CLASSIFYING MODEL-THEORETIC PROPERTIES

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Abstract. In 2004 Csima, Hirschfeldt, Knight, and Soare [1] showed that a set $A \leq_{\mathrm{T}} 0'$ is nonlow$_2$ if and only if A is *prime bounding*, i.e. for every complete atomic decidable theory T, there is a prime model $\mathcal{M}$ computable in A. The authors presented nine seemingly unrelated predicates of a set A, and showed that they are equivalent for Δ_2^0 sets. Some of these predicates, such as prime bounding, and others involving equivalence structures and abelian p-groups come from model theory, while others involving meeting dense sets in trees and escaping a given function come from pure computability theory.

As predicates of A, the original nine properties are equivalent for Δ_2^0 sets; however, they are not equivalent in general. This article examines the (degree-theoretic) relationship between the nine properties. We show that the nine properties fall into three classes, each of which consists of several equivalent properties. We also investigate the relationship between the three classes, by determining whether or not any of the predicates in one class implies a predicate in another class.

§1. Introduction. Given two degree-invariant predicates of a set A there are several ways in which one can study their relationship. One approach is to study the degree-theoretic relationship between the predicates, but restrict the class of degrees with the hope of being able to show that they are indeed equivalent when restricted to the given class. This approach was taken by Csima, Hirschfeldt, Knight, and Soare, [1] who show that nine seemingly unrelated degree-invariant predicates of a set A are in fact equivalent when $A \leq_{\mathrm{T}} 0'$. A different approach is to assume a weak base theory (such as RCA_0), and check to see whether any implications follow. This approach was taken by Hirschfeldt, Shore, and Slaman [2], who show that several similar properties in [1] are not equivalent in the latter context. Yet another approach, which we take, is to consider the degree-theoretic relationship between the properties when A is allowed to range over all sets. In other words, we ask: "if a degree in the computable hierarchy has one property, does it have the other?"

1.1. The main theorem. Two properties examined in [2] we call the strong tree property and the isolated path property. A set A has the *strong tree property* if for any computable tree T with no terminal nodes, and any uniform collection of Δ_2^0 dense sets in T, $\{S_i\}_{i\in\omega}$, there is a function $f(\sigma, y) \leq_{\mathrm{T}} A$ such that, for any node $\sigma \in T$, and any $i \in \omega$, the function f produces a path extending σ as well as each of the dense sets S_i. The strong tree

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property was first introduced by Shinoda and Slaman [9] in the context of effective forcing constructions. The *isolated path property* comes from computable model theory, and says that for every computable tree T with no terminal nodes and isolated paths dense, the set A computes a function that, for any node $\sigma \in T$, produces an isolated path in T extending σ. The isolated path property is natural in the context of computable model theory. Computability theorists build prime models by finding, for every formula $\varphi(\bar{x})$ consistent with T, a principal type containing φ, and since types can be identified with paths in Cantor space, it follows that what is required to build a prime model is exactly the isolated path property. Hence, the isolated path property is equivalent to the *prime bounding property*, which says that for every complete atomic decidable theory T, there is a prime model $\mathcal{M}$ computable in A. Though it is important to recognize that the isolated path property is derived from the prime bounding property, from our point of view it is unnecessary to constantly refer to both, and so we will not discuss the prime bounding property much beyond giving its formal definition in the next section.

Since the isolated nodes of a computable tree form a Π^0_1 set, and the Π^0_1 sets belong to the class of Δ^0_2 sets, it follows that the strong tree property implies the isolated path property in any mathematical context. However, it is not obvious whether or not the reverse implication is true. Csima, Hirschfeldt, Knight, and Soare show that the reverse implication holds if the set A is Δ^0_2, while Hirschfeldt, Shore, and Slaman show that this is not the case if we consider nonomega models of RCA_0. In particular, [2] shows that the isolated path property (which they call the atomic model theorem) is Π^1_1-conservative over RCA_0+$\mathrm{B}\Sigma_2$. $\mathrm{B}\Sigma_2$, or Σ_2 bounding, is a bounding principle for Σ_2 formulas; for the precise definition consult [2]. However, the authors also show that, over RCA_0+$\mathrm{B}\Sigma_2$, the strong tree property implies induction for all Σ_2 formulas ($\mathrm{I}\Sigma_2$). Thus, one can construct a model of RCA_0+$\mathrm{B}\Sigma_2$ that has the isolated path property, but not the strong tree property by starting with a model of RCA_0+$\mathrm{B}\Sigma_2$ + $\neg\mathrm{I}\Sigma_2$ (such models exist, and are clearly nonomega models) and adding to it the isolated path property. Hence, the isolated path property cannot imply the strong tree property in the context of reverse mathematics. Neither of these results answers the degree-theoretic question of whether or not any degree that has the isolated path property also has the strong tree property. Moreover, they do not even provide us with a hypothesis, since in one case the answer is positive, while in the other it is negative.

The main theorem of this paper is to show that from the point of view of computability theory (i.e. degree-theoretically) the isolated path property does in fact imply the strong tree property. One immediate consequence of this surprising result is that the use of nonomega models in showing that the properties differ reverse mathematically is necessary; in other words, the properties are equivalent in every omega model of RCA_0.

We now wish to informally introduce two more properties, which we will show are equivalent to both the strong tree property and the isolated path property. We call the first of two properties the *weak tree property*. This is the same as the strong tree property, except that instead of a uniform collection of Δ^0_2 dense sets, $\{S_i\}_{i\in\omega}$, there is but a single dense subset of T, called S. Thus it is clear that the strong tree property implies the weak tree property. The weak tree property implies the isolated path property, since the isolated nodes of a computable tree form a Π^0_1 set. The other property is called the *escape property*, and says that for any given function $g \leq_{\mathrm{T}} 0'$, the set A can compute a function f that escapes (i.e. is not dominated by) g. Via a theorem of Martin, [1] explains why the escape property is important and why it is degree-theoretic in nature.

We conclude the introduction by first briefly introducing the remaining properties in [1], and then outlining the content of the rest of this paper.

1.2. The monotone property. A set A is said to have the *monotone property* if it can compute, for any infinite Δ^0_2 set S, a function $f(x, y)$ that is nondecreasing in y, and satisfies $\hat{f}(x) = \lim_y f(x, y) \in S$. Monotone functions were originally used by Khisamiev [3], [4], [5], to examine computability theoretic aspects of p-groups. Khoussainov, Nies, and Shore [6], and Nies [8], studied the monotone functions in the context of $\aleph_1$-categorical theories; Hirschfeldt studied them in the context of linear orderings; and [1] examines them in the context of both group theory and equivalence relations.

1.3. Low$_2$. The final property that we mention in the introduction says that the set A is nonlow$_2$; in other words, $A'' >_T \emptyset''$. We will show that this property is not implied by, nor does it imply any of the other properties.

1.4. The three classes. As was stated in the abstract, the overall aim of this paper is to determine which of the implications between the nine properties are true in general. In [1], Csima, Hirschfeldt, Knight, and Soare show that a few of the implications are valid in general, because some of their proofs do not require the hypothesis $A \leq_{\mathrm{T}} 0'$. This serves as our starting point and is outlined in section 2.2. The overall goal of this article is to prove that the nine properties in [1] fall into three equivalence classes under logical implication. The first class consists of the strong tree, weak tree, isolated path, and escape properties; we introduced these properties in section 1.1. The second class contains the monotone property, as well as two other properties; one is related to p-groups and the other deals with equivalence relations. The third class contains the property nonlow$_2$ (i.e. $A'' > 0''$). Furthermore, we go on to show that the third class is independent from the first two, and that the first class implies, but is not implied by the second. This settles all questions of the form "does every set A with property Pi also have property Pj?", $0 \leq i, j \leq 8$.

§2. The properties.

In this section we begin by giving precise definitions of all of the properties (P0)–(P8) defined in [1], and conclude with a diagram of the implications that they were able to show in general (i.e. without the assumption that $A \leq_T 0'$).

We use the notation of [1] and [10] throughout, except that we denote the set to which the properties may or may not hold of by A instead of X, and we write $\sigma \in T$ instead of $x \in T$ when $T \subseteq 2^{<\omega}$ is a tree.

2.1. Definitions. The properties (P0)–(P8) are as follows:

(P0) *The escape property.* $(\forall g \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} A)\,(\exists^\infty x)\,[\ g(x) \leq f(x)\ ]$, where "$(\exists^\infty)$" denotes "there exist infinitely many."

(P1) *Nonlow*$_2$. A is not low$_2$ (namely, $A'' >_{\mathrm{T}} 0''$).

(P2) *Prime bounding.* A is prime bounding. That is to say, for every complete atomic decidable theory T, there is a prime model $\mathcal{A}$ of T decidable in A.

(P3) *The isolated path property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall \sigma \in \mathcal{T})\,[\,g_\sigma \in [\mathcal{T}_\sigma] \;\;\&\;\; g_\sigma \text{ is isolated}\,].$$

(P4) *The strong tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every uniformly Δ^0_2 sequence of subsets $\{S_i\}_{i\in\omega}$ all dense in $\mathcal{T}$, there exists an A-computable function $g(\sigma, y)$ such that for every $\sigma \in \mathcal{T}$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending σ and entering all S_i, namely in our notation,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall \sigma \in \mathcal{T})\,(\forall i)(\exists z \in S_i)[\,\sigma \subseteq z \subset g_\sigma \quad \& \quad g_\sigma \in [\mathcal{T}]\,].$$

(P5) *The weak tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every Δ^0_2 set S dense in $\mathcal{T}$, there exists an A-computable function $g(\sigma, y)$ such that for every $\sigma \in \mathcal{T}$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending σ and extending some $z \in S$ above σ, namely in the notation of CHKS,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall \sigma \in \mathcal{T})\,(\exists z \in S)[\,\sigma \subseteq z \subset g_\sigma \quad \& \quad g_\sigma \in [\mathcal{T}]\,].$$

(P6) *The monotone property.* For any infinite Δ^0_2 set S,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall x)\,(\forall y)\,[\,x \leq g_x(y) \leq g_x(y+1) \;\;\&\;\; \widehat{g}(x) \in S\,].$$

An *equivalence structure* is a structure of the form $\mathcal{A} = (A, E)$, where E is an equivalence relation on A.

(P7) *The equivalence structure property.* For any Δ^0_2 set $S \subseteq \omega - \{0\}$, there is an A-computable equivalence structure with one class of size n for each $n \in S$, and no other classes.

A reduced Abelian p-group is determined, up to isomorphism, by its ulm sequence. Here we restrict our attention to reduced Abelian p-groups $\mathcal{G}$ of length ω, such that for all $n \in \omega$, $u_n(\mathcal{G}) \leq 1$. Define $S(\mathcal{G}) = \{n : u_n(\mathcal{G}) \neq 0\}$.

(P8) *The abelian p-group property.* For any infinite Δ^0_2 set S with $0 \notin S$, there is an A-computable reduced Abelian p-group $\mathcal{G}$, of length ω, and with $u_n(\mathcal{G}) \leq 1$ for all n, such that $S(\mathcal{G}) = S$.

It should be noted that our numbering is the same as [1], except for property (P5), which we take to be the weak tree property, but [1] took to be the *omitting types property* ([1] does not discuss the weak tree property). The authors Csima, Hirschfeldt, Knight, and Soare showed that the omitting types property is equivalent to the strong tree property (P4), and therefore any implication that applies to the strong tree property also applies to the omitting types property.

2.2. Known results. The following is shown in [1], and serves as our starting point:

$$\text{(P0)} \implies \text{(P4)} \implies \text{(P5)} \implies \text{(P3)} \iff \text{(P2)}. \tag{1}$$

$$\text{(P0)} \implies \text{(P6)} \iff \text{(P7)} \iff \text{(P8)} \tag{2}$$

For simplicity, we shall no longer refer to properties (P2), (P7), and (P8), since each is equivalent to one of the remaining properties.

2.3. Helper Properties. The following properties will be used in section 3 to prove the main theorem, which says that the properties (P0), (P2), (P3), (P4), and (P5) are mutually equivalent. A consequence of the main theorem is that this group is also equivalent to either of the following properties.

(OP5) *The open weak tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every Δ^0_2 set S dense in $\mathcal{T}$, there exists an A-computable function $g(\sigma, y)$ such that for every $\sigma \in \mathcal{T}$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending σ and extending some $z \in S$ (z need not extend σ), namely,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall \sigma \in \mathcal{T})\,(\exists z \in S)[\,\sigma \subset g_\sigma \;\;\&\;\; z \subset g_\sigma \;\;\&\;\; g_\sigma \in [\mathcal{T}]\,].$$

(Π^0_1-P5) *The open Π^0_1 weak tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every Π^0_1 set S dense in $\mathcal{T}$, there exists an A-computable function $g(\sigma, y)$ such that for every $\sigma \in \mathcal{T}$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending σ and extending some $z \in S$, namely in our notation,

$$(\exists g \leq_{\mathrm{T}} A)\,(\forall \sigma \in \mathcal{T})\,(\exists z \in S)[\,\sigma \subset g_\sigma \;\;\&\;\; z \subset g_\sigma \;\;\&\;\; g_\sigma \in [\mathcal{T}]\,].$$

2.4. The plan of the paper. In the next section we prove our main theorem. It says, surprisingly, that properties in group 1 {(P0),(P4),(P5),(P3)} are equivalent. Now, by (2), we know that (P0) implies (P6), and that (P6), (P7), and (P8) are equivalent; so one immediate consequence of the main theorem is that the properties of group 1 imply the properties of group 2 {(P6),(P7),(P8)}. In section 4, we show that the reverse implication is not true by constructing a set A which has the monotone property (P6), but does not have the escape property (P0). In section 5, we consider the property (P1) (nonlow$_2$), and show that it is independent from any of the other properties. We then replace (P1) by the stronger predicate which says that a set is not generalized low$_2$, written $A \notin \mathrm{GL}_2$ (i.e. $A'' \not\leq_T (A \oplus \emptyset')'$), and check to see whether any implications hold of this new property. By the end of the paper we will have proved the following set of implications.

$$[\text{(P0)} \iff \text{(P4)} \iff \text{(P5)} \iff \text{(P3)} \iff \text{(P2)}] \implies$$

$$\implies [\text{(P6)} \iff \text{(P7)} \iff \text{(P8)}]$$

$$[\text{(P1)}]$$

§3. The main theorem. The main theorem of this article is Theorem 3.9. It says that the isolated path property implies the escape property. First, however, we prove two theorems (3.1, 3.5) that will motivate the proof of the main theorem and serve as necessary results in its proof. We begin with Theorem 3.1, which says that the strong tree property implies the escape property.

THEOREM 3.1. *(P4) $\iff$ (P0) – i.e. the strong tree property implies the escape property (note that the implication (P0) $\Longrightarrow$ (P4) is given by (1)).*

First we need a definition.

DEFINITION 3.2. An *n-m dominating sequence* is a string of the form 101^n001^m000. A *dominating sequence* is an *n-m* dominating sequence, for some $n, m \in \omega$.

An important property of *n-m* dominating sequences is that if $\sigma \in 2^{<\omega}$ ends in such a sequence (for some $n, m \in \omega$), then any other dominating sequence contained in σ must come before the *n-m* dominating sequence (i.e. dominating sequences cannot overlap). The reason for this is that each consecutive run of ones in a given dominating sequence is separated by one, two, and three zeros, respectively.

The general outline of the proof is as follows. For a given $g \leq_{\mathrm{T}} 0'$, we define a dense open set $S \subseteq 2^{<\omega}$, $S \leq_{\mathrm{T}} 0'$, which codes information about the values $\{\langle x, g(x)\rangle\}_{x\in\omega}$. We then define the sets S_k to be the set S, minus its nodes of length at most k, and, applying the strong tree property to the $\{S_k\}_{k\in\omega}$, we get a function, $f \leq_{\mathrm{T}} A$, that escapes g.

PROOF OF THEOREM 3.1. Given any function $g \leq_{\mathrm{T}} 0'$, we define a uniform sequence of dense open sets $S_k \subseteq 2^{<\omega}$, $k \in \omega$, $S_k \leq_{\mathrm{T}} g$, as follows: for every $\sigma \in 2^{<\omega}$, put all $\tau \supseteq \tau_\sigma$ in S_k where $\tau_\sigma = \sigma^\wedge 101^n001^m000$ and $n = |\sigma| + 10 + k,\ m = g(n) + 1$. By construction it follows that the sets S_k are each open and dense, and they are uniformly computable in $0'$ (since $g \leq_{\mathrm{T}} 0'$).

The basic module of our construction is as follows. Fix a number $k \in \omega$. If we wanted to construct a partial function F that escaped g on a single input x, we could do it using the weak tree property applied to the set S_k. The weak tree property gives us a path, starting from the root $\emptyset$ of $2^{<\omega}$ that extends a node of S_k. Call this path f. Now, we can go from f to F via the reduction procedure that defines $F(k) = m$ for all $k \leq n$ not yet in the domain of F whenever we read an n, m dominating sequence in f.

Note that for f to extend a node of S_k it must contain an n, m dominating sequence for some $n, m \in \omega$. If σ is the smallest initial segment of f that lies in S_k, then $\sigma = \sigma_0^\wedge 101^n001^m000$ ends in an *n-m* dominating sequence, and so any other dominating sequence contained in σ must actually be contained in σ_0 (by the remark after the definition above). But, since the construction of S_k requires $n = |\sigma_0| + 10 + k$, it follows that σ_0 cannot contain any *k-l* dominating sequences for $k \geq n$. Thus, when we read σ, n is not yet in the domain of F, so we will define $F(n) = m$, where $m = g(n) + 1$, and hence $F(n) > g(n)$ for some $n \in \omega$, as required.

For the general case let $f \in 2^{\omega}$ meet all S_k, $k \in \omega$; we give a Turing reduction that produces a total function F that escapes g using f as an oracle. We construct F from f inductively in stages. At stage 0 we let $F = \emptyset$. At stage $s+1$, we consider only the first $s+1$ bits of f, call this string σ, and check to see whether σ ends in an n-m dominating sequence (for some $n, m \in \omega$). If not, then go to stage $s+2$. If so, then check to see if $F(n)$ is defined at the current stage; if $F(n)$ is defined then proceed to stage $s+2$, otherwise set $F(k) = m$ for all arguments $k \leq n$ on which F is undefined and go to stage $s+2$.

Lemma 3.3. *The function F described above is total and escapes the given function $g \leq_{\mathrm{T}} 0'$.*

Proof. Suppose that F does not escape g. Then F is dominated by g, and so there is some number n such that $(\forall m \geq n)[F(m) \leq g(m)]$. We shall construct a number $i > n$ such that $F(i) > g(i)$, thus obtaining a contradiction. Let s be the stage at which we defined $F(n)$. Since the reduction procedure at stage s considers only the first s bits of h, it follows, by the reduction procedure, that at stage s we have $dom(F) \subseteq \{0, ..., s\}$. Therefore, $F(s+1)$ is undefined at stage s. However, we are under the assumption that f meets all S_k, and so f meets S_{s+1}. Let σ be the smallest initial segment of f that meets S_{s+1}; then $\sigma = \tau^\wedge 101^i 001^j 000$ (this defines i), for some $\tau \in 2^{<\omega}$, $|\tau| + 11 + s = i > s$, $j = g(i) + 1$.

We claim that $F(i) > g(i)$; by the reduction procedure and the construction of S, it suffices to show that at stage $k = |\sigma|$, $F(i)$ is still undefined (in which case it will be defined and greater than $g(i)$ at stage $k+1$; this also shows that F is indeed a total function). So suppose for contradiction that $F(i)$ is defined at some stage $z < k$. By the reduction procedure and the fact that dominating sequences cannot overlap, this implies that τ has a substring of the form $\rho^\wedge 101^x 001^y 000$ for some $x \geq i, y \in \omega$. But $i \geq |\tau| + 11$, which is a contradiction. Hence, $F(i) > g(i)$. ⊣

⊣

The following lemma is useful because it helps to simplify several of the following proofs.

Lemma 3.4. *(OP5) $\iff$ (Π^0_1-P5) – i.e. the open weak tree property (OP5) is equivalent to the open Π^0_1 weak tree property.*

Proof. It is obvious that open weak tree implies open Π^1_0 weak tree. For the opposite implication, suppose that a set A has the open Π^1_0 weak tree property, and let $S \subseteq 2^{<\omega}$ be a Δ^2_0 dense set. Define a Π^1_0 dense set $P \subseteq 2^{<\omega}$ as follows. A node σ belongs to P if and only if it extends τ, and has length equal to $\langle \tau, k \rangle$ for some $\tau \in S$, $k > m(\tau)$, where m is the modulus associated to some (fixed) computable approximation of S (which exists by the limit lemma). Note that P is dense since S is dense, and m is total. Now, suppose that $g(\sigma, y) \leq_{\mathrm{T}} A$ extends all $\sigma \in 2^{<\omega}$ to meet P, then g must also extend all $\sigma \in 2^{<\omega}$ to meet S since (by construction) a path in $2^{<\omega}$ meets S if and only if it meets P. ⊣

The next theorem is the second key ingredient in the proof of the main theorem.

Theorem 3.5. *(OP5) $\iff$ (P3) – i.e. the open weak tree property is equivalent to the isolated path property.*

Proof. By the previous lemma, it suffices to show that (P3) implies (Π^0_1-P5). So assume that A satisfies (P3), let $P \leq_{\mathrm{T}} 0'$ be a dense open Π^1_0 subset of $2^{<\omega}$, and $h(\sigma, s)$ will denote a fixed computable Π^1_0 approximation to P. For simplicity, we take the computable tree in which we work to be the full binary tree $2^{<\omega}$ (the general case is similar, and is discussed afterwards). We show that A computes a function which extends all $\sigma \in 2^{<\omega}$ to nodes in P. To achieve this, we construct a computable tree, T, and a 1-1 computable function $F : T \to 2^{<\omega}$ such that $(\forall \sigma, \tau \in T)[\sigma \subseteq \tau \implies F(\sigma) \subseteq F(\tau)]$ – i.e. paths in our tree T correspond to paths in $2^{<\omega}$ via F.

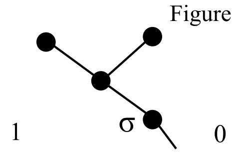
The idea of this proof is to construct a computable tree T, and a partial computable 1-1 onto function F which labels the nodes of T with labels from the full binary tree $2^{<\omega}$, in such a way that paths in T correspond (via F) to paths in $2^{<\omega}$. We build T so that if $\tau \in T$ and $F(\tau) \in 2^{<\omega}$ is not in P, then τ eventually splits. Also, if $F^{-1}(\hat{\rho})$ splits for some $\hat{\rho} \in 2^{<\omega}$, then we impose the condition that the inverse image of every $\hat{\sigma} \supseteq \hat{\rho}$ (under F) must extend $F^{-1}(\hat{\rho}) \in T$. This condition allows us to exploit the fact that P is dense in $2^{<\omega}$ to show that the set of isolated paths in T are dense. Next, we use the fact that A has the isolated path property to obtain $g(\sigma, y) \leq_{\mathrm{T}} A$ which extends all $\sigma \in T$ to isolated paths, and finally, using both g and F, one easily constructs a corresponding function for the dense Π^1_0 set $P \subseteq 2^{<\omega}$.

First some definitions.

Definition 3.6. The *ledge above* a node $\sigma \in 2^{<\omega}$ is the set of nodes $\{\sigma, \sigma^\wedge 1, \sigma^\wedge 10, \sigma^\wedge 11\}$, as is depicted in Figure 1 below.

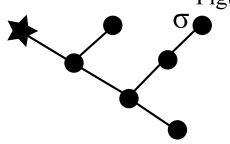
In our diagrams $\sigma^\wedge 0$ is to the right of σ, and $\sigma^\wedge 1$ is to the left of σ.

Figure 1



Definition 3.7. Given a (finite) tree, T, the *ledge below* a node $\sigma(\in T)$ on T is the unique node on T of the form $\sigma'^\wedge 1^n$, where σ' is the largest substring of σ ending in a 1 (if such a string exists), and n is the largest integer such that the resulting node is on the finite tree T. In Figure 2 (below), the node labeled with a star is the ledge below the node labeled σ.

Figure 2

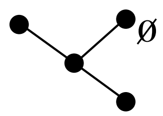


Since the reader may find it unusual to refer to a single node as a ledge, the author would now like to inform the reader that he thinks of the ledge below σ to be the set of nodes extending σ' which end in a 1 (hence it follows that the ledge below σ, as defined above, corresponds uniquely to a set of nodes which make up to actual ledge). The reason for this slightly ambiguous terminology is that it simplifies the explanation of the construction of the computable tree T. Also, a consequence of the construction of T is that if the ledge above ρ is on T, then for any $\tau \supseteq \rho$ on T, then the ledge below τ also extends ρ. This justifies the use of the terms "ledge above", and "ledge below".

The following paragraph gives the intuition behind the construction using the terminology that we have now developed. The construction guarantees that T_s is finite for all s (hence the ledge below a node is always defined), and we build T_{s+1} by extending the leaves of T_s. We have a fixed computable Π^0_1 approximation to the (dense) set P, which we denote as h. Hence, as s increases, a node may leave our approximation to P, but once it leaves it can never return. We are trying to build a tree T with isolated paths dense such that isolated paths in T correspond to paths in $2^{<\omega}$ that extend elements of the dense set P. To accomplish this, we associate to every node in T, an element of $2^{<\omega}$ via a (partial computable, 1-1, onto) function $F : T \to 2^{<\omega}$, such that if $f = \cup_i \tau_i,\ \tau_i \in 2^{<\omega},\ |\tau_i| = i$, is an isolated path in T then $F(f) = \cup_i f(\tau_i)$ is a path in 2^ω extending some element of S. With this in mind, we construct the tree T by creating splittings above nodes $\sigma \in T$ once $F(\sigma)$ leaves the set P, and we ensure that there is a unique path extending σ so long as $F(\sigma)$ appears to be in P. To create a splitting above σ, we insert the ledge above some $\lambda \supseteq \sigma$ into T for some current leaf $\lambda \in T$. To ensure that there is a unique path in T extending σ we do not create any splittings above σ in T; we do this as follows. If λ is the unique leaf of T_s extending σ, then to ensure that F is onto, we must define $F(\rho) = F(\lambda)^\frown 0$, $F(\tau) = F(\lambda)^\frown 1$, for some fresh nodes $\rho, \tau \in T_{s+1}$. However, ρ and τ cannot be comparable, since their images under F are incomparable, and so F would not necessarily take paths in T to paths in $2^{<\omega}$. We resolve this issue by setting $\rho = \lambda^\frown 0$, and taking τ to be a certain extension of the ledge below λ. This is how ledges are used in the construction of T.

The basic module of the construction is as follows. We build a tree T and function F as above such that for any isolated path f of T, $F(f)$ is in the open set defined by S. To do this we first fix a Π^0_1 approximation of P, $h(\sigma, s)$, as described above. The idea is to build T and F in stages, keeping paths in T isolated as long as their isolating nodes stay in the set P. At stage 0 we start by letting T be the ledge above the root of $2^{<\omega}$, and set $F(10) = \emptyset$, as shown in Figure 3.

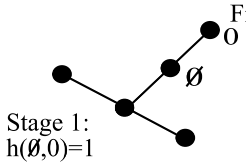
Figure 3



At stage $s+1$, if $h(\emptyset, s) = 1$ we extend the leaves of T_s that end in 0 to leaves of T_{s+1} by adding another 0, and set the images of the new leaves under F_{s+1} to be the images of the leaves they extend, concatenated by 0.

Figure 4 illustrates this portion of the construction of T at stage 1 under the assumption that $h(\emptyset, 0) = 1$. In this case we are acting as if $\emptyset \in P$, and so we proceed to build an isolated path extending $F^{-1}(\emptyset) = 10 \in T$.

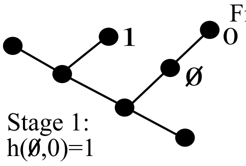
Figure 4



We also include, for each of the leaves of T_s ending in 0, a splitting above the ledge below these leaves. We define the images (under F_{s+1}) of these leaves that end in 0 to be the image of the corresponding leaf, concatenated by 1.

Figure 5 depicts this portion of the construction of T at stage 1 under the assumption that $h(\emptyset, 0) = 1$. At this point of the construction we still believe that $\emptyset \in P$.

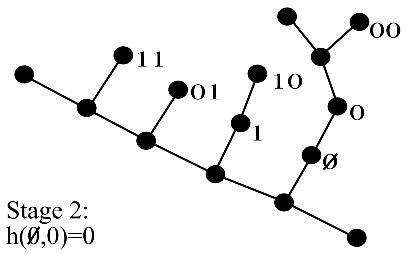
Figure 5



If, on the other hand, $h(\emptyset, s) = 0$ at stage s, then $\emptyset$ is no longer in the set P and so we create a splitting above the node $F_s^{-1}(\emptyset) \in T_s$ by including the ledge above the unique leaf extending $F_s^{-1}(\emptyset)$ in T_s, λ. We define F_{s+1} on the node of the ledge that ends in 0 as follows: $F_{s+1}(\lambda^\wedge 10) = F_s(\lambda)^\wedge 0$ (this follows the same pattern we used to label the ledge above $\emptyset$ at stage 0 of the construction). We extend all other leaves as in the case $h(\emptyset, s) = 1$.

Figure 6 shows what would happen at stage 2 of the construction (of T) if $h(\emptyset, 1) = 0$. Thus, we have just learned via h that $\emptyset \notin P$, and so we now act accordingly and create a splitting in T above the node labeled $\emptyset$.

Figure 6



Now for the general case. We construct $F = \bigcup_s F_s$, $F_{s+1} \supseteq F_s$, and $T = \bigcup_s T_s$, $T_{s+1} \supseteq T_s$, in stages. At stage 0, we let T consist of the ledge above $\emptyset$, and we define $F(10) = \emptyset$. At stage $s+1$ we are given T_s and F_s. Let

$\lambda_0, ..., \lambda_k$ be the leaves of T_s that end in 0, listed in length-lexicographic order (any sequence of uniformly computable orders would suffice) of their images under F_s. It will follow that F_s is defined on all λ_i. Let $\lambda'_0, ..., \lambda'_k$ be the largest substrings of the corresponding λ_i that end in 1, as in the definition of the ledge below a node. Now, we divide stage $s+1$ into $k+1$-many substages as follows. For every $0 \leq i \leq k$, if $h(F_s(\lambda'_i), s) = 1$ then enumerate $\lambda_i^\wedge 0$, $\tau_i^\wedge 0$, $\tau_i^\wedge 1$ into T_{s+1}, where τ_i is the (current) ledge below λ_i; also set $F_{s+1}(\lambda_i^\wedge 0) = F_s(\lambda_i)^\wedge 0$ and $F_{s+1}(\tau_i^\wedge 0) = F_s(\lambda_i)^\wedge 1$. On the other hand, if $h(F_s(\lambda'_i), s) = 0$ then add the ledge above λ_i to T and set $F_{s+1}(\lambda_i^\wedge 10) = F_s(\lambda_i)^\wedge 0$. Also, extend the ledge below λ_i and define F_{s+1} on the new leaves of T_{s+1} that end in 0 as in the case where $h(F_s(\lambda'_i), s) = 1$. That is, enumerate $\tau_i^\wedge 0$ and $\tau_i^\wedge 1$ into T_{s+1} and set $F_{s+1}(\tau_i^\wedge 0) = F_s(\lambda_i)^\wedge 1$. Do this in order, for every $i = 0, \ldots, k$, and then go to stage $s+2$. This ends the construction. Note that T is in fact computable, since if a leaf of T_s is not extended along a given path at stage $s+1$, then it will never be extended along that path at any future stage.

It is worth noting that throughout this proof we use lower case Greek characters (i.e. $\sigma, \tau, \rho, ...$ to represent nodes of T, and we use lower case Greek letters with hats (i.e. $\hat{\sigma}, \hat{\tau}, \hat{\rho}, ...$) to represent the nodes of $2^{<\omega}$, thinking of this as the space where S lives.

Lemma 3.8. *T is a computable tree with no terminal nodes and isolated paths dense. F is an onto, 1-1, partial computable function whose domain is the set of nodes in T that end in 0, and has the following two special properties:*

1. $\tau \subseteq \sigma \implies F(\tau) \subseteq F(\sigma)$, for any τ, σ in the domain of F. This property allows us to go from paths in T to paths in $2^{<\omega}$.

2. If T contains the ledge above ρ then ρ is in the domain of F, and furthermore any $\hat{\tau} \supseteq F(\rho)^\wedge 0$ satisfies $F^{-1}(\hat{\tau}) \supseteq \rho$ (this is a partial converse to 1).

Proof. We provide an algorithm (via the construction of T) for determining which of the extensions of the leaves of T_s are in T_{s+1}; hence T is computable. Similar reasoning shows that F is computable. To show that T has no terminal nodes, first note that at stage s of the construction, we extend the leaves of T_s that end in 0. Also, by induction one can show that every leaf of T_s ending in 1 is the ledge below some leaf $\lambda \in T$. Now, by the construction of T_{s+1}, it follows that this ledge will be extended in T_{s+1} (whether or not $h(\lambda, s) = 1$).

F is onto since at stage $s+1$ the range of F contains all nodes of length s (follows by induction). It is not difficult to show (by induction and the construction) that the domain of F is exactly the set of nodes in T that end in 0. Also, F is 1-1 since $F_s^{-1}(\sigma^\wedge 0)$ is always incomparable to $F_s^{-1}(\sigma^\wedge 1)$, and F preserves proper extensions (by the construction).

In the next two paragraphs we show that F has properties 1 and 2; this follows from the construction, and the author urges any reader who is already convinced of these facts not to spend too much time parsing them.

We prove that F satisfies property 1 by showing that F_s has property 1 for all σ, τ in its domain (for all $s \in \omega$); we do this by induction. The base case is trivial. For the induction step, assume that F_s satisfies property 1 for all σ, τ in its domain; we show that F_{s+1} also has this property. Suppose that $\tau \subseteq \sigma$, for some σ, τ in the domain of F_{s+1}. If both σ and τ are in the domain of F_s we are done. So assume that σ is not in the domain of F_s (if τ is not in T_s then it follows by the construction that any extension of τ cannot be in T_s, and so σ is not in T_s as well). Then σ was added to T_{s+1} at stage $s+1$ and so it must be a leaf that ends in 0 (by the construction). Now, $\tau \subseteq \sigma$ and so τ must be in T_s, since τ is in the domain of F_{s+1} and the construction does not define F_{s+1} on two comparable nodes unless one of them is in the domain of F_s, since we extend F_s to F_{s+1} by defining it on the leaves of T_{s+1} that end in 0 (so they are mutually incomparable). But the construction defines the values of F_{s+1} on the leaves of T_{s+1} (which end in 0) as extensions of the values of F_s on the leaves of T_s. So let ρ be the leaf of T_s extended by σ. Then it follows (since $\tau \subseteq \sigma$, and $\tau \in T_s$) that $\tau \subseteq \rho$. But it follows by the induction hypothesis that $F(\tau) \subseteq F(\rho)$, and by the construction $F(\rho) \subseteq F(\sigma)$, hence $F(\tau) \subseteq F(\sigma)$.

F satisfies the first part of property 2, since we only construct ledges above nodes which end in 0 (by the construction). We show that F satisfies the second part of property 2 by induction on the length of $\hat{\tau}$. First let $\rho \in T$ be such that the ledge above ρ is in T, then we must show that for any $\hat{\tau} \supseteq F(\rho)^\wedge 0$ we have that $F^{-1}(\hat{\tau}) \supseteq \rho$. If $\hat{\tau} = F(\rho)^\wedge 0$ then this follows since by the construction of T we have that $\hat{\tau} = F(\rho^\wedge 10)$, and it is clear that $\rho^\wedge 10 \supseteq \rho$. Suppose the claim is true for all $\hat{\tau} \supseteq F(\rho)^\wedge 0$ of length up to $k \geq |F(\rho)|$, and consider a $\hat{\tau} \supseteq F(\rho)^\wedge 0$ of length $k+1$. We can write $\hat{\tau} = \hat{\tau_0}^\wedge 0$ or $\hat{\tau} = \hat{\tau_0}^\wedge 1$, for some $\hat{\tau_0}$ of length k such that $\hat{\tau_0} \supseteq F(\rho)^\wedge 0$. Now, since $\hat{\tau_0}$ is of length k, by the induction hypothesis we have that $F^{-1}(\hat{\tau_0}) \supseteq \rho$. If $\hat{\tau} = \hat{\tau_0}^\wedge 0$ then by the construction we have either $F^{-1}(\hat{\tau}) = F^{-1}(\hat{\tau_0})^\wedge 0$ (if the ledge above $F^{-1}(\tau_0)$ is not in T) or $F^{-1}(\hat{\tau}) = F^{-1}(\hat{\tau_0})^\wedge 10$ (in the case where the ledge above $F^{-1}(\hat{\tau_0})$ is in T). However, in either case we have that $F^{-1}(\hat{\tau}) \supseteq \rho$, since $F^{-1}(\hat{\tau_0}) \supseteq \rho$. If $\hat{\tau} = \hat{\tau_0}^\wedge 1$ then by the construction $F^{-1}(\hat{\tau})$ extends the ledge below $F^{-1}(\hat{\tau_0})$ at some stage s (here we are using the fact that $\hat{\tau_0} \supseteq F(\rho)^\wedge 0$). But $F^{-1}(\hat{\tau_0}) \supseteq \rho$, and since the ledge above ρ is in T it follows that the ledge below $F^{-1}(\hat{\tau_0})$ extends ρ.

To show that T has its isolated paths dense, let $\sigma \in T$. Now, either there is a unique path extending σ, or else the path above σ splits at some point. By construction, the only way that this can occur is if at some stage we included the ledge above some $\sigma_0 \supseteq \sigma$ in T. Now, there is some $\hat{\rho_0} \supseteq F(\sigma_0)^\wedge 0$ in $2^{<\omega}$ that lies in P (by density), and by property 2 we have that $F^{-1}(\hat{\rho_0}) \supseteq \sigma_0 \supseteq \sigma$. Now, suppose that $F^{-1}(\hat{\rho_0})$ splits at some $\sigma_1 \supseteq F^{-1}(\hat{\rho_0})$. Then $F(\sigma_1) \supseteq \hat{\rho_0}$, and since P is open and $\hat{\rho_0} \in S$, we have $F(\sigma_1) \in P$. Hence by the construction T has a unique path, namely $\sigma_1^\wedge 10^\infty$ extending $\sigma_1^\wedge 10$. But the set P is open, and so we have that $\hat{\rho_1} \supseteq F(\sigma_1) \supseteq \hat{\rho_0} \in S$, hence $\hat{\rho_1} \in P$. Hence, every node including and above $F^{-1}(\sigma_1)^\wedge 0$ is in P, and therefore, by the construction of T, T has a unique path extending the node $\sigma_1^\wedge 10$. Thus, the isolated paths in T are dense, as claimed. $\dashv$

Note that in the construction of T, we only created a splitting at a node $\sigma \in T$ when the inverse image of one of its initial segments under F left the set P. Therefore, an isolated path in T means that there are only finitely many initial segments whose images under F are not in P, hence there must be a node on the path whose image is in P, and so the image of an isolated path in T under

F produces a path in $2^{<\omega}$ that meets P. Thus, it follows that the isolated path property implies the weak tree property.

It is not necessary to consider the general case when P is a dense subset of a computable tree $T' \neq 2^{<\omega}$. This is a consequence of the next theorem (3.9), as well as the known implications proven in [1], and outlined in the introduction. However, it is worth noting that the proof of Theorem 3.9 uses the construction we have just given for the case $T' = 2^{<\omega}$. In other words, our reasoning is sound. This ends the proof of Theorem 3.6. ⊣

Next we state the main theorem. We then state and prove two simple corollaries before giving the proof of the main theorem.

THEOREM 3.9. *(P3) implies (P0) – i.e. the isolated path property implies the escape property.*

COROLLARY 3.10. *(P3) implies (P6) – i.e. the isolated path property implies the monotone property.*

PROOF OF COROLLARY. By (2), we know that (P0) $\Longrightarrow$ (P6). Hence we have that (P3) $\Longrightarrow$ (P0) $\Longrightarrow$ (P6), and so (P3) $\Longrightarrow$ (P6). ⊣

COROLLARY 3.11. *The properties (P0), (P4), (P5), (OP5), (Π^0_1-P5), and (P3) are mutually equivalent.*

PROOF OF COROLLARY. We know that

$$(P0) \implies (P4) \implies (P5) \iff (OP5) \iff (\Pi^0_1 - P5) \iff (P3)$$

and so by the theorem all of these properties are equivalent. ⊣

Before diving into this proof, we make the following simple observation.

REMARK 3.12. *Given a function g, let F_k, $k \in \omega$, be a uniform collection of total functions such that for every $k \in \omega$ there exists a number x_k with the property that $F_k(x_k) > g(x_k)$ and $x_k > k$. Then the function $F(x) = \max_{k\leq x} F_k(x)$ escapes the function g.*

Therefore, to prove Theorem 3.9 we shall give, for every function $g \leq_{\mathrm{T}} 0'$, a uniform procedure for constructing total functions F_k as above.

As in the proof that the isolated path property implies the open Π^0_1 weak tree property, we use lower case Greek letters to denote the nodes of T, and lower case Greek letters with hats to denote elements of the full binary tree, in which the set S lives.

PROOF OF THEOREM 3.9. The proof uses the methods of previous proofs. Let $g \leq_{\mathrm{T}} 0'$ be the function that we are trying to escape using the isolated path property, and recall the sets $S_k \leq_{\mathrm{T}} g$ from the proof of Theorem 3.1 (note that S_k is dense in $2^{<\omega} = T'$ for all $k \in \omega$). Let $S = S_0$ and P be the corresponding Π^0_1 set given in the proof of Lemma 3.4. Using P, build the computable tree T as in the proof of Theorem 3.5.

DEFINITION 3.13. An *isolated node* is one that does not immediately split.

Claim 3.14. *The number of isolated nodes extended by* $F^{-1}(\hat{\sigma}^\frown 1) \in T$ *is at most that of* $F^{-1}(\hat{\sigma})$.

Proof. The claim follows by the construction of T. Let $\sigma = F^{-1}(\hat{\sigma}) \in T_s$ be a leaf. Now, recall that by the construction of T_{s+1}, if $F(\rho) = \hat{\sigma}^\frown 1$, then ρ is an extension of the ledge below σ. More precisely, ρ is of the form $\rho = \sigma'^\frown 1^k 0$, where k is a natural number and σ' is the longest substring of σ that ends in a 1. From this observation it follows that ρ has at most as many zeros as σ. Now, by the construction of T it follows that nodes of T ending in 1 split immediately, and so the isolated nodes of T must end in a 0. The result now follows. $\dashv$

Corollary 3.15. *The number of isolated nodes extended by* $F^{-1}(\hat{\sigma}^\frown 1^n) \in T$ *is at most that of* $F^{-1}(\hat{\sigma})$.

More generally, if we define $iso(\hat{\sigma})$ to be the number of isolated nodes properly extended by $F^{-1}(\hat{\sigma})$, and $Z(\hat{\sigma})$ to be the number of zeros in $\hat{\sigma}$, then

Claim 3.16. $iso(\hat{\sigma}) \leq Z(\hat{\sigma}) + 1$

Proof. By induction. First suppose that $|\hat{\sigma}| = 0$, i.e. $\hat{\sigma} = \emptyset$. By the construction of the tree T in the proof of Theorem 3.5 we have that $F(10) = \hat{\sigma}$ and 10 extends exactly one isolated node in T, namely the root of T. Therefore, in this case we have that $iso(\hat{\sigma}) = 1$ and $Z(\hat{\sigma}) = 0$ and so the claim holds. For the inductive step, suppose that the claim holds for all $|\hat{\sigma}| = k$, and let $|\hat{\rho}| = k + 1$. Then $\hat{\rho} = \hat{\sigma}^\frown 0$ or $\hat{\rho} = \hat{\sigma}^\frown 1$, for some $|\sigma| = k$. The latter case is trivial by Claim 3.14. For the former case, we know by the construction of T that either $F^{-1}(\hat{\rho}) = F^{-1}(\hat{\sigma})^\frown 0$ or $F^{-1}(\hat{\rho}) = F^{-1}(\hat{\sigma})^\frown 10$ (in the first case we didn't add the ledge above σ, while in the second case we did). In either case however, we have that $iso(\hat{\sigma}) \leq Z(\hat{\sigma}) + 1 = Z(\hat{\rho})$, since $\hat{\rho} = \hat{\sigma}^\frown 0$. $\dashv$

Now, let f be an isolated path in T starting at 0 (which our set A can compute, since it has the isolated path property). By Remark 3.12, it is sufficient to construct a total function F that escapes g at a *single* $x \in \omega$. Let σ_i be the first i bits of f. We compute F from f inductively as follows. At stage 0 set $F = \emptyset$. At stage $s + 1$ we consider σ_s. First we check to see if σ_s ends in an $n - m$ dominating sequence. If so, set $F(k) = m$ for all $k \leq n$ not yet in the domain of F. If not, then set $F(k) = 0$ for every $k < iso(\sigma_s)$. From this it follows that F is a total function, since f is an isolated path and so $lim_{s\to\infty} iso(\sigma_s) = \infty$.

Lemma 3.17. *There is an* $x \in \omega$ *such that* $F(x) > g(x)$.

Proof. We know that isolated paths in T correspond to hitting P in $2^{<\omega}$, which, in turn, corresponds to hitting S in $2^{<\omega}$. So some $\sigma_k \subseteq f$ must be of the form $\sigma_k'^\frown 101^n 001^m 000$ (where $n \geq |\sigma_k'| + 10$), and by the reduction procedure we know that when we read σ_k', we have not yet defined $F(x)$ for values of x larger than $|\sigma_k'|$. Now, since $iso(\sigma) \leq Z(\sigma) + 1$, we know that every time we read a 0 we may expand the domain of F by 1, and we also know by a previous remark (stated immediately after the definition of a dominating sequence), the only dominating sequence we will read between σ_k' and σ_k is σ_k itself. Therefore, when we read $\sigma_k'^\frown 101^n 001^m 000$ we have that the domain of F is contained in $\{0, ..., |\sigma_k'| + 7\}$, which means that F is undefined at $x = |\sigma_k'| + 10$. Hence, by the construction we will set $F(x) = m = g(n) + 1$. Therefore, F escapes g at x. $\dashv$

Finally, instead of using the single set $S = S_0$ we work with the family of dense sets $\{S_k\}_{k\in\omega}$. Now, we may use the reduction procedure described above to obtain (uniformly) a collection of total functions F_k, which, by Lemma 3.17, each escape g at a single point x_k. Furthermore, by the definition of the sets S_k and the reduction procedure for obtaining the corresponding functions F_k, it follows that $x_k > k$. Hence, by Remark 3.12, the function $F(x) = \max_{k\leq x} F_k(x)$ exists and escapes the function g. This ends the proof of Theorem 3.9. ⊣

§4. Group 2 does not imply group 1. Since we now know that the properties in group 1 are equivalent, and CHKS showed that the properties in group 2 are equivalent, to show that group 2 does not imply group 1 it suffices to construct a set A with the monotonic property (P6), but without the escape property (P0).

THEOREM 4.1. *(P6) does not imply (P0) – i.e. the monotone property does not imply the escape property.*

PROOF. We will construct a set $A \leq_{\mathrm{T}} 0''$ with the monotone property, but without the escape property. We will code the monotone property into A by first thinking of A as a three-dimensional matrix via a fixed computable pairing function that gives a bijection between ω and $\omega\times\omega$. Thus, we think of the index for the z^{th} bit of A as being given by an ordered triple $z = \langle i, x, n\rangle$. Fix $i, x \in \omega$. We shall henceforth refer to the set $\{\langle i, x, n\rangle | n \in \omega\}$ as the $\langle i, x\rangle - th$ row of A. Thinking of A as an infinite binary string allows us to refer to the $\langle i, x\rangle - th$ rows of finite strings as well. If $\sigma \in 2^{<\omega}$, then the $\langle i, x\rangle - th$ row of σ is the $\langle i, x\rangle - th$ row of $\sigma 0^\infty$. Next we outline the basic idea of the construction of A.

We shall code the monotone property into A as follows. Let $S_i = W_i^{\emptyset'}$ be the i^{th} Σ_2^0 set. Then, if $|S_i| < \infty$, there is no requirement to satisfy. However, if $|S_i| = \infty$, then for every $x \in \omega$ the $\langle i, x\rangle - th$ row of our set A shall contain exactly n ones, for some n such that $x \leq n \in S_i$. Therefore, for any given $i \in \omega$, if Ψ^A is the A-computable function that on input $\langle x, y\rangle$ reads the first y bits of the $\langle i, x\rangle - th$ row of A and outputs the number of ones if it is greater than x, and otherwise outputs x, then it follows that if our coding strategy succeeds A will have the monotonic property for the set S_i via the reduction Ψ. Now, since $\Delta_2^0 \subset \Sigma_2^0$ it follows that A has the monotone property. If a set A has exactly $n \in S_i$ ones on its $\langle i, x\rangle - th$ row, we say that A extends x to meet S_i at n, or, more simply, A meets S_i at n (for some x).

To ensure that A does not have the escape property, we need to construct a function $F \leq_{\mathrm{T}} 0'$ that dominates Φ_i^A for every $i \in \omega$ such that Φ_i^A is total. We will build a tree $T = \cup_s T_s \subseteq 2^{<\omega}$ and a function F, both below $0'$, in stages such that if f is any path in T then F dominates Φ_i^f if Φ_i^A is total. To do this, we use $0'$ to force whether or not $\Phi_i^{\lambda_j}(s)\downarrow$ at stage s, for all leaves $\lambda_j \in T_s$. Hence, if we define $F(s)$ to be greater than the maximum of all $\Phi_i^{\lambda_j}(s)$ that converge, then if A is a path in T we will have that $F \leq_{\mathrm{T}} 0'$ dominates all total functions Φ_i^A. Lastly, we show that there is a path A in T that satisfies our coding procedure from the previous paragraph, and hence has the monotone property. Note that A has the monotone property but does not have the escape property.

The positive and negative requirements that we must satisfy are as follows:

$P_{\langle i,x\rangle}$: if $S \in \Delta^0_2$, $|S_i| = \infty$, extend x to meet $W_i^{\emptyset'} = S_i \in \Delta^0_2$ by having exactly $n > x$, $n \in S_i$ ones on the $\langle i,x\rangle - th$ row of A.

N_j: define $F(j)$ so that it is greater than $\Phi^A_k(j)$, for all $k \leq j$ (if $\Phi^A_k(j)$ exists).

Throughout this proof we use lower case Greek letters to denote both elements of $2^{<\omega}$ as well as the corresponding finite sets, whose characteristic functions are equal to elements of $2^{<\omega}$ extended by the path of all zeros. Thus, when we write $\rho \setminus \sigma$ for some $\rho, \sigma \in 2^{<\omega}$, we mean the set difference between C and B, $C \setminus B$, where C is the finite set whose characteristic function is the infinite string $\rho^\wedge 0^\infty$, and B is the finite set with characteristic function $\sigma^\wedge 0^\infty$.

Constructing the tree T

The construction of $T_1 \leq_{\mathrm{T}} 0'$ is divided into even and odd stages, as follows. At stage $s = 0$, let $T_1 = \emptyset$.

At every stage of the construction we may impose a (computable) set of restraints on nodes $\tau \in T$ which preserve the number of ones on a given row of τ at all later stages. Furthermore, these restraints may overlap for different rows. Hence, it is assumed when we say "search for a string ρ" we actually mean "search for a string ρ that respects the restraints imposed by the construction up to now".

At odd stages $2s+1 > 0$ we act to satisfy the requirement N_s. We are given T_{2s+1}, and let $\lambda_0, \ldots, \lambda_J$ be the leaves of T_{2s+1}. Using $0'$, extend each $\lambda_j^\wedge 0$ to λ'_j so that for all $i \leq s$, if $\Phi_i^{\lambda'_j}(s)\uparrow$, then for all $\rho \supset \lambda_j$ we have that $\Phi_i^\rho(s)\uparrow$. Also, let $F(s) = \max_{i \leq s, j \leq J}\{\Phi_i^{\lambda'_j}(s) : \Phi_i^{\lambda'_j}(s)\downarrow\}$.

At the even stages $2s > 0$ of the construction of T we act to satisfy the monotonicity requirements $P_{\langle i,x\rangle}$, $\langle i,x\rangle \leq s$. The basic idea of how we do this is as follows.

We are given T_{2s}; let $\lambda_0, \ldots, \lambda_J$ be the leaves of T_{2s}. First we extend each leaf λ_j, $j \in \{0, \ldots, J\}$ (acting subject to any previous restrictions) to a node $\lambda'_j \in 2^{<\omega}$ that has at least s-many ones in its $\langle i,x\rangle$-th row for all $\langle i,x\rangle \leq s$. Furthermore, we do not impose any further restraints on the extension λ'_j. Now, for every λ'_j currently unrestricted for $\langle i,x\rangle$ and all $n \leq s$ that are greater than or equal to the number of ones on the $\langle i,x\rangle$-th row of λ'_j, we add a path $\rho_n \supseteq \lambda'_j$ to T_{2s} which has n-many ones in its $\langle i,x\rangle$-th row, and declare ρ_n to be restricted for $\langle i,x\rangle$. Formally, this is written as follows.

Search for $\lambda'_j \supset \lambda_j$ (subject to any previous restraints) so that for every $i, x < s$, λ'_j has at least s ones on the row indexed by $\langle i,x\rangle$. If no such λ'_j exists, then set $\lambda'_j = \lambda_j$. Next fix i, x, j with $\langle i,x\rangle \leq s$, and let B be the $\langle i,x\rangle - th$ row of $\lambda'_j \setminus \lambda_j = \{a^0 < \cdots < a^l\}$, and for every $0 \leq k \leq l$ let $B^k = \{a^k < \cdots < a^l\}$. For every B^k, define $\rho_k = \lambda'_j \setminus B^k$. More explicitly, ρ_k can be defined as follows: $\rho_k(z) = \lambda'_j(z) \in \{0,1\}$ for all $z < |\lambda'_j|$ with $z \notin B^k$, and $\rho_k(z) = 0$ for all $z < |\lambda'_j|$ with $z \in B^k$. Furthermore, we make a promise that all leaves of T_t, $t \geq 2s+1$ extending ρ_k will have the same number of ones on row $\langle i,x\rangle$ as ρ_k — i.e. we restrict our search space from now on when we use $0'$ to make extensions of T.

Repeating this procedure for all i, x, j with $\langle i, x\rangle \leq s$, we let the corresponding ρ_k, and λ'_j, be the leaves of T_{2s+1}.

This ends the construction of T. Note that T has no terminal nodes, since the only restraints imposed by the construction on nodes $\tau \in T$ are those which preserve the number of ones on a current nonempty row of τ. Therefore, it follows that for any node $\tau \in T_s$, $\tau^\wedge 0$ is always a valid extension of τ since it preserves the number of ones on all rows of τ. Also, by the construction of the function F, it is easily verified that $F \leq_{\mathrm{T}} 0'$ and that F dominates every path of T (i.e. if A is a path of T then F dominates all total functions of the form Φ_i^A, for $i \in \omega$). Therefore, every path of T, including the path $A \leq_{\mathrm{T}} 0''$ which we later construct and has the monotone property, cannot have the escape property.

Constructing the set A.

Now, using $0''$ as an oracle, we shall construct a path $f \in [T]$, such that if A is the set with characteristic function f, then A has the monotone property. Note that constructing f is equivalent to constructing A.

We construct $f = \cup_s f_s$ in stages. At stage $s = 0$, let $f = \emptyset$. At stage $s + 1$, we are given f_s, which we extend to a leaf of T_{2s+2}. Consider the least $\langle i, x\rangle < s$ such that there is an $n \in S_i$ greater than x, and x has not yet been extended to meet S_i (note that $0''$ can determine, for any given $i, x \in \omega$, whether or not there is an $n \in S_i$ such that $n > x$). Fix the least such n. If there is a leaf of T_{2s+2} extending f_s which extends x to meet S_i at n, and is equal to some ρ_k described above (for some fixed i, x, j such that $\langle i, x\rangle \leq s$) then let f_{s+1} be this ρ_k. Otherwise, let f_{s+1} be equal to the λ'_j extending f_s (as denoted in the construction above). This ends the construction of f; note that by the construction f is total, since T has no terminal nodes.

Verifying that A has the monotone property.

To show that f has the monotone property, let $S_i \in \Delta^0_2$ be an infinite set (we could also take $S_i \in \Sigma^0_2$ and the proof would work just as well) and $x \in \omega$. We will verify that there is indeed a stage s at which f extends x to meet S_i at some $n \in S_i$. Suppose that s_0 is large so that all $P_{\langle i', x'\rangle}$, $\langle i', x'\rangle < \langle i, x\rangle$ have been satisfied by stage s_0, and let n be the smallest number in S_i that is greater than the number of ones on row $\langle i, x\rangle$ of f_{s_0}.

Now, the construction of f says to keep extending to λ'_j until we see a ρ_k that extends x to meet S_i at n. If we never see such a ρ_k, then, by the construction of T, f has infinitely many ones on its row indexed by $\langle i, x\rangle$, since, at large enough stages the construction of T adds a one to every nonempty row of the (finite) paths through T_s, unless at some stage we make a promise to put only zeros on this row. By the construction of T, the only way that such a promise is made is if at some point we extend f_s to some ρ_k. But extending f_s to some ρ_k means that we extended x to meet S_i at n and therefore have satisfied $P_{\langle i,x\rangle}$. By the construction of T, once we have done this, we promise to put no more ones on row $\langle i, x\rangle$ of f_t, for all $t > s$, and so f extends x to meet S_i at n, as required.

On the other hand, if row $\langle i, x\rangle$ of f contains infinitely many ones, then, by the construction of T, there must have been a smallest stage $t \geq s_0$ at which the number of ones was greater than n. Then, by definition of the ρ_k, it follows that some ρ_k on T_t has exactly n ones on its $\langle i,x\rangle - th$ row. Now, by the construction of f, we will extend f_t so that it passes through this ρ_k. Furthermore, the construction of T makes a promise at this point which guarantees that any path $g \in [T]$ extending ρ_k has exactly n ones on its $\langle i,x\rangle - th$ row. Therefore, f extends x to meet S_i at n, and has therefore satisfied $P_{\langle i,x\rangle}$. So f has the monotone property. ⊣

§5. Nonlow$_2$. The first theorem in this section is an unpublished result of Soare. It says that there exists a set A that is low$_2$ and has the escape property. This proves that none of the properties in group 1 imply property (P1). However, we also know from section 4 that the properties in the first group imply the properties in the second group, and therefore a consequence of the following theorem is that none of the other properties imply (P1).

THEOREM 5.1. *There exists A such that*

$$A'' \equiv_{\mathrm{T}} 0'' \quad \& \quad (\exists f \leq_{\mathrm{T}} A)(\forall g \leq_{\mathrm{T}} 0')(\exists^{\infty} x)\,[g(x) \leq f(x)].$$

Namely, A has properties $\neg$(P1) and (P0). Hence, (P0) $\not\Longrightarrow$ (P1).

PROOF. We use a $0''$ forcing construction of a sequence of finite strings $\{h_s : s \in \omega\}$ such that $A = \bigcup_s \{h_s\}$ is low$_2$ and satisfies (P0). Let $\sigma_0 = \emptyset$, and $x_{-1} = 0$.

Even stages : At stage $s = 2e$ we extend h_s to h_{s+1} to force whether Φ_e^A is total or not, by asking $0''$ whether

$$(\exists \sigma \supseteq h_s)(\exists x)(\forall \tau \supseteq \sigma)[\Phi_e^{\tau}(x)\uparrow].$$

If so, extend h_s to σ, and if not, then at each subsequent stage $t \geq s$ extend h_t so that $\Phi_e^A(x)\downarrow$ for one more x. Thus, the set A is low$_2$ since $0''$ can compute whether or not Φ_e^A is total.

Odd stages : At stage $s = 2\langle e,z\rangle + 1$ we first compute a value $x_s > x_{s-2}$ large enough so that, viewing h_s as a two-dimensional array via a computable pairing function, the $x_s - th$ row of h_s is currently empty. We then use the oracle $0''$ to compute a number, N, which is an upper bound for the set $\{\Phi_u^{\emptyset'}(t) : u, t \leq x_s, \Phi_u^{\emptyset'}(t)\downarrow\}$. Then we extend h_s to h_{s+1} such that the $x_s - th$ row of h_{s+1} begins with a string of N ones, followed by a zero, and all rows of h_{s+1} whose index is less than x_s end in a zero. This ends the construction of A.

To verify that the construction works, we need to check that the set A is both low$_2$ and has the escape property. It is easy to see that A is low$_2$ since at the even stages $s = 2e$ of the construction $0''$ forces whether or not Φ_e^A is a total function. A also has the escape property, since if u is an index so that Φ_u is the Turing functional that, when given an oracle B and input x, computes $\Phi_u^B(x)$ by outputting the position of the first zero in the string $x - th$ row of B, then by the construction, Φ_u^A is total, and dominates every $0'$-computable function on the infinite set $\{x_s : s \in \omega\}$.

⊣

Note that we actually proved a uniform version of the escape property for A – i.e. there is a single A-computable function f that escapes every $0'$-computable function.

THEOREM 5.2. *(P1) does not imply (P6) – i.e. there is a nonlow$_2$ set that does not have the monotone property.*

PROOF. The idea is to construct a perfect tree, $T \subseteq 2^{<\omega}$, and an infinite set $S \leq_{\mathrm{T}} 0'$, such that none of the paths in T have the monotone property for S. This proves the theorem because T has uncountably many paths, and since there are only countably many low$_2$ sets (since any low$_2$ set must lie below $0''$), T has a path that is nonlow$_2$ and does not have the monotone property.

The proof that 0 does not have the monotone property.

Recall that we want to construct a set $S \leq_{\mathrm{T}} 0'$ with the property that for every $e \in \omega$, if $\Phi_e(x, y)$ is a total computable function that is nondecreasing in its second variable, then there is an $x \in \omega$ such that either $\widehat{\Phi_e}(x) = \lim_{y\to\infty} \Phi_e(x, y) = \infty$ or $\widehat{\Phi_e}(x) \notin S$.

At stage $s \geq 0$ we are given numbers $x_0, ..., x_{s-1}$ (uniformly computable in $0'$) and a large number N_s (the N_s are uniformly computable) such that $N_s - N_{s-1} > s + 1$. First we define $x_s = N_{s-1}$ ($N_{-1} = 0$). Now, using $0'$ as an oracle, we determine for which of the x_i, $0 \leq i \leq s$ there is a $y \in \omega$, such that $\Phi_i(x_i, y)\downarrow > N_s$, and for the ones that do not, evaluate the corresponding limits $z_i = \widehat{\Phi_i}(x_i)$, $i \geq 0$ (if they exist). Now, there are at most s values z_i, and so by the pigeonhole principle, there is a number $c_s \in \{N_{s-1} + 1, ..., N_s\}$ that is not equal to any z_i; put the least such c_s into S at stage s. This ends the construction.

Note that S is infinite because at every stage s in its construction, we add c_s to S and c_s is larger than any element of S at the end of stage $s - 1$. Furthermore, if there were a computable function, $\Phi_e(x, y)$, that is nondecreasing in its second variable and satisfies $x \leq \widehat{\Phi_e}(x) \in S$, for all $x \in \omega$, this would lead to a contradiction as follows. At stage e of the construction we define x_e, and from then on, at stage $s \geq e$ we ask if there is a $y \in \omega$ such that $\Phi_e(x_e, y)\downarrow > N_s$. Now, for all s we have $N_s - N_{s-1} > s + 1$, and so $\lim_{s\to\infty} N_s = \infty$, and since $\widehat{\Phi_e}(x_e) = \lim_{y\to\infty} \Phi_e(x_e, y)$ exists, it must be the case that at some stage $s \geq e$ we learn that there is no $y \in \omega$ for which $\Phi_e(x_e, y)\downarrow > N_s$. Furthermore, we determine what $\widehat{\Phi_e}(x_e)$ is at stage s, and by the construction, we keep this value out of the set S, which is a contradiction, since we assumed that for every $x \in \omega$ we have $\widehat{\Phi_e}(x) \in S$. This ends the verification.

Recall that we will construct a perfect tree T, and an infinite set $S \leq_{\mathrm{T}} 0'$, such that every path in T does not have the monotone property for the set S (this is in fact the *tree version* of the proof that 0 does not have the monotone property). To do this, we shall satisfy the condition that says if f is a path of $T = \cup_{i\in\omega} T_i$, and $\Phi_e^f(x_e, y)$ is total and nondecreasing in its second variable and comes to a limit, then $X_e = \lim_y \Phi_e^f(x_e, y)$ is not in the set $S = \cup_{i\in\omega} S_i$. Therefore no path of T has the monotone property. The construction of T and

S follows.

Constructing T and S.
At stage $s = 0$ set $T_0 = \{\emptyset\}$ and $S_0 = \emptyset$.

At stage $s + 1$ we are given T_s; let $\lambda_1, ..., \lambda_k$ denote the leaves of T_s, set $x_s = 1 + \max\{S_s\}$, and $N = (s + 1)2^{s+1} + x_s$. Now, for each leaf λ_i and every $e \leq s$, $0'$ can find a string $\rho \supset \lambda_i$ that satisfies one of the following three properties:

1. $\Phi_e^\rho(x_e, y)\downarrow > N$ for some $y \in \omega$.
2. For all $f \in 2^\omega$ extending ρ either $\Phi_e^f(x_e, \cdot)$ is total, nondecreasing in its second variable, and reaches a limit that is $\leq N$, or $\Phi_e^f(x_e, \cdot)$ defines a partial function (i.e. there is a $y \in \omega$ for which $\Phi_e^f(x_e, y)\uparrow$).
3. $\Phi_e^\rho(x_e, \cdot)$ is not nondecreasing in its second variable.

The string ρ can be found using a $0'$ oracle (for fixed i and e) by first using $0'$ to search for a string $\rho_1 \supset \lambda_i$ such that $\Phi_e^{\rho_1}(x_e, y)\downarrow > N$ for some $y \in \omega$. If there is such a ρ_1, then set $\rho = \rho_1$ and we are done since ρ_1 satisfies condition 1. Otherwise it follows that for every string $\rho \supset \lambda_i$ and every $y \in \omega$, if $\Phi_e^\rho(x_e, y)\downarrow$ then $\Phi_e^\rho(x_e, y) < N$. Now, in the case where no ρ_1 exists use $0'$ to look for a string $\rho_3 \supset \lambda_i$ such that there exist $y_1, y_2 \in \omega$, $y_1 < y_2$, with the property that $\Phi_e^{\rho_3}(x_e, y_1)\downarrow > \Phi_e^{\rho_3}(x_e, y_2)\downarrow$. If such a ρ_3 exists then we are done since ρ_3 satisfies condition 3. Otherwise it follows that for every string $\rho \supset \lambda_i$ the (possibly partial) function given by $\Phi_e^\rho(x_e, y)$ is nondecreasing in its second variable.

Now suppose that neither ρ_1 nor ρ_3 exist. Then it follows by earlier remarks that if $f \in 2^\omega$, $\lambda_i \subset f$, we have that either $\Phi_e^f(x_e, y)$ is total, nondecreasing, bounded by N, and therefore has a limit that is less than or equal to N, or else $\Phi_e^f(x_e, y)$ is a partial function — i.e. there is a $y \in \omega$ for which $\Phi_e^f(x_e, y)\uparrow$. In other words, we have satisfied condition 2. In this case we ask $0'$ to find a string $\rho_2 \supset \lambda_i$ and $y_2 \in \omega$ such that for all $\rho \supset \lambda_i$ and all $y \in \omega$ such that $\Phi_e^\rho(x_e, y)\downarrow$ we have $\Phi_e^\rho(x_e, y) \leq \Phi_e^{\rho_2}(x_e, y_2)$. First of all, since we are under the assumption that neither ρ_1 nor ρ_3 exist, $0'$ can determine if no such string ρ_2 exists. In this case we set $\rho_2 = \lambda_i$ and note that any $f \supset \rho_2$, $f \in 2^\omega$, cannot be a total function. On the other hand, if ρ_2 and y exist, then $0'$ can compute them since it knows whether or not there is a ρ_2 such that $\Phi_e^{\rho_2}(x_e, y_2)\downarrow = N$. If there exist such ρ_2, y then it outputs these values. Otherwise, it asks for ρ_2, y_2 such that $\Phi_e^{\rho_2}(x_e, y)\downarrow = N - 1$, and so on. Note that if $f \in 2^\omega$ extends ρ_2, then by construction of ρ_2 and the fact that we are in case 2 above, we have that either $\lim_y \Phi_e^f(x_e, y) = \Phi_e^{\rho_2}(x_e, y_2)$ or else $\Phi_e^f(x_e, y)$ is a partial function. Hence, for any $\rho_2 \subset f \in 2^\omega$, $0'$ can compute $\lim_y \Phi_e^f(x_e, y)$ (if this limit exists) by computing $\Phi_e^{\rho_2}(x_e, y_2)$.

Note that nowhere have we used any assumptions about λ_i other than $\lambda_i \in 2^{<\omega}$; hence, for every $1 \leq i \leq k$ there is a node $\rho_i \supset \lambda_i$ that satisfies one of the conditions 1, 2, 3, above for every $0 \leq e \leq s$. Such a ρ_i is obtained by iterated extensions of λ_i. Add the nodes $\rho_i^\wedge 0$ and $\rho_i^\wedge 1$ to the tree T_{s+1}. This ends the construction of T.

Note that in case 3 we have vacuously satisfied the $e - th$ requirement for nonmonotonicity – i.e. there is an x_e such that $\Phi_e^A(x_e, \cdot)$ is not a total monotonic (i.e. nondecreasing) function. Also, if for a single e we are able to satisfy the first condition at all stages s, we are happy because the limit $\lim_{y\to\infty} \Phi_e^A(x_e, y) = \infty$, and therefore we have satisfied the $e - th$ requirement for nonmonotonicity.

The only case that causes us concern is the second. We add the nodes $\rho_i^\frown 0, \rho_i^\frown 1$ to T (for every i). By the construction of T it follows that T_{s+1} has 2^{s+1}-many leaves, and (at stage $s + 1$) each leaf is currently trying to diagonalize against s-many functionals. Therefore, by the pigeonhole principle, there is a number $c_s \in \{x_s, \ldots, N\}$ that is not the limit of any $\Phi_e^{\lambda_i}(x_e, y)$, for any $1 \le i \le k$ and $0 \le e \le s$. Furthermore, $0'$ can compute the least such number c_s since $0'$ can tell which pairs $\langle i, e\rangle$ satisfy condition 2 above and also, for each of these pairs, what is the only possible value of $\lim_y \Phi_e^f(x_e, \cdot)$, $\lambda_i \subset f \in 2^\omega$. Now, put the least such c_s into S_{s+1}. This ends the construction of S.

No path in T has the monotone property.
The verification is similar to that of the proof that 0 does not have the monotone property. Note that, by the construction, we have $|S_s| = s$, and so S is infinite. To show that none of the paths in T have the monotone property for the set S, assume, for a contradiction, that there is a path $f \in T$ that has the monotone property for S, and by the same reasoning as before (i.e. at stage s we put c_s into S, which is not the limit of any $\Phi_e^{\rho_i}(x_e, \cdot)$ at stage s) this leads to a contradiction by the way we constructed S and T. This ends the verification.

Now Theorem 5.2 follows by a simple cardinality argument. There are 2^ω many paths in T, and only countably many low$_2$ sets. Hence, T contains a path that is nonlow$_2$. ⊣

Corollary 5.3. *(P1) does not imply any of the other properties.*

Proof. Suppose not, then (P1) $\Longrightarrow$ (Pk), for some $k \neq 1$. But we have shown that if $k \neq 1$ then (Pk) $\Longrightarrow$ (P6), and therefore we have that (P1) $\Longrightarrow$ (Pk) $\Longrightarrow$ (P6), which contradicts the theorem. ⊣

§6. NonGL$_2$. We have now settled all implications between the properties (P0)–(P8), as well as the helper properties introduced in section 2.3. However, in the general computable (i.e. degree-theoretic) context, it is not even clear what the definition of low$_2$ should be. There are three competing definitions:

1. A set A is low$_2$ if $A \le_T 0'$ and $A'' = 0''$.
2. A set A is low$_2$ if $A'' = 0''$.
3. A set A is low$_2$ if $A'' = (A \oplus 0')'$.

The first definition was considered in [1], while the second definition is the one we have been working with so far in this article. The third definition is more general than the first two, and is referred to in the literature as generalized low$_2$.

Definition 6.1. A is generalized low$_2$ (written $A \in GL_2$) if A satisfies 3.

One could criticize our proof of Theorem 5.2 by arguing that our definition of low$_2$ is too restrictive, since it implies that there are only countably many low$_2$ sets. However, there are uncountably many generalized low$_2$ sets, and so the proof of Theorem 5.2 would not go through if we wanted to show the existence of a set $A \notin GL_2$ that does not have the monotone property. The next well-known theorem (see [7] Corollary IV.3.4) shows that no such set exists.

Theorem 6.2. *If $A \notin GL_2$, then A has the escape property.*

Proof. Assume that $A \notin \mathrm{GL}_2$, so that $A'' \not\leq_T (A \oplus \emptyset')'$. It then follows that the set $A \oplus \emptyset'$ is not high above A, and by the relativized version of Martin's domination theorem (see [10] Theorem XI.1.3) it follows that the degree of A escapes the degree to which $A \oplus \emptyset'$ belongs. In other words, for every function $g \leq_{\mathrm{T}} A \oplus \emptyset'$ there is a function $f \leq_{\mathrm{T}} A$ that escapes g. Now, let $h \leq_{\mathrm{T}} 0'$, it follows that $h \leq_{\mathrm{T}} A \oplus \emptyset'$, since $0' \leq_{\mathrm{T}} A \oplus \emptyset'$, and therefore we have that there is an $f \leq_{\mathrm{T}} A$ which escapes h. Hence, for any function $h \leq_{\mathrm{T}} 0'$ there is a function $f \leq_{\mathrm{T}} A$ that escapes h. Thus, A has the escape property (P0). ⊣

Corollary 6.3. *If $A \notin GL_2$, then A has all properties (P0)–(P8).*

Proof. If $A \notin GL_2$, then by Theorem 6.2, it follows that A satisfies (P0). But we have shown that (P0) implies all the other properties except (P1). But it is easy to verify that if $B'' = 0''$ (i.e. B doesn't satisfy (P1)) then B is not generalized low$_2$. Hence, if $B \notin GL_2$ then $B'' \neq 0''$, and so B has property (P1). ⊣

Furthermore, note that since GL_2 is a more general notion than low$_2$ (i.e. low$_2$ sets are also GL_2) and we have shown that there is a low$_2$ set A that has none of the other properties (P0),(P2)–(P8), it follows that this set A is also an example of a GL_2 set with none of the properties (P0),(P2)–(P8). Hence, none of the properties (P0),(P2)–(P8) implies $A \notin GL_2$.

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