}$ then $M \geq n$. Now, for any $\rho' \in \mathbf{C}_n^\lambda$ we have

$$(1-2^{-r-1})|\rho'| \leq (1-2^{-r-1})(n+r) \leq (1-2^{-r-1})(M+r) = M+r-2^{-r-1}(M+r).$$

Therefore, if $n \leq M$ is chosen large enough so that $r - 2^{-r-1}(M+r) < 0$ (recall that r is a known quantity and so the inequality can be effectively solved for n), then we have that

$$M \geq M + r - 2^{-r-1}(M+r) \geq (1-2^{-r-1})|\rho'|.$$

Hence, by definition of M, there is a string $\rho \in \mathbf{C}_n^\tau$ such that $C(\rho) \geq (1-2^{-r-1})|\rho|$. But we also have that $K(\rho) \geq C(\rho)$, and so $K(\rho) \geq (1-2^{-r-1})|\rho|$. Once n has been effectively determined, then for all $\sigma \in \mathbf{C}_n^\lambda$, put σ into T_{s+1}. Define τ_{r+1}^{s+1} to be the least length-lexicographic proper extension $\sigma \supset \tau = \tau_r$ on T_{s+1} such that $K_s(\sigma) \geq (1-2^{-r-1})|\sigma|$, where K_s is a (fixed) computable approximation to K.

Next, let $j > 0$ be the smallest number such that $K_s(\tau_j^s) \neq K_{s+1}(\tau_j^s)$, and set τ_i^{s+1} to be undefined for all $i \geq j$, while setting $\tau_i^{s+1} = \tau_i^s$ for all $i < j$. If such a j does not exist, do nothing. Finally, put all nodes of the form $\lambda' 0$ into T_{s+1}, where λ' ranges over the leaves of T_s not equal to λ. This ends the construction of T.

Lemma 3.2. *For every i, $\tau_i = \lim_s \tau_i^s$ exists. Furthermore, $\tau_i \in T$, and $\tau_{i-1} \subset \tau_i$ for every $i \geq 1$.*

Proof. The proof is by induction. Note that $\tau_0^s = 0$ for all stages of the construction, and so the base case holds. For the induction step, let s_0 be the last stage such that $\tau_{j-1}^{s_0}$ is undefined or 0 if no such stage exists, and suppose (as part of the inductive hypothesis) that at all stages $s > s_0$ $\tau_{j-1}^s = \tau_{j-1}^{s+1}$. Thus, for all $s > s_0$ it makes sense to write $\tau_{j-1} = \lim_s \tau_{j-1}^s$ instead of τ_{j-1}^s.

First note that (by the construction of T) $\tau_j^{s_0}$ is undefined, and furthermore, if $s > s_0$ is a stage at which τ_j^s is defined, then it must properly extend τ_{j-1} and it must also be a node of T_s. Hence, if $\tau_j = \lim_s \tau_j^s$ exists (as we will show in the next paragraph) then $\tau_j \supset \tau_{j-1}$ and $\tau_j \in T$.

Now, by the construction of T_{s+1}, there is a $\sigma \in T_{s+1}$ properly extending τ_{j-1} such that $K(\sigma) \geq (1-2^{-j})|\sigma|$. Let ρ be the length-lexicographically least such

σ, and let $s_1 > s_0$ be a stage by which $K_s(\rho)$ has settled. There are two cases to consider. First, if $\tau_j^{s_2}$ is undefined at some stage $s_2 > s_1$, then (by the construction of T) $\tau_j^{s_2+1}$ will be defined and set equal to ρ, and will remain equal to $\rho \supset \tau_{j-1}$ at all later stages. Otherwise, τ_j is defined at all stages $s \geq s_1$, and the limit of τ_j must exist since (by the construction of T) if $\tau_j^{s_1} \neq \tau_j^{s_2}$ for some $s_2 > s_1$, then there is a stage s, $s_1 < s < s_2$, such that τ_j^s is undefined. Note that in both cases we have shown that there is a final stage s such that τ_j^s is undefined, and for every $s' > s$ we have that $\tau_j^{s'} = \tau_j^{s'+1}$. This proves the lemma. □

Proposition 3.3. Let $t = \cup_i \tau_i \in 2^\omega$, then $t \in X$ and t has effective packing dimension 1.

Proof. The fact that $t \in X$ is trivial, by the definition of X and the previous lemma. Note that t has effective packing dimension 1 since for all i we have that $K(\tau_i) \geq (1 - 2^{-i})|\tau_i|$. Otherwise, if $K(\tau_i) < (1 - 2^{-i})|\tau_i|$, then there must be a least stage t_0 such that for all stages $t > t_0$, $K_t(\tau_i) < (1 - 2^{-i})|\tau_i|$. However, this implies that $\tau_i \neq \lim_{s\to\infty} \tau_i^s$, since the construction guarantees that at all stages $t > t_0$ we have that $\tau_i^s \neq \tau_i$, which is a contradiction. □

Proposition 3.4. Every $f \in X$ other than t is computable.

Proof. Let f be a path in X that is not equal to t. Then there is a number i such that $f \not\supseteq \tau_i$. It now follows that if s is large enough so that τ_i has settled by stage s and λ is the unique leaf of T_s such that $\lambda \subseteq f$, then by the construction we have that $f = \lambda 0^\infty$. □

This ends the proof of the theorem. □

4. Upper box dimensions of Π^0_1-classes

This section contains two theorems that deal with the upper box dimensions of Π^0_1-classes. The first theorem says that if X is a Π^0_1-class, then cDim$(X) \leq \overline{\dim}_{\mathrm{B}}(X)$. The second theorem says that there is a countable Π^0_1-class X that has effective packing dimension 0 and upper box dimension 1. An already known corollary of this result (see [4]) is that there are countable subsets of $[0, 1]$ that have nonzero upper box dimension (recall that upper box dimension is a *classical* notion). This illustrates one way in which the notion of upper box dimension is mathematically badly behaved.

Theorem 4.1. *For every* Π^0_1*-class* X*,* cDim*(*X*)*$\leq \overline{\dim}_{\mathrm{B}}(X)$.

Proof. Let $X \in \Pi^0_1$, and let $s > \overline{\dim}_{\mathrm{B}}(X)$. It suffices to show that $s \geq$ cDim(X). To show that $s \geq$ cDim(X), we will show that for all $f \in X$, cDim$(f) \leq s$.

Fix an $f \in X$, and let $r_n = \log(N_n(X))$. Since $s > \overline{\dim}_{\mathrm{B}}(X)$, there are cofinitely many n such that $ns > r_n$. Let W be the set of all such n, and for $n \in W$ let $F(n)$ be the first n bits of f. To prove the theorem we show that $\limsup_{n\in W} \frac{K(F(n))}{n} < s$. This suffices by theorems 2.7 and 2.8, and the fact that $\limsup_{n\in\omega} \frac{K(F(n))}{n} = \limsup_{n\in W} \frac{K(F(n))}{n}$, since $W \subseteq \omega$ is a cofinite set.

Now, since X is a Π^0_1-class, and can therefore be computably approximated, for any $n \in W$ we can give a prefix-free description of $F(n)$ by giving descriptions for

n, r_n, and a string of length r_n that indicates the position of f in the lexicographic listing of $N_n(X)$. Therefore,

$$K(F(n)) \leq K(n) + K(r_n) + r_n \leq 2\log(n) + 2\log(ns) + ns, \text{ since } r_n < ns,$$

and so $\limsup_{n\in W} \frac{K(F(n))}{n} \leq s$. □

Theorem 4.2. *There is a countable* Π^0_1*-class* X *such that* $\mathrm{cDim}(X) = 0$ *and* $\overline{\dim}_{\mathrm{B}}(X) = 1$.

Proof. We will construct a computable tree T in stages $T = \cup_s T_s,\ T_s \subset T_{s+1}$, such that every path of T is computable and $\limsup_n \frac{\log(N_n(X))}{n} = 1$, where X is the set of paths in T. The fact that every element of X is computable ensures that X is countable, and by theorems 2.7 and 2.8 also ensures that the corresponding Π^0_1-class $X \subseteq 2^\omega$ has effective packing dimension 0.

The construction of T

Stage 0: Enumerate σ into T_0, for all $\sigma \in \{\emptyset, 0, 1\}$.

Stage $s+1$: First, enumerate T_s into T_{s+1}. Then, for every leaf $\lambda \in T_s$ that is not of the form 0^n, enumerate $\lambda 0$ into T_{s+1}. On the other hand, for the unique leaf λ of T_s of the form 0^n for some $n \in \omega$, enumerate σ into T_{s+1} for all $\sigma \in \mathbf{C}^\lambda_n$, where $n \geq 1$ is chosen large so that $\frac{n}{n+|\lambda|} \geq 1 - 2^{-s-1}$. This ends the construction.

To see that T is indeed computable, note that for all $\sigma \in 2^{<\omega}$, if $\sigma \notin T_{|\sigma|}$, then $\sigma \notin T$.

Proposition 4.3. $\overline{\dim}_{\mathrm{B}}(X) = 1$.

Proof. At stage s of the construction we produce a number $m (= n + |\lambda|)$ such that $\frac{\log(N_m(X))}{m} \geq 1 - 2^{-s}$. This implies that $\limsup_{n\in\omega} \frac{\log(N_n(X))}{n} = 1$, and so X has upper box dimension 1. □

Proposition 4.4. Every element of X is computable.

Proof. Let $f \in X$. If $f = 0^\infty$ then f is computable, so assume that $f \neq 0^\infty$. Then there is a least $n \in \omega$ such that $f(n) = 1$. Let $\sigma \in 2^{<\omega}$ represent the first n bits of f. Let s be the smallest stage such that $\sigma \in T_s$ (such an s exists since $f \in X$) and let $\sigma' \supseteq \sigma$ be the unique leaf of T_s that is extended by f. Then, by the construction of T it follows that $f = \sigma' 0^\infty$, and hence f is computable. □

This ends the proof of the theorem. □

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