THE COMPUTABILITY, DEFINABILITY, AND PROOF THEORY OF ARTINIAN RINGS

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Abstract. We show that, in the context of Reverse Mathematics, WKL_0 (Weak König's Lemma) implies the statement ART_0 which says that every Artinian ring is Noetherian, over RCA_0 (Recursive Comprehension Axiom). To achieve this goal we prove a general Computable Full Structure Theorem for computable Artinian rings similar to the classical version found in most Algebra texts.

1. Introduction

Computable Mathematics is the subfield of Computability Theory that focuses on the algorithmic content of mathematical constructions and structures. Generally speaking, computable mathematicians ask questions like "which sets $B \subseteq \mathbb{N}$ can compute an isomorphic copy of a given structure?" and "which computability strengths are sufficient to carry out a given construction?" For example, one could ask "which finitely presented groups have a computable (i.e. decidable) word problem?" or "relative to which Turing degrees are the word problems of *all* computable groups computable?" or "for which Diophantine equations is the set of solutions computable?" Interesting results are typically those that establish deep interactions between the other branches of Mathematics (i.e. Group Theory, Ring Theory, Analysis, etc.) and Computability Theory.

Computable Algebra is one of the oldest branches of Computable Mathematics. Its roots can be found in the works of algebraists such as Galois, Gauss, Dedekind, Kronecker, van der Waerden, and many others (see [SHT, pages 369–371] for more details and historical remarks). More recently, however, the subject formally began after Turing and others gave a precise definition of algorithm, with the early work of Post [Pos47] and Turing [Tur50] on the decidability of the Word Problem for semigroups; the more well-known solution of the World Problem for groups by Novikov [Nov55] and Boone [Boo66]; the work of Davis, Putnam, and Robinson [DPR61] on Hilbert's Tenth Problem; the work of Fröhlich and Shpherdson [FS56] on computable fields; and finally Matyasevich's solution to Hilbert's Tenth Problem [Mat93].

This article is a contribution to Computable Algebra and Reverse Mathematics that ultimately deals with the computable structure of computable Artinian rings, as well as the proof-theoretic consequences of this structure. In short, we will develop and employ some novel algebraic techniques to prove a computable structure theorem for computable Artinian rings that mirrors the classical Structure Theorem for Artinian Rings found in most introductory Algebra texts. To put it even more succinctly:

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we will (essentially) "rewrite" the traditional literature on Artinian rings from the point of view of Computability, Definability, and Reverse Mathematics.

Along the way, we will establish many interesting and beautiful interactions between computable rings that satisfy the Descending Chain Condition (DCC) on ideals (which all Artinian rings satisfy, by definition) and their computability-theoretic structure. We stress that our main theorems are obtained via a *deep algebraic analysis* of Artinian Ring Theory. We will explicitly state and discuss the significance of our main theorems in the next section. The remainder of this section gives a more general overview of our main results and their mathematical, logical, computability-, and proof-theoretic significance.

1.1. **A General Overview of our Results.** This article is a sequel to [Con10], and as such will assume that the reader is familiar with most of the material presented in [Con10], as well as the basics of Computability Theory (see [DH10, Nie09, Soa87, Soa] for more details) and Reverse Mathematics (see [Sim09] for more details). Although in Section 3 we will review much of what we need from these sources. Throughout this article the reader should always bear in mind that all structures that we consider are either finite or countable, and we will assume all rings to be commutative with identity.

Reverse Mathematics began mainly with the work of H. Friedman [Fri75] and others [FSS83, FSS85] and, generally speaking, attempts to classify the strengths of mathematical theorems by determining the weakest axioms that prove them. More specifically, in Reverse Mathematics one typically attempts to classify the strengths of "set-existence theorems" from Second-Order Arithmetic by determining the smallest subsystem of Second-Order Arithmetic in which that theorem has a proof. Over the years five axiom systems have played the most prominent role in Reverse Mathematics. Indeed, it seems that "most" theorems from Mathematics are equivalent to one of the following five subsystems of Second-Order Arithmetic (listed in strictly increasing order of strength): RCA_0 (Recursive Comprehension Axiom), WKL_0 (Weak König's Lemma), ACA_0 (Arithmetic Comprehension Axiom), ATR_0 (Arithmetic Transfinite Recursion Axiom), and $\mathsf{\Pi}^1_1-\mathsf{CA}_0$ (Π^1_1-Comprehension Axiom); for more information on the "Big Five" subsystems of Second-Order Arithmetic, including their precise definitions, see [Sim09]. Generally speaking, RCA_0 is the subsystem of Second-Order Arithmetic that most closely resembles Computable Mathematics; WKL_0 is the smallest subsystem of Second-Order Arithmetic in which compactness arguments are valid; ACA_0 is the smallest subsystem of Second-Order Arithmetic in which Turing's Halting Set and its finite iterations exist; ATR_0 is the smallest subsystem of Second-Order arithmetic that has a "reasonable" theory of ordinals and in which any two ordinals are comparable; and $\mathsf{\Pi}^1_1-\mathsf{CA}_0$ is the smallest subsystem of Second-Order Arithmetic in which Π^1_1-definable sets and finite iterates of the Turing Hyperjump exist. Recall that, in the context of Reverse Mathematics, we take RCA_0 as our base theory and hence we will always work over RCA_0.

Recently, Montalbán [Mon11] has called a theorem of Mathematics *nonrobust* whenever small "perturbations" of that theorem are equivalent to the original theorem over RCA_0 (in the context of Reverse Mathematics). He also points out that, usually, nonrobust theorems correspond to statements not equivalent to any of the "Big Five" in the context of Reverse Mathematics. In [Con10] the author constructs many natural statements about Artinian rings, each of which is equivalent to WKL_0 over RCA_0 and/or $\mathsf{RCA}_0 + \mathsf{I\Sigma}_2$ (see [Con10, Theorem 3.4] and its proof for more details), and also shows that the statement "every Artinian ring is of finite length" is equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{B\Sigma}_2$ (we will review ω-models and the induction schemes $\mathsf{B\Sigma}_2$ and $\mathsf{I\Sigma}_2$ in Section 3 below). This is evidence that the statement "every Artinian ring is Noetherian" is nonrobust, and thus may not be equivalent to any of the "Big Five" subsystems. If this were the case then "Artinian implies Noetherian" would be the first algebraic statement with this property. However, this is not the case because we will present a new and unexpected proof that every Artinian ring is Noetherian in the axiom system $\mathsf{WKL_0}$, which, in the context of [?, Theorem 4.1], makes $\mathsf{WKL_0}$ equivalent to the statement "Artinian implies Noetherian." More specifically, first we will present two proofs that "Artinian implies Noetherian" via $\mathsf{WKL_0}+\mathsf{I}\Sigma_2$, and then we will combine these two proofs in the presence of a Computable Structure Theorem for Artinian Rings (mentioned above) to show that "Artinian implies Noetherian" follows from $\mathsf{WKL_0}$ without any additional assumptions regarding induction/pigeonhole principles.

Recall that the Jacobson radical of a (possibly noncommutative) ring R is the intersection of all maximal ideals of R, and if R is commutative then for any given $x \in R$ the *annihilator* of x is the ideal in R defined by

$$Ann(x) = \{y \in R : x \cdot y = 0\}.$$

As we already said, we will present *two* new proofs that all Artinian rings are Noetherian via $\mathsf{WKL_0}+\mathsf{I}\Sigma_2$ (in the context of Reverse Mathematics). Both proofs consider annihilators of various finite sequences of elements of the Jacobson radical and use the key fact that if $x_0, x_1, \ldots, x_k \in R$, $k \in \mathbb{N}$, are finitely many elements of a commutative ring R, then the annihilator

$$Ann(x_0, x_1, \ldots, x_k) = \bigcap_{i=0}^{k} Ann(x_i) \subset R$$

is Δ^0_1-definable (i.e. computable), uniformly in the parameters $x_0, x_1, \ldots, x_k, k$. However, we consider our first proof to be the more significant one, because it uses the surprising key lemma that says that the nilpotence of the Jacobson radical of an Artinian ring follows from $\mathsf{WKL_0}$. The most interesting part about all of our new proofs is that the new key ingredients are purely algebraic, and not logical or computability-theoretic. In other words, we will present *a new algebraic approach/paradigm* to proving the nilpotence of the Jacobson radical in commutative Artinian rings. Moreover, our paradigm is different than anything presented in the standard algebraic literature to which we are familiar, such as [DF99, Eis95, Lam01, Lan93, AM69, Mat04]. We will also use our proof of the nilpotence of the Jacobson radical in (commutative) Artinian rings via $\mathsf{WKL_0}$ to go on to prove the Full Structure Theorem for commutative Artinian rings (i.e. every commutative Artinian ring is isomorphic to a finite direct product of local Artinian rings) via $\mathsf{WKL_0}$. Again, our approach is different than anything we have seen in the traditional algebraic literature. In particular, our proofs of the "Noetherianess," structure, and nilpotence of the Jacobson radical of commutative Artinian rings each *avoid* constructing any sort of minimal ideal (which is the key idea found throughout the standard literature on Artinian rings), and that is the main reason that our proof avoids using the full strength of $\mathsf{ACA_0}$ and is valid in (the much weaker system) $\mathsf{WKL_0}$. More precisely, the reader will see that our use of annihilators (which are computable in a computable ring) rather than the more traditional approach of constructing minimal ideals as the *key difference* between our (weaker) proof and the other (more powerful) techniques that pervade the standard algebraic literature.

A particularly interesting proof-theoretic consequence of our new proof is that there is a model of $\mathsf{RCA_0}$ in which every commutative Artinian ring is Noetherian, but not every commutative Artinian ring is of finite length (as a module over itself). More specifically, we will eventually see that every model of $\mathsf{WKL_0}+\neg\mathsf{ACA_0}$ has this property. This is interesting in light of the fact that every proof that Artinian implies Noetherian given in the traditional literature such as [DF99, Eis95, Lan93, AM69, Mat04] actually proves that every Artinian ring is of finite length, which, by the author's results in [Con10], is *strictly stronger* in the context of Reverse Mathematics. In other words, we will give several proofs that Artinian implies Noetherian that do not filter through the traditional/standard proof that actually shows Artinian implies finite length.

We now introduce several subsystems of Second-Order Arithmetic, some of which we have previously discussed and will play major roles throughout the rest of this article. We take each of the following statements to imply RCA_0, as well as the statements written beside them. For more information on subsystems of Second-Order Arithmetic, consult [Sim09].

ART_0: Every Artinian ring is Noetherian.
$\mathsf{ART}^{\mathsf{l}}_0$: Every local Artinian ring is Noetherian.
$\mathsf{ART}^{\mathsf{s}}_0$: Every Artinian ring is a finite direct product of local Artinian rings.
NIL_0: The Jacobson radical of an Artinian ring exists and is nilpotent.

The main reverse mathematical theorems of this article show that $\mathsf{ART}^{\mathsf{l}}_0$, ART_0, $\mathsf{ART}^{\mathsf{s}}_0$, and NIL_0 (introduced above) are equivalent to WKL_0 over RCA_0, thus completing the analysis of the Reverse Algebra of Artinian rings begun in [Con10] and answering [Mon11, Question 13]. However, we should mention that we consider the most significant result of this article to be our Full Computable Structure Theorem for Artinian Rings, which we mentioned earlier and we will discuss in greater detail in the next section. In other words, we consider the *methodology of our proof* that ART_0 follows from WKL_0 to be much more interesting than the statement of the theorem itself, because it represents a significant paradigm shift for analyzing Artinian rings in the context of Computability, Definability, and Proof Theory.

Now, for those readers that are familiar with the current state of Reverse Mathematics, we mention that NIL_0, ART_0, $\mathsf{ART}^{\mathsf{l}}_0$, and $\mathsf{ART}^{\mathsf{s}}_0$ are some of the first nontrivial examples of Π^1_2-statements[1] from Second-Order Arithmetic of the form

$$(*) \qquad \forall(\forall \to \forall) \quad \text{or} \quad \forall(\forall \to \exists\forall),$$

where the $\forall, \exists$ quantifiers range over second-order (i.e. set) variables and we have omitted the first-order quantifiers and most of the other aspects of the formulas, to have their reverse mathematical strengths *completely determined* in terms of every other subsystem of Second-Order Arithmetic whose strength is currently determined.[2] Other statements of this form $(*)$ have been examined by Montalbán [Mon06] and by Downey, Hirschfeldt, Lempp, and Solomon [DHLS03] (see these references for more details) in the context of extendibility of orderings ω (the natural numbers) and η (the rational numbers), and in each of these cases the precise reverse mathematical strengths of the theorems considered is still open. One of the reasons why statements of the form $(*)$ are difficult to classify in the context of Reverse Mathematics is because, generally speaking, it is difficult to deduce consequences from (i.e. code information into) such statements. This is because the implication $\to$ is actually a disjunction, and to obtain a useful logical conclusion from a disjunction $\mathsf{A} \vee \mathsf{B}$, one usually proves the negation of one of the disjuncts, say $\neg\mathsf{A}$, and then concludes B. In our case, since we are working over RCA_0, one must prove $\neg\mathsf{A}$ in RCA_0 to conclude B, and, since RCA_0 is a very weak axiom system, this can be quite difficult or even impossible. More generally, however, if a mathematical logician is in the nontrivial case when the standard proof of a given theorem is not obviously valid in RCA_0, or, more generally, the theorem does not easily reverse to the standard proof (as in our case), then he or she has the daunting task of either:

[1] Generally speaking, a Π^1_2 statement is a statement of the form $\forall X \exists Y \cdots$, where the variables $X, Y \subseteq \mathbb{N}$ represent *sets* and the $\cdots$ only mention X, Y and other *number* variables $a, b, c, \ldots \in \mathbb{N}$. For more information on Second-Order Arithmetic and Π^1_2-statements, see [Sim09]. Mathematical statements studied in the context of Reverse Mathematics are usually Π^1_2-statements.

[2] Richard Shore was the first person to point this out to the author who thanks Shore for this astute observation.

(1) Deducing a nontrivial consequence from the theorem that reverses to a standard proof; or
(2) Finding a *new* proof of the theorem that is logically different from any of the standard proofs that appear in the literature.

Sometimes, as in our case, the logician must accomplish *each* of these tasks. In our case task (1) was essentially achieved in [Con10, Theorem 4.1], while task (2) is one of the main goals of this article.

2. The Significance of our Main Results

2.1. **The Computable Structure of Computable Artinian Rings.** Initially, our result in Section 5 below that ART_0 follows from $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ came as quite a surprise, because, generally speaking, on the surface ART_0 appears to be a kind of dependent choice axiom for constructing infinite strictly descending chains of ideals in non-Noetherian rings. This is essentially why the statement "every Artinian ring has finite length" is equivalent to ACA_0, and before we proved the results in Section 5, our intuition led us to believe that ART_0 was more likely to be equivalent to ACA_0 than to WKL_0. Moreover, even after proving all of the theorems in Sections 5 and 6 below, our intuition for why ART_0 should follow from the seemingly feeble axiom $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$, although somewhat clearer, was still somewhat hazy. However, it was clear that the answer had something to do with annihilator ideals.

The main goal of Section 7 below is to give a deep analysis of the computable structure of computable Artinian rings and prove a Computable Full Structure Theorem for Artinian Rings that:

(1) sheds significant light on the computability structure of a computable Artinian ring,
(2) allows us to show that WKL_0 implies ART_0 over RCA_0 without the added assumption of $\mathsf{I}\Sigma_2$ (this is done in Section 8 below),
(3) demystifies and gives a good intuition for why ART_0 *should* follow from a weak axiom like WKL_0, and
(4) explains why the theory of the annihilators[3] of an Artinian ring R essentially determines the theory of R.[4]

One thing that was clear from the author's initial algebraic analysis of ART_0 in Section 5 was that annihilator ideals play a central role and are extremely important to the theory of Artinian rings. Section 7 below extends that analysis and essentially shows that annihilators play *the* central role in the theory of Artinian rings.

The Computable Full Structure Theorem for Artinian Rings is a beautiful interaction between Algebra and Computability Theory. It gives the complete structure of an Artinian ring R (one can also think of it as a "map/blueprint" or the "spine" of R) in terms of finitely many annihilator ideals in R, and essentially allows one to think of R as finitely many computable towers (or "layers") of vector spaces as in the classical version of the theorem (more details are given below). Before we explicitly state the Computable Full Structure Theorem for Artinian Rings, we state the Classical Full Structure Theorem for Artinian Rings, which can be found in most standard Algebra texts. Throughout this article we will use $\omega = \{0, 1, 2, \ldots\}$ to denote the standard natural numbers, and $\mathbb{N}$ to denote the natural

[3]Recall that if R is a commutative ring with identity then the *annihilator of* $x_1, x_2, \ldots, x_n \in R$, $n \in \mathbb{N}$, is the proper ideal

$$I = \{y \in R : (\forall 1 \leq i \leq n)[y \cdot x_i = 0]\} \subset R.$$

[4]In other words, when working with an Artinian ring, rather than working with every ideal in the ring, it suffices to mainly work with *annihilator ideals*.

numbers in some (possibly nonstandard) model of First-Order Arithmetic (see [Sim09] for more details).

Theorem 2.1 (Classical Full Structure Theorem for Artinian Rings). *Let R be a local Artinian ring, and let $M \subset R$ be the unique maximal ideal of R. Then M is nilpotent, i.e. there exists $n \in \omega$ such that $M^n = 0$, and this gives a finite tower/filtration of ideals*

$$0 = M^n \subset M^{n-1} \subset \cdots \subset M \subset R,$$

where $n \in \omega$ is least such that $M^n = 0$. Moreover, every successive quotient in the tower/filtration, M^k/M^{k-1}, is an R/M-vector space.

Now, let R be an Artinian ring. Then R is a finite direct product of local Artinian rings, each with a finite tower/filtration as above.

We think of the second part of the Classical Full Structure Theorem as being the Classical Structure Theorem for Artinian rings, and we use the word "full" in our description of these structure theorems to imply both the first and second parts. This makes sense from the point of view of Computability Theory and Reverse Mathematics because, in general (i.e. in an arbitrary ring), ideals of the form M^i, $0 \leq i \leq n$, $M \subset R$ maximal, are not necessarily computable and therefore may not exist in the context of Reverse Mathematics.

Let $n \in \omega$, $n > 1$, be given, and let

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

be the unique prime factorization of n given by the Fundamental Theorem of Arithmetic. Recall the standard fact from undergraduate Algebra (more specifically the Chinese Remainder Theorem) that says

$$\mathbb{Z}/n\mathbb{Z} \cong \prod_{i=1}^{k} \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}.$$

With this simple example in mind, the Classical Structure Theorem for Artinian Rings says that, roughly speaking, every Artinian ring looks like $\mathbb{Z}/n\mathbb{Z}$, for some $n \in \omega$, $n \geq 1$. This is what makes the Classical Full Structure Theorem so useful and beautiful.

We now state the Full Computable Structure Theorem for Artinian rings, which is key to proving our Main Theorem (described below). Moreover, we consider it to be the central and most significant result contained in this article.

Theorem 7.3 (Full Computable Structure Theorem for Artinian Rings). *Let R be a computable local Artinian ring. Then the unique maximal ideal of R, $M \subset R$, is computable and nilpotent. Moreover, there is a finite tower/filtration of computable (annihilator) ideals*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M \subset R,$$

where $n \in \omega$ is least such that $M^n = 0$.

Now, let R be a computable Artinian ring. Then R is a finite direct product of local computable Artinian rings, each with a finite tower/filtration as above.

From the perspectives of Computability, Definability, and Proof Theory, the finite tower/filtration of annihilator ideals in each local factor of an Artinian ring R given in our Full Computable Structure Theorem for Artinian Rings are *the most important ideals in R* and essentially determine the theory of R, as we shall see in the proof of our Main Reverse Mathematical

Theorem below, which we discuss further in the next subsection.[5] We also note that in general this tower/filtration is *different from* the powers of maximal ideals of R, and this highlights an *important difference* between the Classical Full Structure Theorem and the Full Computable Structure Theorem for Artinian Rings. In other words, our results *do not follow the traditional paradigms* for analyzing the algebraic structure of Artinian rings.

2.2. **Our Main Reverse Mathematical Theorem.** We now state our Main Reverse Mathematical Theorem (Theorem 8.3 below) and an immediate corollary in the context of [Con10, Theorem 4.1]. We consider it to be an important milestone in Reverse Mathematics, and is the end result of roughly five years of work.

Theorem 8.3 (Main Reverse Mathematical Theorem). WKL_0 *proves* ART_0 *(over* RCA_0*).*

Corollary 8.4. WKL_0 *is equivalent to* ART_0 *(over* RCA_0*).*

In Section 5 below we will give *two* different proofs of ART_0 from $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$, and the hypothesis $\mathsf{I}\Sigma_2$ will be crucial to each of the proofs. Thus, from an empirical point of view, it is somewhat surprising that we are able to remove the $\mathsf{I}\Sigma_2$ assumption from both of our proofs in Section 5. However, with the Full Computable Structure Theorem in mind, it is actually not that surprising at all. We will use key ideas from *both* of our proofs in Section 5 below to prove Theorem 8.3. We also note that in most Algebra textbooks, the Classical Full Structure Theorem for Artinian Rings is proved *after* ART_0[6], but here we are using the Full Structure Theorem to prove ART_0. In other words, from the traditional algebraic point of view, we are doing things *backwards*.

We will prove our Main Reverse Mathematical Theorem in steps, by first proving the following lemmas. Recall that in Section 5 we will prove that $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ implies ART_0 (and thus $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ implies $\mathsf{ART}_0^{\mathsf{I}}$). Also note that $\mathsf{WKL}_0 + \mathsf{B}\Sigma_2$ is a strictly weaker theory than $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ (we will discuss this further in the next section; see [Sim09] or [KP77] for more details).

Lemma 8.1. $\mathsf{WKL}_0 + \mathsf{B}\Sigma_2$ *implies* $\mathsf{ART}_0^{\mathsf{I}}$*.*

Lemma 8.2. WKL_0 *implies* $\mathsf{ART}_0^{\mathsf{I}}$*.*

The Computable Full Structure Theorem for Artinian Rings will play a major role in the proof of our Main Reverse Mathematical Theorem, as well as the two lemmas above. Also, in going from the proof of Theorem 5.2 below, which says that $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ implies ART_0, to the proof of Lemma 8.1, we will use the Full Computable Structure Theorem to replace our use of $\mathsf{I}\Sigma_2$ by a use of $\mathsf{B}\Sigma_2$ or, equivalently, the Infinite Pigeonhole Principle[7] [Hir]. Then, to go from the proof of Lemma 8.1 to the proof of Lemma 8.2, we will replace our use of the Infinite Pigeonhole Principle (i.e. $\mathsf{B}\Sigma_2$) with an application of the Finitary Pigeonhole Principle[8] which follows from RCA_0, essentially via Σ^0_1-induction. In going from the proof of

[5]The main reason why these towers are the most important ideals of R is because they readily give a tower/filtration of ideals of R whose successive quotients are R/M-vector spaces, for some maximal ideal $M \subset R$. Thus every computable Artinian ring is essentially a finite tower of computable vector spaces. This will be discussed further later on.

[6]This is probably done because classically ART_0 is essentially a direct corollary of NIL_0, and the texts want to prove NIL_0 as soon as possible. NIL_0 is probably the most important fact about Artinian rings and is used in essentially all proofs of theorems about Artinian rings that do not apply to all Noetherian rings as well.

[7]The Infinite Pigeonhole Principle says that if an infinite set is partitioned into finitely many pieces, then one of those pieces must be infinite.

[8]The Finitary Pigeonhole Principle says that if a finite set of $n+1$ elements is partitioned by at most n-many sets, then one of the sets in the partition has at least 2 elements. It is well-known that, over RCA_0

Lemma 8.2 to the proof of our Main Theorem, we use another application of the Finitary Pigeonhole Principle on top of the previous application in the last sentence. So, to prove ART_0 from WKL_0 we essentially take our two proofs that $\mathsf{WKL}_0+\mathsf{I}\Sigma_2$ implies ART_0 in Section 5, and use the Full Computable Structure Theorem for Artinian Rings to allow ourselves to use the Finitary Pigeonhole Principle instead of $\mathsf{I}\Sigma_2$ or the Infinitary Pigeonhole Principle ($\mathsf{B}\Sigma_2$).

3. Background, Definitions, and Notation

In this section we introduce our basic notation and definitions, as well as the author's and others' previous results that we will require from [Con10, DLM07] and basic Commutative Algebra.

3.1. **The Basics.**

3.1.1. *Computability Theory.* We briefly review the basic definitions and notation that we require and will use throughout the rest of this article. For more information on the basics of Computability, including our definitions and notation described below, consult [DH10, Sim09, Soa87]. Most of what follows in this subsection can also be found in [Con10]. Recall that $\omega = \{0, 1, 2, \ldots\}$ denotes the standard Set of Natural Numbers. On the other hand, $\mathbb{N}$ will denote the first-order part of a (possibly nonstandard) model of RCA_0. We will say that a property holds *for almost all* $n \in \mathbb{N}$ whenever that property holds for all but finitely many $n \in \mathbb{N}$. Throughout this subsection all of our definitions will be made in RCA_0. In other words, throughout this subsection we will assume that we are working in a possibly nonstandard model of RCA_0. Lower case roman letters $a, b, c, \ldots$ will usually denote first-order (i.e. number) variables, while capital roman letters $A, B, C, \ldots$ will usually denote second-order (i.e. set) variables. Let $A, B \subseteq \mathbb{N}$. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *total* whenever the domain of f is $\mathbb{N}$, and we say that f is *partial* to indicate that the domain of f may not be all of $\mathbb{N}$. Also, we say that A is *computable* whenever there is an algorithm that decides, for each $x \in \mathbb{N}$, whether $x \in A$. It is well-known that A is computable iff A is Δ^0_1-definable, and we will use the term computable to *mean* Δ^0_1-definable when working in a nonstandard model of arithmetic. More information on definability and the complexity of formulas can be found in [Soa87, Soa, Sim09]. One can also define what it means for a set A to be computable *relative to an oracle* B [Soa87, Chapter III]. In this case one usually writes $A \leq_T B$, and it follows that $\leq_T$ is a quasi-ordering on $\mathcal{P}(\mathbb{N})$ while the relation

$$A \equiv_T B, \text{ i.e. } A \leq_T B \ \&\ B \leq_T A,$$

is an equivalence relation on $\mathcal{P}(\mathbb{N})$. The equivalence classes of the $\equiv_T$ relation are called *Turing degrees*. It follows that A is computable relative to B iff A is Δ^0_1-definable relative to the parameter B. Saying that a set $A \subseteq \mathbb{N}$ is computable relative to $B \subseteq \mathbb{N}$ is equivalent to saying that A is Δ^0_1-definable relative to B. We should also mention that, strictly speaking, in the context of Reverse Mathematics and nonstandard models of arithmetic, Δ^0_1-definability is the *actual definition* of (relative) computability, though we will use these terms interchangeably. It follows that a set A is computable (relative to B) iff its characteristic function is computable (relative to B). Via a computable 1-1 and onto pairing function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we may speak of computable subsets of

$$\mathbb{N}^n = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{n},\ n \in \mathbb{N},$$

and Σ^0_1-induction, the Finitary Pigeonhole Principle is equivalent to the more general principle that if a set of size $nk + 1$, $n, k \in \mathbb{N}$, is partitioned into at most n-many sets, then one of the sets in the partition has at least $(k + 1)$-many elements.

and it follows that a function $f : \mathbb{N} \to \mathbb{N}$ is computable (relative to B) iff the graph of f is computable (relative to B). We will write $A \oplus B$ to denote the disjoint union of A and B, i.e.

$$A \oplus B = A \times \{0\} \cup B \times \{1\}.$$

Similarly, we define

$$A_0 \oplus A_1 \oplus \cdots \oplus A_k = \bigcup_{i=0}^{k} \left(A_i \times \{i\}\right),$$

and

$$\bigoplus_{i=0}^{\infty} A_i = \bigcup_{i=0}^{\infty} (A_i \times \{i\}).$$

Recall that an infinite set A is *computably enumerable* (c.e.) iff

(1) There is an algorithm that lists the elements of A (not necessarily in increasing order);
(2) There is a 1-1 computable function $f : \mathbb{N} \to \mathbb{N}$ such that A is the range of f;
(3) A is Σ^0_1 definable.

One may also speak of c.e. *relative to* (the oracle/parameter) B. Recall that there is a uniformly computable listing of the partial computable functions, $\{\varphi_e\}_{e\in\mathbb{N}}$, and that the Halting Set $\emptyset'$ is defined as follows:

$$\emptyset' = \{e \in \mathbb{N} : \varphi_e(e)\downarrow\},$$

where $\varphi_e(e)\downarrow$ means that the e^{th} partial computable function *halts* on input $e \in \mathbb{N}$. For any given oracle B, one can construct an effective listing of the partial computable functions relative to B, $\{\varphi^B_e\}_{e\in\mathbb{N}}$, and define the Halting Set relative to B, B', in an analogous fashion. Let $0'$ denote the Turing degree of $\emptyset'$. It is well-known that $\emptyset$ does not compute $\emptyset'$ (i.e. $\emptyset'$ is not computable relative to $\emptyset$), and, more generally, for all oracles B, B does not compute B'. We say that a set A is *low* whenever $A' \equiv_T \emptyset'$. By our previous remarks it follows that if A is low then A does not compute $\emptyset'$. Also, it is well-known that incomputable low sets exist, and that there is a low model of $\mathsf{WKL_0}$ that is not a model of $\mathsf{ACA_0}$ (showing that $\mathsf{ACA_0}$ is not implied by $\mathsf{WKL_0}$). For any given oracle A, a set B is said to be A-*low* whenever $A' \cong_T B'$.

Fix a number $n_0 \in \mathbb{N}$, and let $n_0^{<\mathbb{N}}$ denote the set of finite strings formed from elements in (the finite set) $\{0, 1, \ldots, n_0 - 1\} \subset \mathbb{N}$. We will typically use lower case Greek letters to denote the elements of $n_0^{<\mathbb{N}}$. For all $\sigma \in n_0^{<\mathbb{N}}$, let $|\sigma| \in \mathbb{N}$ denote the length of σ (i.e. the number of character bits of σ) and let $\sigma(k)$, $0 \le k < |\sigma|$, denote the k^{th} character bit of σ. Furthermore, for any given $l \in \mathbb{N}$, let

$$n_0^{=l} = \{\sigma \in n_0^{<\mathbb{N}} : |\sigma| = l\}$$

and

$$n_0^{\geq l} = \{\sigma \in n_0^{<\mathbb{N}} : |\sigma| \geq l\}.$$

For all $\sigma, \tau \in n_0^{<\mathbb{N}}$, we write $\sigma \subset \tau$ to mean that τ is a proper extension of σ (i.e. σ is a proper initial segment of τ) and we write $\sigma \subseteq \tau$ to mean that either $\sigma = \tau$ or $\sigma \subset \tau$. Also, for all $\sigma, \tau \in n_0^{<\mathbb{N}}$ we write $\sigma\tau \in n_0^{<\mathbb{N}}$ to denote the concatenation of τ to the right of σ, and for all $k \in \{0, 1, \ldots, n_0 - 1\}$ we write $\sigma k \in n_0^{<\mathbb{N}}$ to be the unique string of length $|\sigma| + 1$ that has σ as an initial segment and rightmost character bit k. We say that $T \subseteq n_0^{<\mathbb{N}}$ is a *tree* whenever T is closed under $\subset$ - i.e. for all $\tau \in T$ and $\sigma \subset \tau$ we have that $\sigma \in T$. Let $n_0^{\mathbb{N}}$ denote the set of *infinite* strings formed from elements in (the finite set) $\{0, 1, \ldots, n_0 - 1\} \subset \mathbb{N}$. We will typically use the lower case roman letters f, g, and h, to denote elements of $n_0^{\mathbb{N}}$. For all $f \in n_0^{\mathbb{N}}$ and $l \in \mathbb{N}$, let $f{\restriction}l \in n_0^{<\mathbb{N}}$ denote the first l bits of f (i.e. $f{\restriction}l$ is the unique initial segment of f of length l). Also, for all $f \in n_0^{\mathbb{N}}$ and $\sigma \in n_0^{<\mathbb{N}}$, we write $\sigma \subset f$ to mean that σ is an initial segment of f. Now, if $T \subseteq n_0^{<\mathbb{N}}$ is a tree, then we let

$$[T] = \{f \in n_0^{\mathbb{N}} : (\forall n \in \mathbb{N})[f \upharpoonright n \in T]\}$$

and we say that $[T] \subseteq n_0^{\mathbb{N}}$ is *the set of infinite paths through* T. Recall that $\mathsf{WKL_0}$ is equivalent to saying "for all $n_0 \in \mathbb{N}$, every infinite tree in $n_0^{<\mathbb{N}}$ has an infinite path," which, over $\mathsf{RCA_0}$, is equivalent to saying that "for any set $A \subseteq \mathbb{N}$ there exists a set $B \subseteq A$ that is of PA Turing degree relative to A." Recall that a set B is of PA Turing degree relative to a set A iff every infinite A-computable tree $T_A \subseteq n_0^{<\mathbb{N}}$, $n_0 \in \mathbb{N}$, (where the finite strings in $n_0^{<\mathbb{N}}$ are coded as natural numbers via some fixed Gödel numbering) has a B-computable infinite path $f_B \in [T_A] \subseteq n_0^{\mathbb{N}}$. For nonobvious reasons this is equivalent to saying that "for every disjoint pair of A-computably enumerable sets $C_0^A, C_1^A \subseteq \mathbb{N}$, there is a B-computable set $D^B \subseteq \mathbb{N}$ such that $C_0^A \subseteq D^B$ and $D^B \cap C_1^A = \emptyset$." We will primarily use this (latter) characterization of "PA Turing degree relative to A" in all that follows. It is well-known that low PA Turing degrees exist, and, more generally, for any oracle A there is a PA Turing degree that is A-low. For more information on PA Turing degrees and their connection to $\mathsf{WKL_0}$, consult [Sim09]. Finally, recall that $\mathsf{ACA_0}$ is equivalent to the statement "for every set X, the Halting Set relative to X, X', exists" and that this is equivalent to the Σ_n-Comprehension Scheme for all $n \in \mathbb{N}$.

3.2. **Commutative Algebra.** Recall that all of our rings R will be commutative and have an identity element $1 \in R$. Also recall that a *computable ring* is a computable subset of $\mathbb{N}$, endowed with the structure of a ring such that the addition $+_R$ and multiplication $\cdot_R$ operations on R are computable functions. By *local* ring, we mean a ring R with a unique maximal ideal $M \subset R$. Let R be a ring. Then, if $S \subseteq R$, we write $\langle S \rangle_R \subseteq R$ to denote the ideal generated by S in R. Recall that $\langle S \rangle_R$ is the set of finite sums of the form

$$\sum_{i=0}^{k} r_i x_i,$$

where $k \in \mathbb{N}$, $r_i \in R$, and $x_i \in S$. For any $k \in \mathbb{N}$ and $x_0, x_1, \ldots, x_k \in R$, we write $\langle x_0, x_1, \ldots, x_k \rangle_R$ to denote the ideal generated by $\{x_0, x_1, \ldots, x_k\} \subseteq R$. We sometimes omit the subscript R in $\langle S \rangle_R$ if it is clear which ring R the set S belongs to. For all $A, B \subseteq R$, let

$$S_{AB} = \{x \in R : x = ab,\ a \in A,\ b \in B\},$$

$$S_{A,B} = \{x \in R : x = a + b,\ a \in A,\ b \in B\},$$

and define

$$A \cdot B = AB = \langle S_{AB} \rangle_R \ \text{ and } \ A + B = \langle S_{A,B} \rangle_R.$$

Note that $AB \subseteq A, B$, while $A, B \subseteq A + B$. We will write $A + B = C$ to mean that $(A + B) = C$; in other words, every element of C can be expressed as a finite R-linear combination of elements in $A \cup B$. Also, for all sets $S \subseteq R$ and ideals $I \subseteq R$ let

$$(S : I) = \{r \in R : (\forall s \in S)[r \cdot s \in I]\}.$$

It follows that $(S : I) \subseteq R$ is always an ideal (since I is an ideal). We will write $(x : I)$ to mean $(\{x\} : I)$, $x \in R$, $I \subseteq R$ an ideal. For all multiplicatively closed subsets $S \subseteq R$ not containing $0 \in R$, let $R[S^{-1}]$ be the ring of fractions given by

$$R[S^{-1}] = \left\{ \frac{r}{s} : r \in R,\ s \in S \right\}.$$

Recall that $x \in R$ is nilpotent if there exists $n \in \mathbb{N}$ such that $x^n = 0$. Similarly, we say that $A \subseteq R$ is *nilpotent* whenever there exists $n \in \mathbb{N}$ such that

$$A^n = \underbrace{A \cdot A \cdot A \cdot \dots \cdot A}_{n} = 0.$$

This is equivalent to the existence of $n \in \mathbb{N}$ such that for any $a_0, a_1, \dots, a_n \in A$ we have that

$$\prod_{i=0}^{n} a_i = 0.$$

Recall that for all $x \in R$, the annihilator of x, denoted by $Ann(x) \subset R$, is the ideal of R given by

$$Ann(x) = \{y \in R : xy = 0\} = (\{x\} : \langle 0 \rangle).$$

Similarly, for all $x_0, x_1, \dots, x_k \in R$, $k \in \mathbb{N}$, we have that

$$Ann(x_0, x_1, \dots, x_k) = \{y \in R : (\forall i \leq k)[x_i y = 0]\} = (\{x_j : 0 \leq j \leq k\} : \langle 0 \rangle).$$

If R is a quotient ring of the form R_0/I, for some commutative ring R_0 and ideal $I \subseteq R_0$, then we will write $\overline{x} \in R$, $x \in R_0$, to denote that $\overline{x} \in R$ is the image of $x \in R_0$ under the canonical map $R_0 \to R$. If $I_0 \supseteq I$ is an ideal of R_0, then we will also write $\overline{I}_0$ to represent the unique ideal corresponding to I_0 under the canonical map $R_0 \to R$. Given two ideals $I, J \subseteq R$, let

$$(I : J) = \{r \in R : (\forall x \in I)[rx \in J]\}$$

and similarly, for all $x_0 \in R$ and $J \subseteq R$ an ideal, let

$$(x_0, J) = \{r \in R : rx_0 \in J\}.$$

Let

$$\mathbb{Z}_\infty = \mathbb{Z}[X_0, X_1, X_2, \dots]$$

be a computable presentation of the standard free commutative polynomial ring over $\mathbb{Z}$ with infinitely many indeterminates $X_0, X_1, X_2, \dots$ and no relations between them. Let F denote the field of fractions of $\mathbb{Z}_\infty$, and let $\mathbb{Z}_k = \mathbb{Z}[X_0, X_1, \dots, X_k]$, $k \in \mathbb{N}$.

3.3. **Induction Schemes in Reverse Mathematics.** We now review some basic facts about induction schemes in Reverse Mathematics. First of all, recall that a formula ψ in the language of First-Order Arithmetic is Σ^0_n,[9] $n \in \omega$, $n \geq 1$, whenever it is of the form

$$\psi = (\exists x_1)(\forall x_2)(\exists x_3) \cdots (\exists / \forall x_n)[\varphi(x_1, \dots, x_n)],$$

where φ contains only bounded quantifiers. Similarly, a formula ψ is said to be Π^0_n whenever it is of the form

$$\psi = (\forall x_1)(\exists x_2)(\forall x_3) \cdots (\forall / \exists x_n)[\varphi(x_1, \dots, x_n)],$$

where φ contains only bounded quantifiers. It follows that the negation of a Σ^0_n formula is Π^0_n and vice versa. Now, recall that a model of Second-Order Arithmetic is called an *ω-model* if its first-order part is the standard Set of Natural Numbers. It follows that any such model satisfies Mathematical Induction for all predicates. Now, non-ω-models are those models of Second-Order Arithmetic whose first-order parts are not the standard Set of Natural Numbers. These models may satisfy Mathematical Induction for some predicates, but not for others. With this general idea in mind we introduce the following first-order axiom schemes in Second-Order Arithmetic.

($\mathsf{I\Sigma_n}$) $[\psi(0) \wedge (\psi(n) \to \psi(n+1))] \to [(\forall n)\psi(n)]$, where $\psi(n)$ is a Σ^0_n predicate with free variable $n \in \mathbb{N}$ (and possibly with other first/second-order parameters).

[9] The superscript 0 indicates that ψ contains no set (i.e. second-order) variables, only number (i.e. first-order) variables.

$\mathsf{I\Sigma_n}$ is the induction scheme for Σ^0_n formulas (also referred to as Σ^0_n-induction). It is well-known that, over $\mathsf{RCA_0}$, for each $n \in \mathbb{N}$, $\mathsf{I\Sigma_n}$ is equivalent to $\mathsf{I\Pi_n}$–the induction scheme for Π^0_n formulas. It is also well-known that, over $\mathsf{RCA_0}$, $\mathsf{I\Sigma_n}$ is equivalent saying that every Σ^0_n-definable set of natural numbers has a least element (i.e. the Σ^0_n-Well-Ordering Principle), and this is equivalent to saying that every Π_n-definable set of natural numbers has a least element (i.e. the Π^0_n-Well-Ordering Principle). It is also well-known that for each $n \in \mathbb{N}$, $n \geq 1$, $\mathsf{I\Sigma_{n+1}}$ is strictly stronger than $\mathsf{I\Sigma_n}$. In this article we will be most concerned with $\mathsf{I\Sigma_2}$ (i.e. Σ_2-induction). More information on $\mathsf{I\Sigma_n}$ can be found in [KP77, Sim09].

We will also be concerned with a first-order bounding principle called $\mathsf{B\Sigma_2}$, although we will use the well-known fact that $\mathsf{B\Sigma_2}$ is equivalent to the Infinite Pigeonhole Principle over $\mathsf{RCA_0}$ (see [Hir] for more details). Thus, we will essentially write $\mathsf{B\Sigma_2}$ to mean the Infinite Pigeonhole Principle (this is valid since we will always be working over $\mathsf{RCA_0}$). Moreover, it is well-known that $\mathsf{B\Sigma_2}$ is strictly weaker than $\mathsf{I\Sigma_2}$ but strictly stronger than $\mathsf{I\Sigma_1}$–one of the axiom schemes included in $\mathsf{RCA_0}$ (and therefore $\mathsf{I\Sigma_1}$ is always assumed throughout this article). One can also define a hierarchy of first-order bounding principles, usually denoted $\mathsf{B\Sigma_n}$, that is equivalent to $\mathsf{I\Delta_n}$ (the induction scheme for Δ_n formulas[10]); see [Sla04] for more details. For more information on $\mathsf{B\Sigma_n}$ consult [Sim09, KP77].

A well-known theorem of Harrington says that $\mathsf{WKL_0}$ is Π^1_1-conservative over $\mathsf{RCA_0}$. In other words, $\mathsf{WKL_0}$ proves no new arithmetic formulas over $\mathsf{RCA_0}$. Hence, $\mathsf{WKL_0}$ proves neither $\mathsf{I\Sigma_2}$ nor $\mathsf{B\Sigma_2}$ over $\mathsf{RCA_0}$. Therefore the systems $\mathsf{WKL_0}$, $\mathsf{WKL_0}+\mathsf{B\Sigma_2}$, and $\mathsf{WKL_0}+\mathsf{I\Sigma_2}$ have strictly increasing strengths over $\mathsf{RCA_0}$. Some other useful well-known facts regarding $\mathsf{RCA_0}$ are that it proves the Finitary Pigeonhole Principle and Bounded Σ^0_1-Comprehension. In other words, if $A \subseteq \mathbb{N}$ is Σ_1-definable, then it follows from $\mathsf{RCA_0}$ that every initial segment of A exists. It is well-known that, over $\mathsf{RCA_0}$, Bounded Σ^0_n-Comprehension is equivalent to Bounded Π^0_n-Comprehension (defined similarly to Bounded Σ^0_n-Comprehension), for all $n \in \mathbb{N}$.

3.4. **The Plan of the Paper.** In the next section we shall review some preliminary results and prove a technical proposition that will help us in the proofs of our results in Sections 5-8. The main goal of Section 5 is to give two proofs that $\mathsf{ART_0}$ follows from $\mathsf{WKL_0}+\mathsf{I\Sigma_2}$ over $\mathsf{RCA_0}$. However, along the way we will also show that both $\mathsf{ART^s_0}$ and $\mathsf{NIL_0}$ follow from $\mathsf{WKL_0}$. In Section 6 we take a brief intermission from our main objectives to prove some reversals that say $\mathsf{ART^s_0}$ and $\mathsf{NIL_0}$ each imply $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$ (recall that [Con10, Theorem 4.1] shows that $\mathsf{ART_0}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$). Thus it will follow that $\mathsf{NIL_0}$ and $\mathsf{ART^s_0}$ are equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$, and $\mathsf{ART_0}$ is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}+\mathsf{I\Sigma_2}$. In Section 7 we prove the Full Computable Structure Theorem for Artinian Rings, which essentially shows that, from the logical perspective of Definability, annihilator ideals are the most important kind of ideal in an Artinian ring and essentially determine the theory of these rings. We will prove the Full Computable Structure Theorem both classically and in any model of $\mathsf{WKL_0}$. Although Sections 5 and 6 are technically not prerequisites for Section 7, some of the crucial ideas used in Section 7 come from our initial results in Section 5. Finally, in Section 8 we use the Full Computable Structure Theorem of Section 7, along with the Finitary Pigeonhole Principle and some key ideas from our previous proofs of Theorem 5.2 in Section 5, to prove our Main Reverse Mathematical Theorem which says that $\mathsf{ART_0}$ is equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$ (Theorem 8.3). Thus, our results about $\mathsf{ART_0}$ in Section 8 will supercede our results about $\mathsf{ART_0}$ in Section 5.

[10] A Σ_n formula ψ is Δ_n whenever it is equivalent to a Π_n-formula.

4. Some Preliminary Results

We now proceed to collect various known results that will be useful later on in this paper, beginning with Sections 5 and 6. The first lemma (Lemma 4.1) is a standard fact from Commutative Algebra (regarding fraction rings); the second lemma (Lemma 4.2) and its corollary (Corollary 4.3) are results in Computable Algebra and Reverse Mathematics that have essentially appeared in the literature [Con10, DLM07, FSS83, Sim09] several times but never been explicitly stated as we will now state them below. The next two theorems are [Con10, Theorem 3.4, Theorem 4.1], and Lemma 4.7 reviews a standard technique for constructing computable rings (the"pullback technique") that, in a general sense, is analogous to constructing free objects in the context of Category Theory. Afterwards, we will collect a few more results that will be useful in Sections 7 and 8.

Let R be a commutative ring (with identity), and $U \subseteq R$ a multiplicatively closed subset not containing zero. We begin this section by proving an easy and well-known lemma that we will use in our second proof of Theorem 5.2 in the next section. We use the notation $R[U^{-1}]$ to denote the ring of fractions obtained from R by adding elements of the form $\frac{1}{u}$, for all $u \in U$.

Lemma 4.1. *Let $\varphi : R \to R[U^{-1}]$ be the natural map given by $r \to \frac{r}{1}$, and let $I_0 \subset I_1 \subseteq R$ be ideals of R such that*

$$x \cdot u \notin I_0,$$

for all $x \in I_1$ and $u \in U$. Then $\varphi(I_0)R[U^{-1}] \subset \varphi(I_1)R[U^{-1}]$, as ideals in $R[U^{-1}]$.

Proof. Let $x_1 \in I_1 \setminus I_0$, and suppose (for a contradiction) that

$$\frac{x_1}{1} = \frac{x}{u} \in I_0 R[U^{-1}] \subseteq R[U^{-1}],$$

for some $x \in I_0$ and $u \in U$. It follows that we have

$$u \cdot x_1 = x \in I_0 \subset R,$$

a contradiction (by hypothesis). □

The following lemma and its corollary are from Computable Algebra has essentially appeared in the literature several times, but never been explicitly stated as follows. Lemma 4.2 and Corollary 4.3 are both well-known by computable algebraists, and we will use them without necessarily saying so; we have included them mainly for the nonexpert's convenience.

Lemma 4.2. *Suppose that R is a computable commutative ring and $\{x_k\}_{k\in\mathbb{N}}, S \subset R$, are such that, for all $n \in \mathbb{N}$, x_{n+1} is not an R-linear combination of elements in $S \cup \{x_0, x_1, \ldots, x_n\}$. Then every PA Turing degree computes an infinite sequence of ideals $\{I_k\}_{k\in\mathbb{N}}$ of R such that, for all $n \in \mathbb{N}$, $S \cup \{x_0, x_1, \ldots, x_n\} \subseteq I_n$, but $x_{n+1} \notin I_n$.*

We will not prove Lemma 4.2, but the proof can essentially be found in [Con10, DLM07, FSS83, Sim09]; it was first essentially proven in [FSS83]. The main idea of the proof of Lemma 4.2 is to construct an infinite computable tree T such that every infinite path through T computes an infinite sequence of ideals $\{I_k\}_{k\in\mathbb{N}}$ as in the statement of the lemma. The conclusion of the lemma then follows by one of the characterizations of PA Turing degrees that we gave above. The following corollary is the reverse mathematical analog of the previous lemma. It assumes $\mathsf{WKL_0}$ and the proof is similar to that of Lemma 4.11 below, which we will explicitly give later on in this section.

Corollary 4.3. ($\mathsf{WKL_0}$) *Suppose that R is a commutative ring and $\{x_k\}_{k\in\mathbb{N}}, S \subseteq R$, are such that, for all $n \in \mathbb{N}$, x_{n+1} is not an R-linear combination of elements in $S \cup \{x_0, x_1, \ldots, x_n\}$. Then there is an infinite sequence of ideals $\{I_k\}_{k\in\mathbb{N}}$ of R such that, for all $n \in \mathbb{N}$, $S \cup \{x_0, x_1, \ldots, x_n\} \subseteq I_n$, but $x_{n+1} \notin I_n$.*

Before we proceed we need to recall the following definition from Algebra.

Definition 4.4. *Let R be a ring with identity. Then we say that a subset $S \subset R$ is t-nilpotent if for every infinite sequence of elements $x_0, x_1, x_2, \dots, x_n, \dots \in S$ (with possible repetitions) there exists $N \in \mathbb{N}$ such that*

$$\prod_{i=0}^{N} x_i = 0.$$

Also recall that the *nilradical* of a commutative ring R is the intersection of all prime ideals of R, while the *Jacobson radical* is the intersection of all maximal ideals of R. If R is Artinian it follows that these two radicals are equal.

For the reader's convenience we now state the two main theorems that we will use from [Con10].

Theorem 4.5. [Con10, Theorem 3.4] *Let R be a commutative ring with identity. The following statements are equivalent over $\mathsf{RCA}_0 + \mathsf{I}\Sigma_2$.*

(1) WKL_0
(2) *If R is an Artinian integral domain, then R is a field.*
(3) *If R is Artinian, then every prime ideal of R is maximal.*
(4) *If R is Artinian, then the Jacobson radical J and the nilradical N of R exist and $J = N$.*
(5) *If R is Artinian, then the Jacobson radical of R is t-nilpotent.*
(6) *If R is Artinian and the nilradical of R exists, then R/N is Noetherian.*

In fact, it follows from the proof of [Con10, Theorem 3.4] that (1)-(5) above are equivalent over RCA_0. We will refer to and use [Con10, Theorem 3.4] in Section 5 below.

Theorem 4.6. [Con10, Theorem 4.1] *There exists a computable integral domain R containing an infinite uniformly computable strictly ascending chain of ideals*

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R,\ n \in \mathbb{N},$$

and such that every infinite strictly descending chain of ideals in R,

$$R \supseteq J_0 \supset J_1 \supset J_2 \supset \cdots \supset J_n \supset \cdots,\ n \in \mathbb{N},$$

contains a member (i.e. an ideal) of PA Turing degree.

Moreover, in the proof of [Con10, Theorem 4.1] the author proves that there is an infinite uniformly computable strictly ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R,\ n \in \mathbb{N},$$

in R such that

$$I_\infty = \bigcup_{n\in\mathbb{N}} I_n$$

is also computable and every ideal $I \subset R$ that is not of PA Turing degree is computable and equal to I_n, for some $n \in \mathbb{N} \cup \{\infty\}$. We will use [Con10, Theorem 4.1] in Section 4 below.

Finally, we recall the following useful result in Computable Algebra that allows one to construct computable rings that are computably isomorphic to computably enumerable subrings of a larger computable ring. We will use the following lemma in Section 6 below, and its proof can be found in [DLM07, Section 2.3]. Its proof is essentially the computable version of the construction of universal objects in Category Theory.

Lemma 4.7. *Suppose that Q is a computable ring, and $R_0 \subseteq Q$ is a computably enumerable subring of Q. Then R_0 is computably isomorphic to a computable ring R.*

We now discuss the more advanced background material required for Sections 7 and 8.

It is known to most reverse mathematicians that most of finite dimensional Linear Algebra follows from RCA_0, essentially because all of finite dimensional Linear Algebra is computable. The proofs of the next three lemmas (from Linear Algebra) in RCA_0 are essentially the same as the classical proofs, and thus can essentially be found in any standard Linear Algebra textbook. The first lemma is used to prove the second, and the second lemma is used to prove the third. We will use the third lemma to prove an important proposition (in this section) that will be useful later on in proving our Main Reverse Mathematical Theorem (i.e. Theorem 8.3) that says ART_0 follows from WKL_0 over RCA_0. We leave the proofs of the lemmas to the reader.

Lemma 4.8 (RCA_0)**.** *Every $m \times n$ matrix A has a reduced row echelon form.*

Lemma 4.9 (RCA_0)**.** *If S is a system of $m \in \mathbb{N}$ linear equations in $n \in \mathbb{N}$ unknowns, with $n > m$, then S has a nontrivial (i.e. nonzero) solution.*

Lemma 4.10 (RCA_0)**.** *Let V be a vector space over a field F. If $\{v_0, v_1, \dots, v_n\}$, $n \in \mathbb{N}$, is a spanning set for V, then any collection of m vectors in V, where $m > n$, is linearly dependent.*

We will use the previous lemma in the proof of the following proposition, which we will use to prove our Main Reverse Mathematical Theorem in Section 8.

Proposition 4.11 (WKL_0)**.** *Let R be a ring that is a finite product of fields,*

$$R = F_0 \times F_1 \times \cdots \times F_n, n \in \mathbb{N},$$

and let

$$M = V_0 \times V_1 \times \cdots \times V_n$$

be an R-module such that the action of M on R is given naturally by the actions of each F_i on the corresponding F_i-vector space V_i, $0 \leq i \leq n$. Also, let $\{S_k\}_{k \in \mathbb{N}}$ be an infinite sequence of finite subsets of M such that for each $k \in \mathbb{N}$ we have that $|S_k| > k$ and S_k is an F_i-linearly independent subset of V_i, for some $0 \leq i \leq n$. Then M contains infinite strictly ascending/descending chains of submodules,

$$I_0 \subset I_1 \subset \cdots \subset I_m \subset \cdots \subset M, \ m \in \mathbb{N},$$

and

$$M \supseteq J_0 \supset J_1 \supset J_2 \supset \cdots \supset J_m \supset \cdots, \ m \in \mathbb{N}.$$

Proof. We give a sketch of the proof with most of the details filled in. The main idea and details behind the proof of Proposition 4.11 can be found in [Sim09, Lemma IV.6.2] and [FSS83, FSS85]; it was also used by the author in [Con10, Section 3] and by Downey, Lempp, and Mileti in [DLM07, Proposition 3.1]. We assume that the reader is familiar with at least one of these sources.

Without any loss of generality we will construct an infinite strictly descending chain of submodules $M \supseteq J_0 \supset J_1 \supset \cdots$; the construction of an infinite strictly ascending chain of submodules is very similar.

First of all, for each $k \in \mathbb{N}$, let $\{s_{k,0}, s_{k,1}, \dots, s_{k,k}\}$ be a listing of the first $(k+1)$-many elements of S_k, let $R = \{r_0, r_1, r_2, \dots\}$ be an effective listing of the elements of R, and $M = \{m_0, m_1, m_2, \dots\}$ be an effective listing of the elements of M. Now, let T_0 be the computable tree with $(k+1)$-many branchings at level k, for all $k \in \mathbb{N}$, and such that $\sigma \in T_0$ if and only if every R-linear combination of vectors from

$$\{s_{0,\sigma(0)}, s_{1,\sigma(1)}, \dots, s_{|\sigma|-1,\sigma(|\sigma|-1)}\}$$

with at least one nonzero R-coefficient from $\{r_0, r_1, \ldots, r_{|\sigma|-1}\}$ is nonzero. Since R is a product of fields and M is a product of corresponding vector spaces, it follows that $\sigma \in T_0$ if and only if for every $0 \leq i < |\sigma|$ we have that $s_{i,\sigma(i)}$ is not an R-linear combination of

$$\{s_{0,\sigma(0)}, s_{1,\sigma(1)}, \ldots, s_{i-1,\sigma(i-1)}, s_{i+1,\sigma(i+1)}, \ldots, s_{|\sigma|-1,\sigma(|\sigma|-1)}\}$$

with coefficients from $\{r_0, \ldots, r_{|\sigma|-1}\}$. Finally, it follows that if $f \in [T_0]$ is an infinite path through T, and

$$S = \{s_{k,f(k)} : k \in \mathbb{N}\},$$

then no $s_{k,f(k)}$ is a finite R-linear combination of $S \setminus \{s_{k,f(k)}\}$.

Now we claim that T_0 is an infinite tree. To prove this we will show that for every $\sigma \in T$ such that $\{s_{k,\sigma(k)} : 0 \leq k < |\sigma|\}$ is R-linearly independent, there exists $0 \leq l \leq |\sigma|$ such that $\{s_{k,\sigma l(k)} : 0 \leq k < |\sigma l|\}$ is also R-linearly independent. Then, along with the fact that any nonzero $s \in S_0$ is linearly independent over R, and Π^0_1-induction, it will follow that for each $n \in \mathbb{N}$ there exists $\sigma \in T$, $|\sigma| = n$. Now, let $0 \leq i \leq n$ be such that $S_{|\sigma|} \subseteq V_i$, then

$$\{s_{|\sigma|,0}, s_{|\sigma|,1}, \ldots, s_{|\sigma|,|\sigma|}\} \subseteq S_{|\sigma|}$$

spans a $(|\sigma|+1)$-dimensional subspace of V_i. On the other hand, $\{s_{i,\sigma(i)} : 0 \leq i < |\sigma|\}$ spans at most a $|\sigma|$-dimensional subspace of V_i. So, by the previous lemma it follows that some $s_{|\sigma|,i_0}$, $0 \leq i_0 \leq |\sigma|$, is not an R-linear combination of $\{s_{i,\sigma(i)} : 0 \leq i < |\sigma|\}$. It now follows that $\sigma i_0 \in T_0$. Therefore (by Π^0_1-induction it follows that) T_0 is an infinite tree, and via $\mathsf{WKL_0}$ it follows that T_0 has an infinite path $f \in [T_0]$ that codes an infinite R-linearly independent set of elements in M. In other words, we have used $\mathsf{WKL_0}$ to show that there exists an infinite sequence $S = \{s_{k,f(k)} : k \in \mathbb{N}\}$ of elements in M such that for each $k \in \mathbb{N}$, $s_{k,f(k)}$ is not a finite R-linear combination of $S \setminus \{s_{k,f(k)}\}$.

Now, let $T \subseteq 2^{<\mathbb{N}}$ be a computable tree whose infinite paths $f \in [T]$ code infinite strictly descending submodules of M,

$$J = \bigoplus_{i=0}^{\infty} J_k, \;\; J_k \supset J_{k+1},$$

such that for each $i \in \mathbb{N}$ we have that $s_{i,f(i)} \in J_i$ but $s_{i,f(i)} \notin J_{i+1}$. More precisely, an infinite path $f \in [T]$ codes an element $x = m_j \in M$, $j \in \mathbb{N}$, into the submodule J_k, $k \in \mathbb{N}$, iff when we write $f = \oplus_{i=0}^{\infty} f_i$, $f_i \in 2^{\omega}$ (each f_i is a(n infinite) column of f that codes J_i), we have that $f_i(j) = 1$. Otherwise, $f(j) = 0$ iff $x = m_j \notin J_i$.

Note that the uniformly computably enumerable strictly descending chain of submodules coded by

$$F = \bigoplus_{i=0}^{\infty} \langle s_{i,f(i)}, s_{i+1,f(i+1)}, \ldots \rangle_M$$

would be an infinite path in T if it existed (but it may not). However, since the sequence of generators $\{s_{i,f(i)}\}_{i=0}^{\infty}$ exists by our previous arguments, and therefore the submodules $\langle s_{i,f(i)}, s_{i+1,f(i+1)}, \ldots \rangle_M$ are Σ^0_1-definable, uniformly in $i \in \mathbb{N}$ (because the corresponding generator sequences are computable, uniformly in $i \in \mathbb{N}$), then via Bounded Σ^0_1-Comprehension, it follows that every finite initial segment of F exists and is on T. Hence, $T \subseteq 2^{<\mathbb{N}}$ is an infinite tree. Finally, $\mathsf{WKL_0}$ says that T has an infinite path, $g \in [T] \subseteq 2^{\mathbb{N}}$, and by our definition of T it follows that g codes an infinite strictly descending chain of subspaces in T, which proves the proposition. $\square$

5. The Utility of Annihilators in Artinian rings

The main purpose of this section is to show that the structure theorem for Artinian rings (i.e. $\mathsf{ART}_0^{\mathsf{s}}$) is provable in the system WKL_0. Along the way we will also show that $\mathsf{WKL}_0+\mathsf{I}\Sigma_2$ proves that every Artinian ring is Noetherian (i.e. ART_0) and that WKL_0 proves that the Jacobson radical of an Artinian ring is nilpotent (i.e. NIL_0). Afterwards we will give an alternate direct proof of ART_0 via $\mathsf{WKL}_0+\mathsf{I}\Sigma_2$.[11] By previous results of the author [Con10] it will follow that ART_0 is equivalent to WKL_0 over $\mathsf{RCA}_0+\mathsf{I}\Sigma_2$. In the next section we will examine the consequences of NIL_0, ART_0, and $\mathsf{ART}_0^{\mathsf{s}}$, in the context of Reverse Mathematics. In the last section we will show that ART_0 follows from WKL_0, superceding some of the results in this section.

We begin by showing that WKL_0 proves NIL_0 over RCA_0.

Theorem 5.1. WKL_0 *implies* NIL_0 *over* RCA_0.

Proof. We reason in WKL_0. Let R be a given Artinian ring. In the proof of [Con10, Theorem 3.4] the author showed that WKL_0 proves that the Jacobson radical of R, $J\subset R$, exists. Let $J=\{z_0,z_1,z_2,\dots\}$ be an enumeration of the elements of J, and for each $n\in\mathbb{N}$, let

$$A_n=Ann(z_0,\dots,z_n).$$

RCA_0 proves that the sequence of ideals $\{A_n\}_{n\in\mathbb{N}}$ exists. Now, suppose that there are infinitely many $k\in\mathbb{N}$ such that $A_{n_k}\supset A_{n_k+1}$. Then, via RCA_0, it follows that there exists an infinite strictly descending chain of ideals of the form $\{A_{n_k}\}_{k\in\mathbb{N}}$ in R, contradicting the fact that R is Artinian. Hence, there must exist a number $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have that

$$A_{n_0-1}=Ann(z_0,z_1,\dots,z_{n_0-1})=Ann(z_0,z_1,\dots,z_n)=A_n.$$

Let $T\subseteq n_0^{<\mathbb{N}}$ be the n_0-branching computable tree defined by

$$T=\{\sigma\in n_0^{<\mathbb{N}}:\prod_{j<|\sigma|}z_{\sigma(j)}\neq 0\}.$$

First, suppose that T is infinite. Then by WKL_0 T has an infinite path $f\in[T]\subset n_0^{\mathbb{N}}$ such that for all $n\in\mathbb{N}$,

$$\prod_{j<n}z_{f(j)}\neq 0.$$

In other words, J is not t-nilpotent (see [Con10] for more details), which contradicts [Con10, Theorem 3.4]. Hence, we must have that T is a finite tree.

Now, suppose that T is finite, and let $m_0\in\mathbb{N}$ be large enough so that there is no string of length m_0 on T. In this case we claim that $J^{m_0}=0$. To see why, assume (for a contradiction) that there exist elements $x_0,x_1,\dots,x_{m_0-1}\in J$ such that $\prod_{j<m_0}x_j\neq 0$ and via the Π^0_1-Well-Ordering Principle (which is equivalent to Σ^0_1-induction, and thus follows from RCA_0) let $i_0\leq m_0$, $i_0\in\mathbb{N}$, be maximal such that there exists $\sigma\in n_0^{=i_0}\subset n_0^{<\mathbb{N}}$ such that

$$\prod_{j<i_0}z_{\sigma(j)}\cdot\prod_{i_0\leq j<m_0}x_j\neq 0.$$

We claim that $i_0=m_0$. Suppose for a contradiction that $i_0<m_0$. Then there exists $\sigma\in n_0^{=i_0}$ such that the product

$$z_{\sigma(0)}z_{\sigma(1)}\cdots z_{\sigma(i_0-1)}x_{i_0}x_{i_0+1}\cdots x_{m_0-1}\neq 0,$$

[11] Recall that classifying the reverse mathematical strength of ART_0 was the main motivation for all of the research in this article.

from which it follows that the product

$$x_{i_0}x_{i_0+1}\cdots x_{m_0-1}z_{\sigma(0)}z_{\sigma(1)}\cdots z_{\sigma(i_0-1)}\neq 0.$$ [12]

Now, since $x_{i_0}\in J$, and by our definition of $n_0\in\mathbb{N}$ and $z_0,z_1,\ldots,z_{n_0}\in J$ above, it follows that there is some $n\in\{0,1,\ldots,n_0-1\}$ such that

$$z_nx_{i_0+1}\cdots x_{m_0-1}z_{\sigma(0)}z_{\sigma(1)}\cdots z_{\sigma(i_0-1)}\neq 0,$$

from which it follows that

$$z_{\sigma(0)}z_{\sigma(1)}\cdots z_{\sigma(i_0-1)}z_nx_{i_0+1}\cdots x_{m_0-1}\neq 0,$$

from which it follows that there is some $\tau\in n_0^{=(i_0+1)}$ properly extending σ, $\tau=\sigma n$, such that

$$\prod_{j<i_0+1}z_{\tau(j)}\cdot\prod_{i_0+1\leq j<m_0}x_j\neq 0,$$

contradicting our definition of i_0. Hence, $i_0=m_0$ and there is a string $\tau\in n_0^{=m_0}$ such that

$$\prod_{j<m_0}z_{\tau(j)}\neq 0,$$

contradicting our definition of m_0. Hence no such sequence $x_0,x_1,\ldots,x_{m_0-1}\in J$ exists and $J^{m_0}=0$. □

In the next section we will use the author's results in [Con10] to prove that NIL_0 implies WKL_0 over RCA_0. We now use the previous theorem (i.e. Theorem 5.1) to show that $\mathsf{WKL}_0+\mathsf{I\Sigma}_2$ implies ART_0 over RCA_0.

Theorem 5.2. $\mathsf{WKL}_0+\mathsf{I\Sigma}_2$ *implies* ART_0 *over* RCA_0*.*

Proof. The proof is very similar to the part of the proof of [Con10, Theorem 3.4] given in [Con10, Section 3.5]. We assume that the reader is familiar with [Con10, Section 3.5] and therefore we will only give a sketch of the proof and leave the details (which can essentially be found in [Con10, Section 3.5]) to the reader. Let R be a ring with an infinite strictly ascending chain of ideals $I_0\subset I_1\subset I_2\subset\cdots$ and a corresponding infinite sequence of elements $\{x_k\}_{k\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$ we have that $x_i\in I_{i+1}\setminus I_i$. We will construct, via $\mathsf{WKL}_0+\mathsf{I\Sigma}_2$, an infinite strictly descending chain of ideals $J_0\supset J_1\supset J_2\supset\cdots$ in R.

First of all, via [Con10, Section 3.5.1] we can assume that R has only finitely many maximal ideals, $M_0,M_1,\ldots,M_{n_0-1}\subset R$, $n_0\in\mathbb{N}$ (otherwise we can construct an infinite strictly descending chain of ideals in R by repeatedly intersecting maximal ideals; see [Con10, Section 3.5.1] for more details). In this case, we have that the Jacobson radical of R, $J\subset R$, is equal to the product $M_0M_1\cdots M_{n_0-1}$ (see [Con10, Section 3.5.2] for more details), and by Theorem 5.1 above it follows that there is a number $m_0\in\mathbb{N}$ such that $J^{m_0}=(M_0M_1\cdots M_{n_0-1})^{m_0}=0$. Now, for all $0<i\leq n_0m_0$, write $i=k_in_0+r_i$, $r_i,k_i\in\mathbb{N}$, $r_i<n_0$, and define

$$A_i=J^{k_i}M_0M_1\cdots M_{r_i}.$$

Although we cannot actually prove that A_i, $0<i\leq n_0m_0$, exist via WKL_0, we have that A_i is Σ^0_1-definable for all $0<i\leq n_0m_0$, and we can use $\mathsf{I\Sigma}_2$ (as in [Con10, Section 3.5.2]) to find the largest number $i_0\in\mathbb{N}$, $i_0\leq n_0m_0$, such that there exist infinitely many $k\in\mathbb{N}$ such that $x_k\in A_{i_0}$. It follows that there is a computably enumerable subset of $\{x_k\}_{k\in\mathbb{N}}$ that live inside $A_{i_0}\setminus A_{i_0+1}$. Let $f:\mathbb{N}\to\mathbb{N}$ be a total computable strictly increasing function such that $x_{f(k)}\in A_{i_0}\setminus A_{i_0+1}$ for all $k\in\mathbb{N}$. Now, note that the quotient A_{i_0}/A_{i_0+1} is an $R/M_{r_{i_0+1}}$-vector space and by our construction of $\{x_k\}_{k\in\mathbb{N}}$ above it follows that for all $k\in\mathbb{N}$ $x_{f(k)}$ is not an

[12] Here we are using the commutativity of R. Whether or not ART_0 follows from WKL_0 in the noncommutative case is still open.

R-linear combination of $\{x_{f(j)}\}_{j>k}$. Now, by an argument similar to the one given in [Con10, Section 3.5.2] that is essentially based on the proof of Corollary 4.3 mentioned above, we can use $\mathsf{WKL_0}$ to construct an infinite strictly descending chain of ideals $J_0 \supset J_1 \supset J_2 \supset \cdots$ such that for all $k \in \mathbb{N}$ we have that $x_{f(k)} \in J_k \setminus J_{k+1}$. □

In the next section we will use the author's results in [Con10] to prove that $\mathsf{ART_0}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0} + \mathsf{I\Sigma_2}$. We now use Theorem 5.1 above to show that $\mathsf{WKL_0}$ implies the Structure Theorem for Artinian Rings (i.e. $\mathsf{ART_0^s}$) over $\mathsf{RCA_0}$. First, however, we need to prove the Chinese Remainder Theorem ($\mathsf{CRT_0}$) in $\mathsf{RCA_0}$. We now state the Chinese Remainder Theorem.

Theorem 5.3 (Chinese Remainder Theorem ($\mathsf{CRT_0}$))**.** *Let $n \in \mathbb{N}$, $n \geq 2$, and R be a commutative ring with ideals $A_1, A_2, \ldots, A_n \subset R$ such that for all $1 \leq i < j \leq n$ we have that $A_i + A_j = R$. Then the map*

$$\varphi : R \to R/A_1 \times R/A_2 \times \cdots \times R/A_n$$

is a surjection with kernel

$$A_1 A_2 \cdots A_n = A_1 \cap A_2 \cap \cdots \cap A_n.$$

Lemma 5.4. $\mathsf{RCA_0}$ *implies* $\mathsf{CRT_0}$.

Proof. The following proof of the Chinese Remainder Theorem in $\mathsf{RCA_0}$ is essentially identical to many of the proofs given in standard Algebra texts, and does not require induction.

First of all, by the proof of [Con10, Proposition 2.16] (which is the same as the classical proof of the same fact), it follows that the kernel of φ is as claimed above.

Suppose now that $n = 2$ in the statement of $\mathsf{CRT_0}$ above. Then there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1 + a_2 = 1 \in R$. Then $a_1 x_2 + a_2 x_1 \in R$ maps to $(\overline{x}_1, \overline{x}_2) \in R/A_1 \times R/A_2$ under φ, which shows that φ is surjective.

Now, let $n \in \mathbb{N}$ (be not necessarily 2). For each $2 \leq i \leq n$ there exist elements $a_i \in A_1$ and $b_i \in A_i$ such that

$$a_i + b_i = 1 \in R.$$

Furthermore, the product $\prod_{2 \leq i \leq n}(a_i + b_i) = 1_R$ and lies in $A_1 + \prod_{2 \leq i \leq n} A_i$, and hence

$$A_1 + \prod_{2 \leq i \leq n} A_i = R.$$

Note that the product of ideals $\prod_{2 \leq i \leq n} A_i$ does not necessarily exist in $\mathsf{RCA_0}$. We can now apply our proof of the Chinese Remainder Theorem in the case $n = 2$ to obtain an element $y_1 \in R$ such that

$$y_1 \equiv 1 \bmod A_1, \quad y_1 \equiv 0 \bmod A_j,\ 2 \leq j \leq n.$$

Similarly, there exist elements $y_2, \ldots, y_n \in R$ such that for all $2 \leq i \leq n$ we have that

$$y_i \equiv 1 \bmod A_i, \quad y_i \equiv 0 \bmod A_j,\ 1 \leq j \leq n, j \neq i.$$

It follows that the element $x = \sum_{1 \leq i \leq n} x_i y_i$ maps to

$$(x_1, x_2, \ldots, x_n) \in R/A_1 \times R/A_2 \times \cdots \times R/A_n$$

under φ, which shows that φ is surjective. This completes the proof of the lemma. □

We will now use the fact that $\mathsf{CRT_0}$ holds in $\mathsf{RCA_0}$ to prove the Structure Theorem for Artinian Rings via $\mathsf{WKL_0}$.

Theorem 5.5. $\mathsf{WKL_0}$ *implies* $\mathsf{ART_0^s}$ *over* $\mathsf{RCA_0}$.

Proof. We reason in $\mathsf{WKL_0}$, and assume that the reader is familiar with the proof of [Con10, Theorem 3.4]. Let R be an Artinian ring. Let $M_0, M_1, \ldots, M_{n_0-1} \subset R$, $n_0 \in \mathbb{N}$, $n_0 \geq 2$, be the maximal ideals of R (see [Con10, Section 3.5.1] or the proof Theorem 5.2 above for more details). Note that if $n_0 = 1$ then R is local and the theorem follows trivially. For all $0 \leq j < n_0$ let

$$x_j \in M_j \setminus \bigcup_{\substack{0 \leq k < n_0 \\ k \neq j}} M_k$$

and let $J = M_0 \cap M_1 \cap \cdots \cap M_{n_0-1} = M_0 M_1 \cdots M_{n_0-1} \subset R$ be the Jacobson radical of R. By [Con10, Theorem 3.4] it follows that $M_0, M_1, \ldots, M_{n_0-1}$ are also the prime ideals of R and J is also the nilradical of R. Via Theorem 5.1 above and the Π^0_1-Well-Ordering Principle, let $\langle m_0, m_1, \ldots, m_{n_0-1} \rangle \in \mathbb{N}$ be the least n_0-tuple such that

$$M_0^{m_0} M_1^{m_1} \cdots M_{n_0-1}^{m_{n_0-1}} = 0.$$

Now, we claim that for every $0 \leq j < n_0$ the ideal $M_j^{m_j}$ exists. To see why, without loss of generality assume that $j = 0$ (the general argument is similar). First, we claim that there exists

$$x \in \prod_{1 \leq i < n_0} M_i^{m_i} \setminus M_0.$$

Otherwise, we would have that $\prod_{1 \leq i < n_0} M_i^{m_i} \subseteq M_0$ and hence the element

$$x = \prod_{1 \leq i < n_0} x_i^{m_i}$$

would satisfy $x \in M_0$, from which it follows that some x_i, $1 \leq i < n_0$, is in M_0 (since M_0 is a prime ideal), a contradiction. Hence $x \in \prod_{1 \leq i < n_0} M_i^{m_i} \setminus M_0$ as claimed. Next, we claim that $Ann(x) = M_0^{n_0}$, from which it follows that $M_0^{n_0}$ exists. By our construction of $m_0, m_1, \ldots, m_{n_0-1} \in \mathbb{N}$ it is easy to see that $M_0^{m_0} \subseteq Ann(x)$. Now, suppose that there is some $y \in Ann(x) \setminus M_0^{n_0}$ and use $\mathsf{WKL_0}$ to construct an ideal I such that $M_0^{n_0} \subseteq I \subset Ann(x)$ and $y \notin I$. Consider the ring R/I. Since R/I is the quotient of an Artinian ring, $\mathsf{RCA_0}$ proves that R/I is Artinian and every ideal in R/I corresponds to an ideal in R containing I via pullback. We claim that R/I is a local ring with unique maximal ideal $\overline{M}_0 \subset R/I$. For suppose that there is another maximal ideal $\overline{M} \neq \overline{M}_0$ in R/I, then $\overline{M}$ must correspond to a maximal ideal M in R containing I. But then we have that

$$M \supset I \supseteq M_0^{m_0}$$

and so it follows that M contains M_0, from which it follows that $M = M_0$, since M, M_0 are maximal and hence prime ideals in R. Therefore, we have that $\overline{M} = \overline{M}_0$, a contradiction. Now, since R/I is a local Artinian ring with unique maximal ideal $\overline{M}_0$ it follows that $\overline{x} \in \prod_{1 \leq i < n_0} \overline{M}_i^{m_i} \setminus \overline{M}_0$ is a unit in R/I. But, on the other hand we have that $\overline{y} \neq 0$ in R/I (by our construction of I above) and (by our construction of y above) $\overline{y} \cdot \overline{x} = 0$ in R/I, a contradiction. Hence, no such $y \in Ann(x) \setminus M_0^{m_0}$ exists, and thus $Ann(x) = M_0^{m_0}$ as claimed.

For all $0 \leq i < j < n_0$ it follows that $M_i^{m_i} + M_j^{m_j} = R$. Otherwise we could use $\mathsf{WKL_0}$ to construct an ideal I such that $M_i^{m_0} + M_j^{m_0} \subseteq I \subseteq M_k \subset R$, $0 \leq k < n_0$, a contradiction since M_k is a prime ideal and so $x_i, x_j \in M_k$. Furthermore, by an argument similar to one given in the last half of the previous paragraph it follows that for every $0 \leq i < n_0$ the ring $R/M_i^{m_i}$ is a local ring with unique maximal ideal $\overline{M}_i \subset R/M_i^{m_i}$. Now, we can apply the Chinese Remainder Theorem ($\mathsf{CRT_0}$) to construct a homomorphism

$$\varphi : R \to R/M_0^{m_0} \times R/M_1^{m_1} \times \cdots \times R/M_{n_0-1}^{m_{n_0-1}}$$

given by

$$\varphi(x) = (\overline{x}_{R/M_0^{m_0}}, \overline{x}_{R/M_1^{m_1}}, \ldots, \overline{x}_{R/M_{n_0-1}^{m_{n_0-1}}})$$

with kernel

$$M_0^{m_0} M_1^{m_1} \cdots M_{n_0-1}^{m_{n_0-1}} = 0.$$

Hence, φ is an isomorphism and the theorem follows. □

5.1. **A second proof of ART_0 via $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$.** We now give a second (different) proof of Theorem 5.2 above. We assume that the reader is familiar with the proofs of Theorems 5.1 and 5.2 above.

A different proof of Theorem 5.2. We reason in $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$. First, let R be a local ring with unique maximal ideal $M \subset R$ and an infinite strictly ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset M \subset R.$$

We aim to show that R has an infinite strictly descending chain of ideals $J_0 \supset J_1 \supset J_2 \supset \cdots$; this suffices to prove the theorem. Let $z_0, z_1, \ldots, z_{n_0-1} \in M \subset R$, $T \subset n_0^{<\mathbb{N}}$, and $m_0 \in \mathbb{N}$, be as in the proof of Theorem 5.1 above, and let $\{x_k\}_{k\in\mathbb{N}}$ be as in the proof of Theorem 5.2 above. Without loss of generality we can assume that T is a finite tree; otherwise the theorem follows as in Theorem 5.1 above. So far our proof has been very similar to that of Theorem 5.2 above, but here is where they start to diverge.

First of all, let

$$U = R \setminus M,$$

and note that U is the set of units in R since R is a local Artinian ring, and we are reasoning in WKL_0 (see [Con10, Theorem 3.4] for more details). Now, via $\mathsf{I}\Sigma_2$ let $l_0 \in \mathbb{N}$, $l_0 < m_0$, be the least number such that there exist infinitely many $k \in \mathbb{N}$ and $u_{0,k}, u_{1,k}, \ldots, u_{k-1,k} \in U = R \setminus M$ such that for all $\sigma \in n_0^{\geq l_0} \cap T$ we have that

$$y_k = x_k + \sum_{j=0}^{k-1} u_{j,k} x_j \in Ann(z_\sigma),$$

where $\sigma \in n_0^{<\mathbb{N}}$,

$$z_\sigma = \prod_{i<|\sigma|} z_{\sigma(i)} \in M \subset R,$$

and $z_\emptyset = 1_R$. Note that $l_0 > 0$ since $z_\emptyset = 1_R$. Let $\sigma_0 \in T$, $|\sigma_0| = l_0 - 1$, be such that there do not exist infinitely many $k \in \mathbb{N}$ and $u_{0,k}, u_{1,k}, \ldots, u_{k-1,k} \in U$ as above (i.e. let $\sigma_0 \in T$ be a witness to the fact that l_0 is minimal). Finally, via our construction of σ_0 and RCA_0, let $\{y_k\}_{k\in\mathbb{N}}$ be an infinite sequence of elements of R as displayed above where σ is any successor of σ_0 in $n_0^{<\mathbb{N}}$.

We claim that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $k \in \mathbb{N}$, we have that $z_{\sigma_0} y_k$ is not an R-linear combination of $\{z_{\sigma_0} y_l\}_{l>k}$. To prove our claim suppose otherwise (for a contradiction). In this case by hypothesis we have that for all $k \in \mathbb{N}$ there exists $j > k$, $j \in \mathbb{N}$, such that $z_{\sigma_0} y_j$ is an R-linear combination of $\{z_{\sigma_0} y_l\}_{l>j}$. Now, first of all note that by our constructions of $\sigma_0 \in T$ and $z_{\sigma_0}, z_0, \ldots, z_{n_0-1} \in M \subset R$ above it follows that for all $k \in \mathbb{N}$ and $x \in M$ we have that

$$x \cdot z_{\sigma_0} y_k = 0$$

since, by our construction of $\{y_k\}_{k\in\mathbb{N}}$ in the previous paragraph, for all $k \in \mathbb{N}$ and successor strings $\sigma \supset \sigma_0$, $\sigma \in n_0^{<\mathbb{N}}$, we have that $z_\sigma y_k = 0$ (here we are using the defining property of $z_0, \ldots, z_{n_0-1}$, as we did in the proof of Theorem 5.2 above). Hence, under our current hypothesis and our remarks in the previous sentence, we can assume that there exist infinitely many $j \in \mathbb{N}$ such that $z_{\sigma_0} y_j$ is a U-linear combination of $\{z_{\sigma_0} y_l\}_{l>j}$. Now, since $U = R \setminus M$ is the set of units in the local ring R, every equation of the form

$$z_{\sigma_0} y_j = z_{\sigma_0} \cdot \sum_{l=j+1}^{l_0} u_l y_l, \; j \in \mathbb{N}, \; l_0 > j, \; u_l \in U,$$

can be manipulated/rearranged (via division by $u_{l_0} \in U$) to read

$$z_{\sigma_0} \left(y_{l_0} + \sum_{j<l_0} u_j y_j \right) = 0.$$

The fact that this can be done for unboundedly many (i.e. arbitrarily large) $l_0, j \in \mathbb{N}$ contradicts our construction of σ_0 in the previous paragraph. Therefore, the claim we made in the first sentence of this paragraph holds, and we can use this claim along with $\mathsf{WKL_0}$ to construct an infinite descending chain of ideals $J_0 \supset J_1 \supset J_2 \supset \cdots$ in R such that for all $k \in \mathbb{N}$ we have that $z_{\sigma_0} y_{k_0+k} \in J_k \setminus J_{k+1}$. This completes the proof of the theorem in the case when R is a local ring. We now turn our attention to proving the theorem when R is not a local ring.

Assume that R is an Artinian ring, with finitely many maximal ideals $M_0, M_1, \ldots, M_{m_0-1} \subset R$, $m_0 \in \mathbb{N}$, and an infinite strictly ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots \subset R$. The fact that R has finitely many maximal ideals is a consequence of [Con10, Theorem 3.4]. Let $x_0, x_1, x_2, \ldots \in R$, $x_i \in I_{i+1} \setminus I_i$, $i \in \mathbb{N}$, be as before. Note that for every $n \in \mathbb{N}$ there exists $0 \leq i < m_0$ such that $(x_n : I_n) \subseteq M_i$, since otherwise (using our previous arguments in [Con10, Theorem 3.4]) we could use this fact along with $\mathsf{WKL_0}$ to construct a maximal ideal $M \subset R$ that is different from $M_0, M_1, \ldots, M_{m_0-1}$, a contradiction. For all $n \in \mathbb{N}$, let $i_n : \mathbb{N} \to \mathbb{N}$, be a uniformly computable sequence of total functions such that the range of i_n is $(x_{n+1} : I_n) \subset R$, and for all $n, k \in \mathbb{N}$ let

$$I_{n,k} = \{i_n(0), i_n(1), \ldots, i_n(k)\}$$

and note that $\{I_{n,k}\}_{n,k\in\mathbb{N}}$ is a uniformly computable listing of finite sets. Now, construct the computable tree $T_1 \subseteq m_0^{<\mathbb{N}}$ such that for all $\sigma \in m_0^{<\mathbb{N}}$ we have that

$$\sigma \in T_1 \Leftrightarrow (\forall \tau \subseteq \sigma)[I_{|\tau|,|\sigma|} \subseteq M_{\sigma(|\tau|-1)}].$$

It follows from bounded Π^0_1-comprehension and our previous remarks in this paragraph that for every $n \in \mathbb{N}$ there is a string of length n on T_1. It is also not difficult to verify that if $f \in m_0^{\mathbb{N}}$ is an infinite path through T_1, then for every $n \in \mathbb{N}$ we have that

$$(x_n : I_n) \subseteq M_{f(n)}.$$

Moreover, via an argument similar to the proof (i.e. standard construction) of Lemma 4.3 above, along with our remarks in the previous paragraph, we can apply $\mathsf{WKL_0}$ to construct an infinite strictly ascending chain of ideals $\{I_{k,0}\}_{k\in\mathbb{N}}$ such that:

(1) $x_k \in I_{k+1,0} \setminus I_{k,0}$;
(2) $I_{k,0} \subseteq I_k$;
(3) $(I_{k+1,0} : I_{k,0}) \subseteq (x_{k+1} : I_k) \subseteq M_{f(k)}$.

So without any loss of generality, we can assume that our original chain $\{I_k\}_{k\in\mathbb{N}}$ satisfies each of these properties.

But, by $\mathsf{WKL_0}$ it follows that there is an infinite path $f \in m_0^{\mathbb{N}}$ through $T_1 \subseteq m_0^{<\mathbb{N}}$, since T_1 is an infinite computable tree. Furthermore, it follows from $\mathsf{I\Sigma_2}$ that there is a number $i_0 \in \mathbb{N}$, $0 \leq i_0 < m_0$, such that there exist infinitely many $n \in \mathbb{N}$ such that $f(n) = i_0$. Let i_0 be as in the previous sentence and use $\mathsf{RCA_0}$ to construct

$$P = \{n \in \mathbb{N} : f(n) = i_0\}.$$

It follows that P is an infinite set. Let

$$P = \{p_0 < p_1 < \cdots < p_k < \cdots,\ k \in \mathbb{N}\}$$

be a listing of the (infinitely many) elements of P. Let

$$R_0 = R_{M_{i_0}} = R[(R \setminus M_{i_0})^{-1}]$$

be the localization of R at the prime ideal M_{i_0} ($\mathsf{RCA_0}$ suffices to construct R_0). By our construction of the infinite set/sequence $P = \{p_k\}_{k\in\mathbb{N}}$ and Lemma 4.1 in Subsection 4 above it follows that the infinite ascending chain of ideals

$$I_{p_0} \subset I_{p_1} \subset I_{p_2} \subset \cdots \subset I_{p_k} \subset \cdots,\ k \in \mathbb{N},$$

in R corresponds to an infinite strictly ascending chain of ideals in R_0. Now we can apply our previous proof (for local rings) to the local ring R_0 to produce an infinite strictly descending chain of ideals $J_0^0 \supset J_1^0 \supset J_2^0 \supset \cdots$ in R_0 corresponding to an infinite strictly descending chain of ideals $J_0 \supset J_1 \supset J_2 \supset \cdots$ in R. □

6. Deriving weak König's lemma from $\mathsf{NIL_0}$, $\mathsf{ART_0}$, and $\mathsf{ART_0^s}$

The main purpose of this section is to establish optimal lower bounds on the reverse mathematical strengths of $\mathsf{NIL_0}$, $\mathsf{ART_0}$, and $\mathsf{ART_0^s}$. More specifically, we will prove that

(1) $\mathsf{NIL_0}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$;
(2) $\mathsf{ART_0}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$; and
(3) $\mathsf{ART_0^s}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$.

The author already proved (2) in [Con10, Theorem 4.1]. Our proof of (1) is based on the same construction (i.e. [Con10, Theorem 4.1]). Our proof of (3) is based on [DLM07, Theorem 3.2, Proposition 3.4].

Theorem 6.1. $\mathsf{NIL_0}$ *implies* $\mathsf{WKL_0}$ *over* $\mathsf{RCA_0}$.

Proof. We reason in $\mathsf{RCA_0}$. Let R be as in [Con10, Theorem 4.1]. Recall that R is an integral domain such that the only computable ideals in R form an infinite strictly ascending chain $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_\infty \subset R$, where $I_\infty = \cup_{n\in\mathbb{N}} I_i$, and every other ideal of R, $J \neq I_p, R$, $p \in \mathbb{N} \cup \{\infty\}$, is of PA Turing degree (for more information see Section 4 above or [Con10, Theorem 4.1]). Now, since R is an integral domain, it follows from $\mathsf{NIL_0}$ that the intersection of the maximal ideals of R must be zero. It follows that R must contain a maximal ideal M that is not equal to any I_p, $p \in \mathbb{N} \cup \{\infty\}$, and therefore M must be of PA Turing degree. Relativizing this argument to every oracle $A \subset \mathbb{N}$ proves that $\mathsf{NIL_0}$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. □

Theorem 6.2. $\mathsf{ART_0^s}$ *implies* $\mathsf{WKL_0}$ *over* $\mathsf{RCA_0}$.

Proof. Our proof uses some elements of the proof of [DLM07, Theorem 3.2], but is also somewhat different. First, we will construct a computable ring R using some of the ideas found in [DLM07, Theorem 3.2], then we will note that our construction relativizes to an arbitrary oracle $X \subset \mathbb{N}$ to produce a ring R_X with various special properties having to do with PA Turing degrees relative to X. Afterwards, we will show that there is no model of $\mathsf{RCA_0}+\neg\mathsf{WKL_0}+\mathsf{ART_0^s}$ by assuming that such a model $\mathfrak{M}$ exists, and deriving a contradiction based on an application of the axiom $\mathsf{ART_0^s}$ to the ring R_X, where $X \subseteq \mathbb{N}$ is an oracle in $\mathfrak{M}$ chosen so that there is no PA Turing degree relative to X in $\mathfrak{M}$ (note that the oracle X exists because $\mathfrak{M}$ satisfies $\neg\mathsf{WKL_0}$ by hypothesis). More precisely, we will apply $\mathsf{ART_0^s}$ to R_X to deduce the existence of a PA Turing degree relative to X, which is a contradiction by the way we chose X.

We now construct $R = R_\emptyset$. First, however, let $C_0, C_1 \subset \mathbb{N}$ be disjoint computably enumerable sets such that any separator for C_0, C_1 is of PA Turing degree. Let $c_0, c_1 : \mathbb{N} \to \mathbb{N}$ be one-to-one total computable functions such that the range of c_i is C_i, $i \in \{0, 1\}$. Let

$$Z = \{z_0 < z_1 < z_2 < \cdots < z_i < \cdots \ , \ i \in \mathbb{N}\} \subset \mathbb{N}$$

be an infinite computable set disjoint from both C_0 and C_1, and for all $k \in \mathbb{N}$ let

$$z_0(k) = z_{c_0(k)} \in Z \text{ and } z_1(k) = z_{c_1(k)} \in Z.$$

It is well-known that Z exists; see [KS07, Lemma 2.6], for example. Recall that

$$\mathbb{Z}_\infty = \mathbb{Z}[X_0, X_1, X_2, \ldots]$$

is the free polynomial ring over $\mathbb{Z}$ with infinitely many indeterminates $X_0, X_1, X_2, \ldots$, that F is the field of fractions of $\mathbb{Z}_\infty$, and that $\mathbb{Z}_k = \mathbb{Z}[X_0, X_1, \ldots, X_k]$. For all $p \in \mathbb{Z}_\infty$ define $s(p) \in \mathbb{N}$ to be the least number s_0 such that $p \in \langle X_{c_0(k)} : k \in \mathbb{N},\ 0 \leq k \leq s_0\rangle_{\mathbb{Z}_\infty}$; let $s(p) = \infty$ if no such s_0 exists. Also let $m_p \in \mathbb{N}$, $p \in \mathbb{Z}_\infty$, be the least $m \in \mathbb{N}$ such that $p \in \mathbb{Z}_m$. Finally, let

$$D_0 = \left\{ \frac{X_{c_0(k)}}{p} : k \in \mathbb{N},\ p \in \mathbb{Z}_\infty,\ k < s(p) \right\},$$

$$D_1 = \left\{ \frac{X_{z_0(k)}}{p} : k \in \mathbb{N},\ p \in \mathbb{Z}_\infty,\ z_0(k) > s(p) \right\},$$

$$D_2 = \left\{ \frac{X_{c_0(s(p))} - X_{z_1(k)}}{p} : k \in \mathbb{N},\ p \in \mathbb{Z}_\infty,\ s(p) < \infty,\ m_p < z_1(k) \right\},$$

and

$$D_3 = \left\{ \frac{1}{X_{c_1(k)}} : k \in \mathbb{N} \right\}.$$

Note that D_0, D_1, D_2, and D_3 are c.e. subsets of the computable field F. It follows that the subring generated by

$$R_0 = \mathbb{Z}_\infty \cup \bigcup_{i=0}^{3} D_i$$

in F is *computably* isomorphic to a computable ring R (see Lemma 4.7 above for more details). Without any loss of generality we identify R with $R_0 \subset F$. Note that $R_0 \cong R$ are integral domains.

For each $n \in \mathbb{N}$, let

$$A_n = \{c_0(0), c_0(1), \ldots, c_0(n)\} \cup Z \subset \mathbb{N}.$$

By our construction of $R \cong R_0$ above it follows that for every $n \in \mathbb{N}$ the ideal

$$I_n = \langle X_k : k \in A_n\rangle_R \subset R$$

is computable, and therefore exists via RCA_0. Let

$$I_\infty = \bigcup_{n \in \mathbb{N}} I_n.$$

Now, by our construction of R above, it follows that for all $n \in \mathbb{N}$ we have that $X_{c_0(n+1)} \notin I_n$ since (by our construction of R above) for all $n \in \mathbb{N}$ we have that the numerator of every element of $I_n \subset R \cong R_0$ is in $\langle X_k : k \in A_n\rangle_{\mathbb{Z}_\infty}$. We leave the easy verification of these facts to the reader.

Now, we claim that R is an Artinian ring unless there is a PA Turing degree. To prove this claim, first of all note that if I is a nontrivial ideal such that $I \not\subseteq I_\infty$ then, by our construction of D_0 and D_3 above, it follows that I has PA Turing degree since we have that

(a) $X_{c_0(k)} \in I$, for all $k \in \mathbb{N}$; and
(b) $X_{c_1(k)} \notin I$, for all $k \in \mathbb{N}$.[13]

A similar (simpler) argument applies in the case $I = I_\infty$. Hence, in a model of $\mathsf{RCA_0}+\neg\mathsf{WKL_0}$ we do not have any such ideals I. Now, by our construction of D_0 above and our previous remarks in this paragraph it follows that in any model of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$ every nontrivial ideal of R is contained in I_N, for some $N \in \mathbb{N}$. Let

$$J_0 \supset J_1 \supset J_2 \supset \cdots$$

be an infinite strictly descending chain of ideals in R, and, via the Σ^0_1-Well-Ordering Principle, let $n_0 \in \mathbb{N}$ be maximal such that $I_{n_0} \subset J_k$, for all $k \in \mathbb{N}$. Note that if n_0 does not exist, then it follows that $I_\infty \subset J_k$, for all $k \in \mathbb{N}$, from which it follows (from our previous remarks) that J_k is of PA Turing degree for all $k \in \mathbb{N}$. Therefore, our claim (that R is Artinian unless there is a PA Turing degree) is valid in the case when n_0 does not exist. On the other hand, if $n_0 \in \mathbb{N}$ exists, then let $m_0 \in \mathbb{N}$ be such that for all $m \geq m_0$ we have that I_{n_0+1} is not contained in J_m (i.e. $m \in \mathbb{N}$ is a witness to the fact that n_0 is maximal). It follows that

(1) $I_{n_0+1} \supseteq J_{m_0} \supset J_{m_0+1} \supseteq I_{n_0}$; and
(2) $X_{c_0(n_0+1)} \notin J_{m_0}$.

Now, because of (1) and (2) above, as well as our constructions of D_1 and D_2 (also above), it follows that we have

(a) $X_{z_0(k)} \in J_{m_0}$, for almost all $k \in \mathbb{N}$; and
(b) $X_{z_1(k)} \notin J_{m_0}$, for almost all $k \in \mathbb{N}$.

By our constructions of $Z \subset \mathbb{N}$ and $z_0, z_1 : \mathbb{N} \to \mathbb{N}$ above it follows that J_{m_0} is of PA Turing degree, which proves our claim that R is Artinian unless there is a PA Turing degree. Note that all of our constructions and arguments thus far can be relativized to any given oracle $X \subseteq \mathbb{N}$. In other words, for any given oracle $X \subseteq \mathbb{N}$ there is an $X-$computable integral domain R_X such that R_X is an Artinian ring unless there exists a PA Turing degree relative to X.

Now, suppose for a contradiction that $\mathsf{ART}^{\mathsf{s}}_0$ does not imply $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. Let $\mathfrak{M}$ be any model of $\mathsf{RCA_0} + \neg\mathsf{WKL_0} + \mathsf{ART}^{\mathsf{s}}_0$, and let $X \in \mathfrak{M}$ be a subset of the universe of $\mathfrak{M}$ such that $\mathfrak{M}$ *does not* contain a PA Turing degree relative to X. Throughout this paragraph we will work within the model $\mathfrak{M}$. Let $R_X \in \mathfrak{M}$ be the $X-$computable integral domain described in the final sentence of the previous paragraph (R_X exists in $\mathfrak{M}$ via $\mathsf{RCA_0}$). Now, by our construction of R_X and our construction of $X \in \mathfrak{M}$ via our hypothesis $\neg\mathsf{WKL_0}$, it follows that R_X is an Artinian integral domain in $\mathfrak{M}$. Hence, we can apply our hypothesis $\mathsf{ART}^{\mathsf{s}}_0$ to R_X to conclude that R_X is isomorphic to a finite direct product of local Artinian rings, i.e. there exists $k_0 \in \mathbb{N}$ and local Artinian rings $R_{0,X}, R_{1,X}, \ldots, R_{k_0,X}$ with respective unique maximal ideals $M_{0,X}, M_{1,X}, \ldots, M_{k_0,X}$, $M_{i,X} \subset R_{i,X}$, $0 \leq i \leq k_0$, such that

$$R_X \cong R_{0,X} \times R_{1,X} \times \cdots \times R_{k_0,X} = Z_X$$

via a given isomorphism $\varphi : R_X \to Z_X$. But, since R_X is an integral domain it follows that $k_0 = 0$ since otherwise R_X contains nontrivial zero divisors. Hence, R_X is a local Artinian integral domain with unique maximal ideal $M_X \subset R_X$, $M_X, R_X \in \mathfrak{M}$. But, by our previous remarks and constructions, and the maximality of M_X, it follows that either M_X is not contained in I_∞, or else $M_X = I_\infty$. But in either case (again, by previous remarks) it follows that M_X is of PA Turing degree relative to X, a contradiction. Therefore we must have that $\mathsf{ART}^{\mathsf{s}}_0$ implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. □

[13]Note that by our construction of D_3 and R we have that $X_{c_1(k)} \notin B$ for any proper ideal $B \subset R$ and $k \in \mathbb{N}$.

7. The Full Computable Structure Theorem for Artinian Rings

We are now ready to prove the Full Computable Structure Theorem for Artinian Rings, which is similar to the Classical Full Structure Theorem for Artinian Rings, except that we will work exclusively with annihilator ideals (rather than powers of maximal ideals), thus ensuring that all ideals in the Full Computable Structure Theorem are computable whenever the ring is computable. First we prove the theorem for local Artinian rings, and then we prove it for arbitrary Artinian rings.

Elements of the proof of the following proposition can be found in Section 5 above.

Theorem 7.1 (Full Computable Structure Theorem for Local Artinian Rings)**.** *Let R be a computable local Artinian ring. Then the unique maximal ideal of R, $M \subset R$, is an annihilator ideal and therefore computable. Furthermore, there is a finite computable strictly descending chain of ideals*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{N-1} = M \subset M_N = R,\ \ N \in \omega,$$

such that for each $0 \leq i < N$ we have that

$$M \cdot M_{i+1} \subseteq M_i.$$

It follows that each factor module M_{i+1}/M_i is a computable R/M-vector space.

Proof. First we show that M is computable. To do this it suffices to show that the sets of units/nonunits in M are computable. To prove this, first note that, because R is Artinian, for every $r \in R$ there exists $k \in R$ and $m \in \omega$ such that

$$r^m = kr^{m+1};$$

otherwise the chain

$$\langle r \rangle \supset \langle r^2 \rangle \supset \cdots$$

would contradict the fact that R is Artinian. It follows that for each $r \in R$ there exist $n, k \in R$ such that

$$r^m(1 - kr) = 0.$$

Now, we claim that r is a unit if and only if $1 - kr = 0$, i.e. k is the inverse of r. First of all, if r is not invertible then we cannot have that $1 - kr = 0$. On the other hand, if r is invertible then it follows that $1 - kr = 0$, since we can divide the equation $r^m(1 - kr) = 0$ by (the unit) r^m. Therefore, to decide whether or not a given $r \in R$ is invertible, simply search for $k \in R$ and $m \in \omega$ such that $r^m(1 - kr) = 0$ (they certainly exist since R is Artinian) and check whether or not $kr = 1$. If so, then r is invertible. Otherwise, r is not invertible. Finally, since M is computable, it follows that the field R/M is computable.

Now, let $M = \{z_0, z_1, z_2, \ldots\}$ be a computable listing of the elements of M. Since R is Artinian it follows that there exists $n \in \omega$ such that

$$Ann(z_0, \ldots, z_n) = Ann(z_0, \ldots, z_n, \ldots, z_{n+k}),$$

for all $k \in \omega$, and each z_j, $0 \leq j \leq n$, is nilpotent. Let n_j be such that $z_j^{n_j} = 0$. By the Finitary Pigeonhole Principle it follows that every product of z_j, $0 \leq j \leq n$, of degree $N_0 = \sum_{j=0}^{n} n_j + 1$ is zero. Now, let $N \in \omega$, $0 \leq N \leq N_0$, be least such that all (finitely many) products of the z_j, $0 \leq j \leq n$, of degree N is zero. For each $0 \leq i \leq N$ let $M_i \subseteq R$ be the annihilator of the products of z_j, $0 \leq j \leq n$, of degree i (here $z^0 = 1_R$ for all $z \in R$). It follows that:

(1) the finite sequence $\{M_i\}_{i=0}^{N}$ is computable,
(2) $M_0 = 0$,
(3) $M_N = R$, and
(4) for each $0 \leq i < N$, $M_i \subseteq M_{i+1}$.

All that is left to show is that $M_{N-1} = M$ and for each $0 \leq i < N$ we have that $M \cdot M_{i+1} \subseteq M_i$. Before we can prove this, however, we must show that if $y, z \in M$ are such that $y \cdot z \neq 0$ then there exists $0 \leq j_0 \leq n$ for which $y \cdot z_{j_0} \neq 0$. To prove this note that by our construction of $n \in \omega$ it follows that $Ann(z_0, z_1, \ldots, z_n)$ is contained in $Ann(z)$, for all $z \in M$ (otherwise we would have added z to the set $z_0, \ldots, z_n$ above). Hence, it follows that if y is not in the annihilator of z then y is not in $Ann(z_0, z_1, \ldots, z_n)$, i.e. there exists $0 \leq j_0 \leq n$ such that y is not annihilated by z_{j_0}.

Now suppose that $y \in M$ and $y \cdot z \neq 0$ for some $z \in M$ that is a product of $\{z_j : 0 \leq j \leq n\}$, of degree $N-1$. Then, by the previous paragraph it follows that we have $z_i \cdot z \neq 0$, for some $0 \leq i \leq n$, a contradiction since all products of z_j, $0 \leq j \leq n$, of degree N are zero, by definition of N. Therefore, for any $y \in M$ we must have that $y \in M_{N-1}$. Hence it follows that $M_{N-1} = M$, since $M_{N-1} \subset R$ is a proper ideal because, by definition of $N \in \omega$, there is a nonzero product of $\{z_j : 0 \leq j \leq n\}$ of degree $N-1$.

We now show that $M \cdot M_{i+1} \subseteq M_i$, for all $0 \leq i < N$. Assume, for a contradiction, that there exists $0 \leq i < N$ such that $M \cdot M_{i+1} \nsubseteq M_i$. Then, in particular, there exists $z \in M$ and $x \in M_{i+1}$ such that $z \cdot x \notin M_i$. More specifically, there exists $z \in M$ and $x \in R$ such that x annihilates all products of $\{z_j : 0 \leq j \leq n\}$, of degree $i+1$ but there exists a product of $\{z_j : 0 \leq j \leq n\}$, call it Z, of degree i, such that $z \cdot x \cdot Z \neq 0$. In this case by our remarks two paragraphs above it follows that there exists $0 \leq j_0 \leq n$ such that $z_{j_0} \cdot x \cdot Z \neq 0$, i.e. $x \cdot (z_j \cdot Z) \neq 0$. In other words, there is a product of $\{z_j : 0 \leq j \leq n\}$, of degree $i+1$, namely $z_{j_0} \cdot Z$, that is not annihilated by x, a contradiction. □

Remark 7.2. *It is not difficult to check that the proof of the previous proposition is valid in* $\mathsf{WKL_0}$. *The only two facts that may require some additional justification for the reader are that M is computable (i.e. Δ^0_1-definable) and every element of M is nilpotent, and the proofs of these two facts via* $\mathsf{WKL_0}$ *can be found in* [Con10, Section 3].

Corollary 7.3 (Full Computable Structure Theorem for Artinian Rings)**.** *Let R be a computable Artinian ring. Then R is a finite direct product of computable local Artinian rings, i.e.*

$$R \cong R_0 \times R_1 \times \cdots \times R_{n_0}$$

where R_i, $0 \leq i \leq n_0$ is a local Artinian ring with unique maximal ideal $M_i \subset R_i$, and for each R_i, $0 \leq i \leq n_0$, there exists $n_i \in \omega$ and a finite chain of computable ideals, $\{M_{i,j}\}_{j=0}^{n_i}$, in R_i, such that:

1. $M_{i,0} = \{0_{R_i}\}$,
2. $M_{i,n_i} = R_i$,
3. $M_{i,n_i-1} = M_i$,
4. $M_{i,j+1} \supseteq M_{i,j}$, $0 \leq j < n_i$, *and*
5. $M_i \cdot M_{i,j+1} \subseteq M_{i,j}$.

Moreover, $n_i \in \omega$ is such that $M^{n_i} = 0$.

Proof. It is well-known that every Artinian ring is isomorphic to a finite direct product of local Artinian rings. If $e_0, e_1, \ldots, e_{n_0} \in R$ are the idempotents corresponding to this direct product decomposition of R and $E = \{e_0, \ldots, e_{n_0}\}$, $E_i = E \setminus \{e_i\}$, $0 \leq i \leq n_0$, then $R_i \cong Ann(E_i) \subset R$ and hence R_i is isomorphic to a computable subring of R. The rest of the corollary follows from the previous proposition. □

Remark 7.4. *In Theorem 5.5 above we showed that* $\mathsf{WKL_0}$ *proves that every Artinian ring isomorphic to a finite direct product of local Artinian rings. From this and our previous remark it follows from the proof of Corollary 7.3 that* $\mathsf{WKL_0}$ *proves the Full Computable Structure Theorem for Artinian Rings.*

An interesting immediate consequence of the Full Computable Structure Theorem for Artinian Rings is the following result of Baur [Bau74].

Corollary 7.5. [Bau74],[SHT, Corollary 4.3.2] *If R is a computable Artinian ring, then R has an ideal membership algorithm.*

Proof. Recall that by ideal membership algorithm we mean an algorithm that decides membership in finitely generated ideals that is uniform in the (finitely many) generators.

Now, if R is Artinian, then it follows that every R/M_i-vector space of the form $M_{i,k+1}/M_{i,k}$, $0 \leq i \leq n_0$, $0 \leq k \leq n_i$, in the Full Computable Structure Theorem above is finite dimensional. Hence, every ideal I in R is finitely generated and moreover is essentially given by a finite union of subspaces of finite dimensional computable vector spaces, which is uniformly computable in the generators of I. □

8. Proving ART_0 via WKL_0 and the Full Computable Structure Theorem for Artinian Rings

We are now ready to begin proving our Main Theorem which says that WKL_0 proves ART_0. Our proof will consist of two lemmas and a theorem, each of which builds on its predecessor. The first lemma proves $\mathsf{ART}_0^{\mathsf{l}}$ via $\mathsf{WKL}_0 + \mathsf{B}\Sigma_2$. The second lemma proves $\mathsf{ART}_0^{\mathsf{l}}$ via WKL_0. The final theorem is our Main Theorem.

Lemma 8.1. $\mathsf{WKL}_0 + \mathsf{B}\Sigma_2$ *proves* $\mathsf{ART}_0^{\mathsf{l}}$.

Proof. We reason in $\mathsf{WKL}_0 + \mathsf{B}\Sigma_2$. Let R be a given local Artinian ring with unique maximal ideal $M \subset R$. Our previous results and remarks in this section say that WKL_0 proves that there is a finite increasing chain of ideals $\{M_i\}_{i=0}^n$ in R such that

(1) $M_0 = 0$,
(2) $M_n = R$,
(3) $M_{n-1} = M$,
(4) $M_{i+1} \supseteq M_i$, $0 \leq i < n$, and
(5) $M \cdot M_{i+1} \subseteq M_i$.

By property (4) it follows that each successive quotient M_{i+1}/M_i, $0 \leq i < n$, is an R/M-vector space. Now, assume (for a contradiction) that R is not Noetherian and let

$$I_0 \subset I_1 \subset \cdots \subset I_m \subset \cdots \subset R$$

be a given infinite strictly ascending chain of ideals in R. Then, for each $m \in \mathbb{N}$ there exists an index $0 \leq i_m \leq n$ such that $I_{m+1} \cap M_{i_m} \supset I_m \cap M_{i_m}$, and the infinite sequence $\{i_m\}_{m=0}^\infty$ is computable in the (given) chain $\{I_m\}_{m=0}^\infty$.

Now, we have that $i_m \in \{0, 1, \ldots, n\}$ for all $m \in \mathbb{N}$, and so via $\mathsf{B}\Sigma_2$ (which is equivalent to the Infinite Pigeonhole Principle) it follows that there exists $n_0 \in \{0, 1, \ldots, n\}$ such that infinitely many $m \in \mathbb{N}$ satisfy $i_m = n_0$. In other words, for all $x \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m > x$, such that $i_m = n_0$. This infinite subsequence of $\{i_m\}_{m\in\mathbb{N}}$ is computable from $\{i_m\}_{m\in\mathbb{N}}$ and $n_0 \in \mathbb{N}$. Therefore, without any loss of generality we may assume that this infinite subsequence is *equal to* or *all of* $\{i_m\}_{m\in\mathbb{N}}$. In this case the infinite ascending chain of ideals $\{I_m\}_{m\in\mathbb{N}}$ corresponds to an infinite strictly ascending chain of subspaces in $V = M_{n_0+1}/M_{n_0}$, and via WKL_0 it is possible to construct an infinite descending chain of subspaces in V corresponding to an infinite strictly descending chain of ideals in R (see [Con10, Lemma 3.5, Corollary 3.6, Subsection 3.5] for more details). Thus, R is not Artinian, a contradiction. Therefore, R must be Noetherian and $\mathsf{ART}_0^{\mathsf{l}}$ follows. □

Most of the work in proving our Main Theorem is contained in the proof of the following lemma.

Lemma 8.2. WKL_0 *proves* $\mathsf{ART}^{\mathsf{I}}_0$.

Proof. The proof of the current lemma picks up where the second to last paragraph of the proof of the previous lemma ends, i.e. we pick up just before we used the hypothesis $\mathsf{B}\Sigma_2$ in the proof of the previous lemma. The goal of the proof of the current lemma is to replace our use of $\mathsf{B}\Sigma_2$ in the proof of the previous lemma with some applications of the Finitary Pigeonhole Principle. In other words, we want to replace our use of the Infinitary Pigeonhole Principle in the last lemma with the strictly weaker Finitary Pigeonhole Principle, which follows from RCA_0 and WKL_0.

First, assume (for now) that there is $0 \leq n_0 \leq n$ such that for infinitely many $m \in \mathbb{N}$ we have that $i_m = n_0$ (as in the previous lemma where we used the Infinite Pigeonhole Principle to help us with the proof). In this case we may assume without any loss of generality that $i_m = n_0$ for *all* $m \in \mathbb{N}$. Also, let $N \in \mathbb{N}$, and $z_0, z_1, \ldots, z_N \in M$, $N \in \mathbb{N}$, be as in the proof of Proposition 7.1 above. Then if $Z \subseteq M \subset R$ denotes the set of nonzero (or all) products $z_0, z_1, \ldots, z_N$ of degree $n_0 - 1$, it follows that $|Z| \leq (N+1)^{n_0}$.

For each $m \in \mathbb{N}$, let

$$v_m \in I_{m+1} \cap M_{i_m} \setminus I_m \cap M_{i_m},$$

and let $v_{m,1}$ be a nonzero Z-multiple of v_m in (the R/M-vector space) $V = M_1$ (such a multiple always exists by our construction of $\{M_k\}_{k=0}^n$ and the fact that $v_m \in M_{i_m} = M_{n_0}$). Now, we claim that there is a computable strictly increasing infinite sequence of numbers $N_0 < N_1 < \cdots < N_k < \cdots$, $k \in \mathbb{N}$, such that at least k of the vectors $v_{0,1}, v_{1,1}, \ldots, v_{N_k,1}$ are linearly independent over R/M, or, equivalently, R-linearly independent (since $M \cdot M_1 = 0$).

By construction of M_{n_0}, for each v_m, $m \in \mathbb{N}$, there is some $z \in Z$ such that $v_{m,1} = z \cdot v_m \neq 0$, $v_{m,1} \in V$. Therefore, by the Finitary Pigeonhole Principle we have that for $v_0, v_1, \ldots, v_{N_k}$, where $N_k = (N^{n_0}+1)k$, $k \in \mathbb{N}$, there will be some $z \in Z$ such that for a set of size k, that we denote $K \subseteq \{0, 1, \ldots, n\}$, there is a *single* $z \in Z$ such that $v_{j,1} = z \cdot v_j$ for all $j \in K$. Furthermore, we claim that these vectors are linearly independent. Suppose, for a contradiction that the vectors $\{v_{j,1} : j \in K\} \subseteq V$ are linearly dependent. Then there exist unital coefficients from R, $u_j \in R \setminus M$, $j \in K$, such that

$$\sum_{j \in K} u_j v_{j,1} = 0,$$

but then

$$z \cdot \sum_{j \in K} u_j v_j = 0$$

and so we have that $\sum_{j \in K} u_j v_j \in M_{n_0 - 1}$, implying that $\{v_j : j \in K\}$ is linearly dependent in the R/M-vector space $V_{n_0} = M_{n_0}/M_{n_0-1}$. But this contradicts our construction of $\{v_m : m \in \mathbb{N}\}$ above since for each $k \in \mathbb{N}$ we have that:

(1) $\{v_0, v_1, \ldots, v_k\} \subseteq I_k$,
(2) $v_{k+1} \in I_{k+1} \setminus I_k$, and
(3) $\{I_k : k \in \mathbb{N}\}$ corresponds to an infinite strictly ascending chain of subspaces in the R/M-vector space V_{n_0}.

This argument is very similar to the main idea of the author's second proof that $\mathsf{WKL}_0 + \mathsf{I}\Sigma_2$ implies ART_0 in Section 5 above.

In the previous paragraph we showed that if $N_k = (N^{n_0}+1)k$, $k \in \mathbb{N}$, then there is a subset of size k, $K \subseteq \{0, 1, \ldots, n\}$, such that the vectors $\{v_{j,1} : j \in K\} \subset V = M_1$ are linearly independent. Our goal now is to prove the current lemma without the assumption that $i_m = n_0$ for infinitely many $m \in \mathbb{N}$.

Let Z be the set of nonzero products of factors from $\{z_0, z_1, \ldots, z_N\}$, and let $N_0 \in \mathbb{N}$ be as in the proof of Theorem 7.1 above, i.e. every product of $\{z_0, z_1, \ldots, z_N\}$ of degree N_0 is zero. Then there are at most

$$N_0^* = (1 + N + N^2 + \cdots + n^{N_0})-$$

many (product) elements in Z. Now, by essentially the same reasoning as in the second to last paragraph above (i.e. by the Finitary Pigeonhole Principle) it follows that if $N_k = (N_0^* + 1)k$ then there is a subset of size k, $K \subseteq \{0, 1, 2, \ldots, N_0^*\}$, and $z \in Z$ such that $z \cdot v_m \neq 0$ and $z \cdot v_m \in M_1 = V$, and furthermore the set

$$\{v_{j,1} = z \cdot v_j : j \in K\} \subseteq V$$

is linearly independent. Finally, by the special case of Proposition 4.11 above in which $n = 1$, it follows that $\mathsf{WKL_0}$ can construct an infinite strictly descending chain of ideals/subspaces in $M_1 = V$, showing that R is not Artinian, a contradiction. Therefore, we must have that R is Noetherian and so $\mathsf{ART}_0^{\mathsf{I}}$ holds. ☐

Theorem 8.3 (Main Reverse Mathematical Theorem)**.** $\mathsf{WKL_0}$ *proves* $\mathsf{ART_0}$ *over* $\mathsf{RCA_0}$.

Corollary 8.4. $\mathsf{WKL_0}$ *is equivalent to* $\mathsf{ART_0}$ *over* $\mathsf{RCA_0}$.

Proof. The corollary follows directly from our Main Reverse Mathematical Theorem and [Con10, Theorem 4.1]. ☐

Proof of the Main Theorem. We reason in $\mathsf{WKL_0}$. Let R be an Artinian ring, and suppose (for a contradiction) that there exists an infinite strictly ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots \subset R,\ k \in \mathbb{N}.$$

By Remark 7.4 above it follows that the Full Computable Structure Theorem for Artinian Rings holds in $\mathsf{WKL_0}$. In other words R is isomorphic to a finite product of local Artinian rings,

$$R \cong R_0 \times R_1 \times \cdots \times R_{m_0} = \hat{R},$$

with unique maximal ideal $M_i \subset R_i$, $0 \leq i \leq m_0$, and finite increasing chains of annihilator ideals $\{M_{i,j}\}_{j=0}^{n_i}$, $n_i \in \mathbb{N}$, in R_i such that:

(1) $M_{i,0} = 0$,
(2) $M_{i,n_i} = R_i$,
(3) $M_{i,n_i-1} = M_i \subset R_i$,
(4) $M_{i,j+1} \supseteq M_{i,j}$, $0 \leq j < n_i$, and
(5) $M_i \cdot M_{i,j+1} \subseteq M_{i,j}$.

Furthermore, every ideal $I \subseteq R$ corresponds to a product of ideals $I_0 \times I_1 \times \cdots I_{m_0}$, $I_l \subseteq R_k$, $0 \leq l \leq m_0$, in $\hat{R}$. For each $k \in \mathbb{N}$, let $0 \leq l_k \leq m_0$ be such that $R_{l_k} \cap I_{k+1} \supset R_{l_k} \cap I_k$. Let $\langle \cdot, \cdot \rangle : (m_0 + 1) \times \mathbb{N} \to \mathbb{N}$ be a computable bijection and for each (fixed) $0 \leq m \leq m_0$ let $\{z_{\langle m,l \rangle} : l \in \mathbb{N}\}$ be a listing of the elements of $M_m \subset R_m$. Now, let $J_0 = \hat{R}$, and for all $k \in \mathbb{N}$, let

$$J_{k+1} = J_k \cap R_0 \times R_1 \times \cdots \times R_{i-1} \times Ann_{R_i}(z_k) \times R_{i+1} \times \cdots \times R_{m_0},$$

where $k = \langle i, l \rangle$, $0 \leq i \leq m_0$, $l \in \mathbb{N}$. Note that for all $k \in \mathbb{N}$ we have $z_k = z_{\langle i,l \rangle} \in M_i \subset R_i$ and so $Ann_{R_i}(z_k)$ makes perfect sense.

Now, since $R \cong \hat{R}$ is Artinian and (by construction) $J_{k+1} \subseteq J_k$ for all $k \in \mathbb{N}$, it follows that there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that $J_n = J_{n_0}$. From this it follows that there exists $n_0 \in \mathbb{N}$ (equal to n_0 in the previous sentence) such that for each $0 \leq i \leq m_0$ we have that

$$Ann_{R_i}(z_{\langle i,0 \rangle}, z_{\langle i,1 \rangle}, \ldots, z_{\langle i,n_0 \rangle}) = Ann_{R_i}(z_{\langle i,0 \rangle}, z_{\langle i,1 \rangle}, \ldots, z_{\langle i,n_0 \rangle}, \ldots, z_{\langle i,n \rangle}),$$

for all $n \geq n_0$. Notice that we have essentially proved a bounding principle for finite strictly descending chains with elements of the form

$$Ann(z_{\langle i,0\rangle}, \ldots, z_{\langle i,k\rangle}),\ 0 \leq i \leq m_0,\ k \in \mathbb{N}.$$

Now, since each of the $z_{\langle i,l\rangle} \in M_i \subset R_i$ is nilpotent for all $0 \leq i \leq m_0$ (see [Con10, Section 3] for more details), $0 \leq l \leq n_0$, it follows that for each $0 \leq i \leq m_0$ there is a number $N_i \in \mathbb{N}$ such that every product of $\{z_{\langle i,0\rangle}, z_{\langle i,1\rangle}, \ldots, z_{\langle i,n_0\rangle}\}$ of degree N_i is zero. Let

$$N = \max_{0 \leq i \leq m_0} N_i,$$

and

$$N^* = 1 + n_0 + n_0^2 + \cdots + n_0^N.$$

Finally, if we set

$$N_k^* = (m_0+1)(N^*+1)k,\ k \in \mathbb{N},$$

then, by the Finitary Pigeonhole Principle, it follows that there is R_i, $0 \leq i \leq m_0$, for which there are at least $(N^*+1)k$-many numbers $0 \leq l \leq (m_0+1)(N^*+1)k$, such that

$$I_{l+1} \cap R_i \supset I_l \cap R_i.$$

Furthermore, via an argument similar to that given in the proof of the previous lemma, we have that the annihilator ideal $V_i = M_{i,1} \subset R_i$ contains a finite subset K such that at least k-many elements of K that are linearly independent vectors when V_i is viewed as a R/M_i-vector space.[14] Now, via Proposition 4.11[15] above it follows that $\mathsf{WKL_0}$ can construct an infinite strictly descending chain of subspaces in

$$V_0 \times V_1 \times \cdots \times V_{m_0} = M_{0,1} \times M_{1,1} \times \cdots \times M_{m_0,1} \subset R_0 \times R_1 \times \cdots \times R_{m_0} = \hat{R} \cong R.$$

This contradicts the fact that R is Artinian, and proves our Main Theorem. $\square$

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[14] Recall that we used the Finite Pigeonhole Principle in the proof of the previous lemma, hence the proof of the Main Theorem actually uses two applications of the finite pigeonhole principle, one on top of the other.

[15] More specifically, apply Proposition 4.11 above with $R = R_0/M_0 \times \cdots \times R_{m_0}/M_{m_0}$ (a product of fields) and $M = V_0 \times \cdots \times V_{m_0}$, and the natural action of R on M given via the R_i/M_i action on V_i, $0 \leq i \leq m_0$.

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