THE STRENGTH OF THE BOLZANO-WEIERSTRASS THEOREM

CHRIS J. CONIDIS

Abstract. In [Fri75], Friedman introduces two statements of hyperarithmetic analysis, $\mathsf{SL_0}$ (sequential limit system) and $\mathsf{ABW_0}$ (arithmetic Bolzano-Weirstrass), which are motivated by standard and well-known theorems from analysis. In this article we characterize the reverse mathematical strength of $\mathsf{ABW_0}$ by comparing it to most known theories of hyperarithmetic analysis.

In particular we show that, over $\mathsf{RCA_0} + \mathsf{I}\Sigma^1_1$, $\mathsf{SL_0}$ is equivalent to $\Sigma^1_1 - \mathsf{AC_0}$, and that $\mathsf{ABW_0}$ is implied by $\Sigma^1_1 - \mathsf{AC_0}$, and implies $\mathsf{weak} - \Sigma^1_1 - \mathsf{AC_0}$. We then use Steel's method of forcing with tagged trees [Ste78] to show that $\mathsf{ABW_0}$ is incomparable with INDEC (i.e. Jullien's Theorem) and $\Delta^1_1 - \mathsf{AC_0}$. This makes $\mathsf{ABW_0}$ the first theory of hyperarithmetic analysis that is known to be incomparable with other (known) theories of hyperarithmetic analysis.

1. Introduction

The main goal of this paper is to examine the reverse mathematical strength of two statements of second order arithmetic first introduced by Friedman in [Fri75], and motivated by standard, well-known theorems from mathematical analysis (for a general reference on the ongoing program of reverse mathematics, see [Sim]). Generally speaking, given a theorem from ordinary mathematics, T, the program of reverse mathematics attempts to assign a strength to T based upon the weakest subsystem of second order arithmetic that proves T. Very frequently the answer to this question is one of the following five subsystems of second order arithmetic: $\mathsf{RCA_0}$ (recursive comprehension axiom), $\mathsf{WKL_0}$ (weak König's Lemma), $\mathsf{ACA_0}$ (arithmetic comprehension axiom), $\mathsf{ATR_0}$ (arithmetic transfinite recursion axiom), and $\Pi^1_1 - \mathsf{CA_0}$ (Π^1_1 comprehension axiom); the system $\mathsf{RCA_0}$ is almost always assumed. In other words, when one makes a reverse mathematical assertion, one usually means that the assertion holds under the blanket assumption of $\mathsf{RCA_0}$.

Generally speaking, $\mathsf{RCA_0}$ (i.e. recursive comprehension) resembles computable mathematics, and T is equivalent to $\mathsf{RCA_0}$, over $\mathsf{RCA_0}$, if there is a proof of T that involves only computable constructions (i.e. the proof can be carried out computably). More specifically, $\mathsf{RCA_0}$ says that if a set $A \subseteq \mathbb{N}$ exists, and $B \subseteq \mathbb{N}$ is computable from A (i.e. B is Turing reducible to A), written $B \leq_T A$, then B also exists. Note that, if we take $A = \emptyset$, then we get that $\mathsf{RCA_0}$ implies that the computable sets exist. Also, one can show that the ω-models that satisfy $\mathsf{RCA_0}$ are exactly those that are closed downwards under $\leq_T$ and closed under disjoint union. Here the *disjoint union* of two sets $A, B \subseteq \mathbb{N}$ is equal to the set $\{2n : n \in A\} \cup \{2n + 1 : n \in B\}$. By ω-model, we mean a model of second order arithmetic whose first order part is equal to the standard natural numbers $(\omega, 0, 1, +, \times)$. Such models are normally identified by their second order parts.

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The behavior of ACA_0 (arithmetic comprehension) is similar to that of RCA_0, but with respect to arithmetic reducibility instead of Turing reducibility. A set $B \subseteq \mathbb{N}$ is arithmetically reducible to a set $A \subseteq \mathbb{N}$ if there exists $n \in \mathbb{N}$ such that B is computable from $A^{(n)}$ (here $A^{(n)}$ denotes the n^{th} Turing jump of A), or, equivalently, if B is definable by an arithmetic formula with A as a parameter (informally, a formula is arithmetic if it has finitely many number quantifiers and no set quantifiers; a more formal definition is given in the Basic Notation and Background subsection at the end of this section). An ω-model is a model of ACA_0 if and only if it is closed under arithmetic reducibility and disjoint union. In this case we have that ACA_0 implies T if there is a proof of T that involves only arithmetical constructions. It is well-known that ACA_0 implies WKL_0, the theory of second order arithmetic which says that RCA_0 holds and that every infinite binary branching tree has an infinite path. This fact will be used in the proof of Theorem 2.1. Next, we define the class of hyperarithmetic sets, a main focus of our study.

The main purpose of this article is to determine the reverse mathematical strength of two standard, well-known theorems of mathematical analysis. These statements were first introduced by H. Friedman in [Fri75]. One of the statements is denoted by SL_0 (sequential limit system), and says that if $A(X)$ is an arithmetic predicate of reals such that $\{X : A(X)\}$ has an accumulation point, Z_0, then there is an infinite subsequence of $\{X : A(X)\}$ that converges to Z_0. The second statement is denoted by ABW_0 (arithmetic Bolzano-Weirstrass), and says that if $A(X)$ is a bounded arithmetical predicate of reals, then $A(X)$ either has finitely many solutions, or else $\{X : A(X)\}$ has an accumulation point Z_0 (Z_0 need not satisfy $A(X)$). More information on hyperarithmetic sets and hyperarithmetic reducibility is given in the next section (i.e. Section 1.1). More information on theories of hyperarithmetic analysis is given in Section 1.2. For more information on SL_0 and ABW_0, consult Section 1.3.

We now (briefly) state our main results. In the next section (i.e. Section 2), we prove that, over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$, SL_0 is equivalent to $\Sigma^1_1 - \mathsf{AC}_0$, and that ABW_0 is implied by $\Sigma^1_1 - \mathsf{AC}_0$, and implies $\mathsf{weak} - \Sigma^1_1 - \mathsf{AC}_0$. Then, in Sections 3 and 4 we employ Steel's method of forcing with tagged trees [Ste78] to show that ABW_0 is incomparable with other theories of hyperarithmetic analysis called INDEC (i.e. Jullien's Theorem) and $\Delta^1_1 - \mathsf{AC}_0$ (see Section 1.2 for more information on theories of hyperarithmetic analysis). A more precise description of the information in this paragraph is given in Section 1.4.

1.1. **Hyperarithmetic Sets.** Let $\mathcal{L} = \langle L, \leq_L \rangle,\ L \subset \omega$, be a presentation of a linear ordering which has least element $0 \in \omega$. Given sets $X, Y \subseteq \omega$, we say that Y is an $H(X, \mathcal{L})$-*set* if $Y^{[0]} = X$, and for every $l \in L \setminus \{0\}$ we have that

$$Y^{[l]} = \bigoplus_{k <_L l} (Y^{[k]})',$$

where $Y^{[j]} = \{n \in \omega : \langle j, n \rangle \in Y\}$ and $\bigoplus_{k \in A} B_k = \{\langle k, n \rangle : k \in A, n \in B_k\}$ (here $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ is a computable pairing function, as defined in [Soa]).

It is not difficult to show that, when $\mathcal{L}$ is an ordinal (i.e. $\mathcal{L}$ has no infinite descending sequences), then there exists a unique $H(X, \mathcal{L})$-set, which we denote by $X^{(\mathcal{L})}$. However, if we let $\mathcal{L}'$ be a different presentation of an isomorphic copy of $\mathcal{L}$, then the Turing degree of $X^{(\mathcal{L}')}$ may not equal that of $X^{(\mathcal{L})}$. But, if we take $\mathcal{L}$ to be a computable ordinal (as defined in [AK]), then, by a result of Spector [Spe55], all $H(X, \mathcal{L})$ sets are Turing equivalent (where $\mathcal{L}$ ranges over different computable presentations of a *fixed* computable ordinal). We denote the least non-X-computable ordinal by ω_1^X, and write

ω_1^{CK} for $\omega_1^{\emptyset}$ (where CK stands for Church-Kleene). It is well-known that the set of computable ordinals is closed downwards [AK].

A formula φ of second order arithmetic is Σ^1_1 (relative to $A \subseteq \omega$) if it is of the form $(\exists X)\psi$, for some arithmetic formula ψ (with A as a parameter), and X is a set variable. A set $B \subseteq \omega$ is Σ^1_1 (relative to $A \subseteq \omega$), written $B \in \Sigma^1_1$ ($B \in \Sigma^1_1(A)$), if it is definable by a Σ^1_1 formula (with parameters from A). We say that $B \subseteq \omega$ is Δ^1_1 (relative to $A \subseteq \omega$), and write $B \in \Delta^1_1$ ($B \in \Delta^1_1(A)$), if there exist Σ^1_1 formulas φ, ψ (with A as a parameter) such that

$$(\forall n)[(n \in B) \Leftrightarrow (\varphi(n) \Leftrightarrow \neg\psi(n))].$$

Theorem 1.1 ([AK]). *For any two sets $A, B \subseteq \omega$, the following are equivalent:*

1. $(\exists \alpha < \omega_1^A) B \leq_T A^{(\alpha)}$.
2. $B \in \Delta^1_1(A)$.
3. *There is a A-computable infinitary formula φ such that $X = \{n : \varphi(n)\}$.*

If $\omega_1^A = \omega_1^{CK}$, then we also have that:

4. *there is a computable infinitary formula φ such that $X = \{n : \varphi(n, Y)\}$.*

A brief explanation of computably infinitary formulas is given at the end of this section (for more information see the subsection called Basic Notation and Background).

Definition 1.2. Whenever $A, B, \subseteq \omega$ satisfy any of conditions (1)–(3) above, we say that B is hyperarithmetically reducible to A, and write $B \leq_H A$. We also define $\mathrm{HYP}(A) = \{X \subseteq \omega : X \leq_H A\}$ and HYP=HYP($\emptyset$).

More information on hyperarithmetic sets and hyperarithmetic reducibility can be found in [AK].

1.2. Theories of Hyperarithmetic Analysis.

Definition 1.3. Let T be a collection of axioms of second order arithmetic. We say that T is a *theory of hyperarithmetic analysis* if

1. T holds in HYP(A), for every $A \subseteq \omega$, where HYP(A) is the ω-model consisting of the sets that are hyperarithmetically reducible to A.
2. All ω-models of T are hyperarithmetically closed.

This is equivalent to saying that, for every $A \subseteq \omega$, HYP(A) is the minimum ω-model of T that contains A, and that every ω-model of T is closed under disjoint unions.

Thus, the theories of hyperarithmetic analysis are those axioms of second order arithmetic that characterize both HYP and hyperarithmetic reduction. They also seem to exhibit a unique and interesting behavior in that small modifications often yield inequivalent theories. In the next section we will use the fact that all the theories of hyperarithmetic analysis mentioned in this section (and, indeed, this article) imply ACA_0. This fact is already known (see, for example, [Mon06]) for those theories of hyperarithmetic analysis that are not equal to ABW_0 and SL_0, which will be defined below in the next subsection (i.e. Section 1.3). We will prove this fact in the case of SL_0 and ABW_0[1] in Section 2 (Lemma 2.2).

Definition 1.4. A sentence S is a *sentence (statement) of hyperarithmetic analysis* if $\mathsf{RCA}_0 + S$ is a theory of hyperarithmetic analysis.

[1] To show that ABW_0 implies ACA_0 we require the additional hypothesis of induction for Σ^1_1 formulas, which we denote $\mathsf{I}\Sigma^1_1$, and we define in Section 1.5.

Before we can state the well-known theories of hyperarithmetic analysis that play a role in this article, we require a definition and a theorem of Jullien [Jul]. Jullien's Theorem was first examined in the context of reverse mathematics by Montalbán in [Mon06].

Definition 1.5. Given a linear ordering $\mathcal{Z} = \langle Z, \leq \rangle$, we define a *cut in* $\mathcal{Z}$ to be a pair of sets $\langle L, R \rangle$ such that $L = Z \setminus R$ is an initial segment of $\mathcal{Z}$. We say that $\mathcal{Z}$ is *indecomposable* if, for every cut $\langle L, R \rangle$, $\mathcal{Z}$ embeds into either L, or else $\mathcal{Z}$ embeds into R (thinking of L and R as sub-orderings of $\mathcal{Z}$). We say that $\mathcal{Z}$ is *indecomposable to the right* if, for every cut $\langle L, R \rangle$ with $R \neq \emptyset$, $\mathcal{Z}$ embeds into R. We define *indecomposable to the left* in a similar fashion. A linear order is called *scattered* if η, the order type of the rational numbers, does not embed in it.

Theorem 1.6 (Jullien's Theorem, [Jul])**.** *Every scattered indecomposable linear order is either indecomposable to the right, or indecomposable to the left.*

The following list consists of the well-known theories of hyperarithmetic analysis that play a significant role in this article.

($\mathsf{\Sigma^1_1 - AC_0}$) If $A(X, n)$ is an arithmetic predicate with a free set variable X and a free number variable n, then we have

$$(\forall n)(\exists Y)[A(Y, n) \Rightarrow (\exists Z)(\forall n)[A(Z^{[n]}, n)]].$$

($\mathsf{\Pi^1_1 - SEP_0}$) If φ, ψ are Σ^1_1 formulas satisfying $(\forall n)[\neg(\neg\varphi(n) \wedge \neg\psi(n))]$, then there exists a set D such that $(\forall n \in D)[\neg\varphi(n) \wedge \psi(n)]$.

($\mathsf{\Delta^1_1 - CA_0}$) If φ, ψ are Σ^1_1 formulas such that $(\forall n)[\varphi(n) \vee \psi(n)]$ and $(\forall n)[\neg\varphi(n) \Leftrightarrow \psi(n)]$, then there exists a set D such that $(\forall n)[n \in D \Leftrightarrow \varphi(n)]$.

($\mathsf{INDEC_0}$) Every scattered indecomposable linear order is either indecomposable to the right or indecomposable to the left. (Jullien's Theorem)

($\mathsf{\Sigma^1_1 - AC^w_0}$) If $A(X, n)$ is an arithmetic predicate with a single free set variable X and a single free number variable n, then we have

$$(\forall n)(\exists! Y)[A(Y, n) \Rightarrow (\exists Z)(\forall n)[A(Z^{[n]}, n)]].$$

Throughout this article we shall denote the weak Σ^1_1 choice scheme by $\mathsf{\Sigma^1_1 - AC^w_0}$ (as above), or $\mathsf{weak - \Sigma^1_1 - AC_0}$. We prefer to use the latter, but use the former at certain times (as above) because it is more compact.

It is known that (over $\mathsf{RCA_0}$)

$$(\mathsf{\Sigma^1_1 - AC_0}) \rightarrow (\mathsf{\Pi^1_1 - CA_0}) \rightarrow (\mathsf{\Delta^1_1 - CA_0}) \rightarrow (\mathsf{INDEC_0}) \rightarrow (\mathsf{weak - \Sigma^1_1 - AC_0}),$$

and none of the arrows is reversible. Each of the above implications are straightforward, except for $\mathsf{\Delta^1_1 - CA_0} \rightarrow \mathsf{INDEC_0}$, proven by Montalbán in [Mon06], and $\mathsf{INDEC_0} \rightarrow \mathsf{weak - \Sigma^1_1 - AC_0}$, shown by Neeman in [Nee]. On the other hand, the fact that the two leftmost arrows cannot be reversed was first established by Montalbán in [Mon], while the irreversibility of the two rightmost arrows was deduced by Neeman in [Nee]. It should also be noted that Steel first showed that $\mathsf{\Delta^1_1 - CA_0}$ does not imply $\mathsf{\Sigma^1_1 - AC_0}$ in [Ste78], and van Wesep was first to show that $\mathsf{weak - \Sigma^1_1 - AC_0}$ does not imply $\mathsf{\Delta^1_1 - AC_0}$ in [vW].

1.3. $\mathsf{SL_0}$ **and** $\mathsf{ABW_0}$**.** We now introduce the two statements of hyperarithmetic analysis, $\mathsf{SL_0}$ (sequential limit system) and $\mathsf{ABW_0}$ (arithmetic Bolano-Wierstrass), that are the main focus of our study in this paper. $\mathsf{SL_0}$ and $\mathsf{ABW_0}$ were first introduced by Friedman in [Fri75], and come from from basic, well-known theorems in mathematical analysis.

Our statement of ABW_0 differs slightly from that of Friedman [Fri75], which mentions predicates of real numbers. We now state Friedman's version of ABW.

(ABW) Consists of RCA (i.e. RCA_0, plus the induction scheme for all formulas) together with the axioms which assert that to every bounded arithmetic predicate of reals there is either a finite sequence of reals that includes all solutions, or a real, every neighborhood of which contains at least two solutions.

It is easy to prove that, over RCA_0, our version of ABW_0 (below) is equivalent to that of Friedman (above) minus the induction scheme for all formulas. Although we prefer to think of ABW_0 in terms of Friedman's definition above, we find our version of ABW_0 (below) to be more convenient for carrying out proofs related to ABW_0, and therefore after this section we will work exclusively with our version of ABW_0, given below.

Let $A(X)$ be an arithmetic predicate (possibly with parameters) of reals with a single free set (i.e. real) variable X and no other free variables. Let Z be a set of reals. Recall that a real Z_0 is said to be an *accumulation point* of the set Z if every neighborhood of Z_0 contains some $X \in Z$ such that $X \neq Z_0$ (Z_0 need not belong to Z). In other words, Z_0 is an accumulation point of Z if

$$(\forall n)(\exists X \in Z)[(X \neq Z_0) \wedge (X \restriction n = Z_0 \restriction n)].$$

Note that this is equivalent to saying that every neighborhood of Z_0 contains at least two elements of Z. We say that a predicate is *bounded* if its solutions all live in Cantor space[2] (we define Cantor space in the next subsection, called Basic Notation and Background). Over RCA_0, our definition of bounded predicate is equivalent to many other standard definitions.

(SL_0) If $A(X)$ has an accumulation point Z_0, then there is an infinite sequence of reals, $\{X_n\}_{n\in\mathbb{N}}$, such that $(\forall n)[A(X_n)]$ and $\lim_n X_n = Z_0$.

(ABW_0) If $A(X)$ is bounded then either $A(X)$ has finitely many solutions, or else the set of solutions to A, $\{X : A(X)\}$, has an accumulation point.

It should be noted that Friedman originally introduced SL_0 and ABW_0 in the context of unrestricted induction, and in doing so did not include the subscript 0.[3]

Definition 1.7. For all $X \subseteq \mathbb{R}$, we say that X is a *$F_\sigma \cap G_\delta$ set of reals* if X can be written in the form $X = Y \cap Z,\ Y, Z \subseteq \mathbb{R}$, where Y is an F_σ set of reals, and Z is a G_δ set of reals.

1.4. **Our Main Results.** Friedman's article [Fri75] serves as our starting point. In [Fri75], Friedman makes the following assertions.

Proposition 1.8. [Fri75, Theorem 2.1] *Over* RCA*,* SL_0 *is equivalent to* $\Sigma^1_1 - \mathsf{AC}_0$*. In other words, if we assume* RCA_0 *and induction for all formulas of second order arithmetic, then* SL_0 *is equivalent to* $\Sigma^1_1 - \mathsf{AC}_0$*.*

Proposition 1.9. [Fri75, page 239] *Over* RCA*, we have that* $\Sigma^1_1 - \mathsf{AC}_0$ *implies* ABW_0*.*

One of our main goals in the next section (Theorem 2.1), is to provide explicit proofs of the previous two propositions. To the author's knowledge, no proof of either proposition has ever been published, but the author believes that Friedman must have had proofs similar to the ones given in the next section before [Fri75].

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[2]It is not difficult to show that, over RCA_0, this is equivalent to the standard definition of a bounded set of reals.

[3]By including the subscript 0 in SL_0 and ABW_0, we indicate that we are only assuming induction for Σ^0_1 formulas. This convention applies to all subsystems of second order arithmetic (i.e. RCA_0, WKL_0, ACA_0, etc.)

In the next section (Theorem 2.1) we prove the following theorem.

Theorem 2.1. [Fri75, page 239] *Over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$ *(here* $\mathsf{I}\Sigma^1_1$ *denotes* Σ^1_1*-induction and is defined in the next subsection),* SL_0 *is equivalent to* $\Sigma^1_1 - \mathsf{AC}_0$*, while* ABW_0 *implies* $\mathsf{weak} - \Sigma^1_1 - \mathsf{AC}_0$ *and is implied by* $\Sigma^1_1 - \mathsf{AC}_0$*. It follows that* SL_0 *and* ABW_0 *are statements of hyperarithmetic analysis.*

In Section 3 we prove the following theorem.

Theorem 3.1. *There is an* ω*-model of* ABW_0 *that is not a model of* $\Delta^1_1 - \mathsf{CA}_0$*. Therefore,* ABW_0 *does not imply* $\Delta^1_1 - \mathsf{CA}_0$*.*

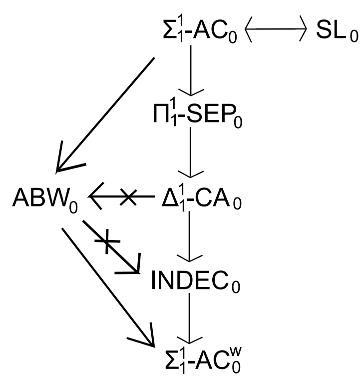
In Section 4 we prove the following theorem.

Theorem 4.1. *There is an* ω*-model of* ABW_0 *that is not a model of* INDEC*. Therefore,* ABW_0 *does not imply* INDEC_0*.*

These results imply that the reverse mathematical strength of ABW_0 lies strictly between those of $\Sigma^1_1 - \mathsf{AC}_0$ and $\mathsf{weak} - \Sigma^1_1 - \mathsf{AC}_0$, and that the strength of ABW_0 is incomparable to those of $\Delta^1_1 - \mathsf{CA}_0$, and INDEC_0. These facts make ABW_0 the first theory of hyperarithmetic analysis that is known to be incomparable to other theories of hyperarithmetic analysis. We leave the following question open, and conjecture that its answer is "yes."

Question 1.10. *Is every* ω*-model of* $\Pi^1_1 - \mathsf{SEP}_0$ *also a model of* ABW_0*? In other words, in the context of* ω*-models, does* $\Pi^1_1 - \mathsf{SEP}_0$ *imply* ABW_0*?*

The following diagram summarizes our results (over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_1$).



1.5. **Basic Notation and Background.**

1.5.1. *Cantor space and Baire space.* We use 2^ω to denote Cantor space, the space of infinite sequences of 0s and 1s. We use $2^{<\omega}$ to denote the full binary tree, or the set of finite sequences of 0s and 1s. If $\sigma \in 2^{<\omega}$, and either $\tau \in 2^{<\omega}$ or $\tau \in 2^\omega$, we write $\sigma \subseteq \tau$ to mean that τ extends σ, or (equivalently) that σ is an initial segment of τ. We define $\sigma \subset \tau$ to mean that $\sigma \subseteq \tau$ and $\sigma \neq \tau$. If $X \in 2^\omega$ and $n \in \omega$, we write $X \upharpoonright n$ to denote the initial segment of X of length n. A set $T \subseteq 2^{<\omega}$ is called a *tree* if T is closed under initial segments, and for any given tree $T \subseteq 2^{<\omega}$ and $f \in 2^\omega$, we say that f is a path through T if for every $n \in \omega$ we have that $f \upharpoonright n = \langle f(0), f(1), \cdots, f(n-1)\rangle \in T$. Furthermore, if $T \subseteq 2^{<\omega}$ is a tree, we define $[T]$ be the set of paths in 2^ω through T.

The standard topology on 2^ω is obtained by defining basic open sets of the form

$$[\sigma] = \{X \in 2^\omega : \sigma \subset X\},$$

for every $\sigma \in 2^{<\omega}$. It follows that a sequence of points $\{X_n : n \in \omega\} \subseteq 2^\omega$ converges to $X \in 2^\omega$ if and only if

$$(\forall k \in \omega)(\exists m \in \omega)(\forall n \geq m)[X_n \upharpoonright k = X \upharpoonright k].$$

Let ω^ω denote the set of infinite strings of natural numbers (Baire space), and let $\omega^{<\omega}$ denote the set of finite strings of natural numbers. For all $X \in \omega^\omega$, $\sigma, \tau \in \omega^{<\omega}$, and $n \in \omega$, we define $\sigma \subseteq X$, $\sigma \subseteq \tau$, $\sigma \subset X$, $\sigma \subset \tau$, $X \upharpoonright n$, tree, path, $[T]$ (where $T \subseteq \omega^{<\omega}$ is a tree), and basic open sets $[\sigma] \subseteq \omega^\omega$ analogously to the case of Cantor space.

Given a tree $T \subseteq \omega^{<\omega}$, and $\sigma \in T$, we define the well-founded rank of σ (relative to T), denoted $|\sigma|_T$, such that $|\sigma|_T = \sup\{|\tau|_T + 1 : \tau \in T, \sigma \subset \tau\}$. It follows that $\sigma \in T$ is not in the well-founded part of $T \subseteq \omega^{<\omega}$ exactly when $|\sigma|_T = \infty$. Lastly, for any given sets $A, B \subseteq \omega$, we will use the notation $A \subset_f B$ to mean that A is a finite subset of B.

We will assume that the reader is familiar with the basics of reverse mathematics, as presented in Simpson's book [Sim]. We will use $\mathsf{I}\Sigma^1_1$ (Σ^1_1-induction) to denote the scheme which says that for any Σ^1_1 formula φ the following holds:

$$(\mathsf{I}\Sigma^1_1) \qquad (\varphi(0) \wedge (\forall n)[\varphi(n) \to \varphi(n+1)]) \to (\forall n)\varphi(n).$$

For the definition of Σ^1_1 formula, or more information on the role of induction in reverse mathematics, consult [Sim].

1.5.2. *Arithmetical and Hyperarithmetical Formulas and Hierarchies.* A formula φ is *arithmetic* if it contains only number quantifiers (i.e. φ contains no set quantifiers). More specifically, a formula is Σ^0_0 or Π^0_0 if it is an open formula (i.e. it contains only *bounded* quantifiers). Now, for any $n \in \omega$, a formula φ is Σ^0_n, $n > 0$, if φ is of the form $(\exists \overline{x})\psi(\overline{x})$, and $\psi(\overline{x})$ is Π^0_{n-1}. A formula φ is Π^0_n, $n > 0$, if φ is of the form $(\forall \overline{x})\psi(\overline{x})$, and $\psi(\overline{x})$ is Σ^0_{n-1}. We can also define the sets of $\Sigma^0_n(A_0, A_1, \ldots, A_k)$ and $\Pi^0_n(A_0, A_1, \ldots, A_k)$ formulas in the same way, except that we also allow the sets $A_0, A_1, \ldots, A_k$ to be used as parameters in these formulas. A formula is *arithmetic* (relative to $A_0, A_1, \ldots, A_n$) if it is Σ^0_n ($\Sigma^0_n(A_0, A_1, \ldots, A_n)$), for some $n \in \omega$. A set $A \subseteq \omega$ is arithmetic (relative to $A_0, A_1, \ldots, A_k$) if it is definable by an arithmetic formula (with parameters $A_0, A_1, \ldots, A_k$). More information on arithmetic formulas and arithmetic sets can be found in [Soa].

Let $A \subseteq 2^\omega$. We say that A is a Σ^0_n (or Π^0_n)-class if there is a Σ^0_n (or Π^0_n) formula φ such that

$$A = \{X \in 2^\omega : \varphi(X)\}.$$

Similarly, one can define the notion of $\Sigma^0_n(B)$-class and $\Pi^0_n(B)$-class, for any parameter $B \subset \mathbb{N}$. It is well-known that A is F_σ if and only if there is a set $B \subseteq \omega$ such that A is a $\Sigma^0_2(B)$-class, and A is G_δ if and only if there is a set $B \subseteq \omega$ such that A is a $\Pi^0_2(B)$-class. Hence, the F_σ and G_δ subsets of Cantor space are arithmetically definable (with parameters).

A set $A \subseteq \omega$ is *computably enumerable* (c.e.) if it is the range of a 1-1 computable function. In other words, A is c.e. if there is an algorithm that lists the elements of A (not necessarily in order). Generally speaking, a formula φ is a computable infinitary formula if it contains (finite or) infinite conjunctions or disjunctions, so long as they are taken over computably enumerable (c.e.) sets of computable infinitary formulas. More specifically, a computable infinitary Σ^0_0 or Π^0_0 formula is an open formula (i.e. a formula with only *bounded* quantifiers). Now, for any given computable ordinal $\alpha > 0$, we define a computable infinitary Σ^0_α formula to be a (possibly) infinite disjunction of a c.e. set of formulas of the form $(\exists \overline{x})\varphi_i(\overline{x})$, where φ_i is a computable infinitary Π^0_β formula, for some $\beta < \alpha$. The definition of a computable infinitary Π^0_α formula is similar, except we replace disjunction by conjunction, $\exists$ by $\forall$, and Π^0_β by Σ^0_β. Also, the sets of computable infinitary $\Sigma^0_\alpha(A_0, A_1, \dots, A_k)$, $\Pi^0_\alpha(A_0, A_1, \dots, A_k)$ formulas are defined similarly, except that we allow the sets $A_0, A_1, \dots, A_k$ to appear as parameters in our formulas. $X \subset \omega$ is Σ^0_α (relative to $A_0, A_1, \dots, A_k$) if it can be defined by a Σ^0_α formula (with parameters $A_0, A_1, \dots, A_k$). A formula φ is *computable infinitary* (relative to $A_0, A_1, \dots, A_k$) if it is Σ^0_α ($\Sigma^0_\alpha(A_0, A_1, \dots, A_k)$), for some computable ordinal $\alpha < \omega_1^{CK}$. For further information on computably infinitary formulas and their relation to the hyperarithmetic hierarchy, consult [AK].

2. $\mathsf{ABW_0}$ and $\mathsf{SL_0}$ are Theories of Hyperarithmetic Analysis

The purpose of this section is the proof of the following theorem (i.e. Theorem 2.1). As we already said in the introduction, implications (1)-(3) below are stated (without proof) in [Fri75, page 239].

Theorem 2.1. *The following implications hold over* $\mathsf{RCA_0} + \mathsf{I}\Sigma^1_1$.

(1) $\Sigma^1_1 - \mathsf{AC_0} \rightarrow \mathsf{SL_0}$.
(2) $\Sigma^1_1 - \mathsf{AC_0} \rightarrow \mathsf{ABW_0}$.
(3) $\mathsf{SL_0} \rightarrow \Sigma^1_1 - \mathsf{AC_0}$. *(and hence* $\Sigma^1_1 - \mathsf{AC_0} \leftrightarrow \mathsf{SL_0}$*)*
(4) $\mathsf{ABW_0} \rightarrow \mathsf{weak} - \Sigma^1_1 - \mathsf{AC_0}$.

More precisely, we have that $\mathsf{SL_0} \leftrightarrow \Sigma^1_1 - \mathsf{AC_0}$ *over* $\mathsf{RCA_0}$ *(without* $\mathsf{I}\Sigma^1_1$*).*

Before we begin the proof of Theorem 2.1, we require the following lemma. Recall that every known theory of hyperarithmetic analysis that is not equal to either $\mathsf{SL_0}$ or $\mathsf{ABW_0}$ implies $\mathsf{ACA_0}$. The following lemma says that the same holds for $\mathsf{SL_0}$ and $\mathsf{ABW_0}$.

Lemma 2.2. *Over* $\mathsf{RCA_0} + \mathsf{I}\Sigma^1_1$ *we have that:*

(1) $\mathsf{SL_0} \rightarrow \mathsf{ACA_0}$.
(2) $\mathsf{ABW_0} \rightarrow \mathsf{ACA_0}$.

Proof. We reason in $\mathsf{RCA_0} + \mathsf{I}\Sigma^1_1$.

To prove that a subsystem of second order arithmetic S implies $\mathsf{ACA_0}$, it suffices to show that S implies the existence of the halting set $\emptyset' \subseteq \mathbb{N}$. To show that $\mathsf{SL_0}$ implies the existence of the halting set $\emptyset'$, let $A(X)$ be the arithmetic predicate which says that there is some $n \in \mathbb{N}$ such that $X \restriction (2n+1)$ is equal to $0^n 1 \emptyset' \restriction n$, and $X = 0^n 1 X \restriction n 0^\infty$. Then, we have that every neighborhood of 0^∞ contains at least two solutions to $A(X)$, and therefore, by $\mathsf{SL_0}$, there is a sequence of reals, $\oplus_{n \in \mathbb{N}} X_n$, such that for all $n \in \mathbb{N}$, $A(X_n)$ holds. Via $\mathsf{RCA_0}$, we can assume without loss of generality that for every $k \in \mathbb{N}$, X_k satisfies $X_k = 0^m 1 \emptyset' \restriction m$ for some $m > k$. Then it is clear that, for any given $n \in \mathbb{N}$, the first n bits of $\emptyset'$ can be computed (uniformly) from X_n (so, via $\mathsf{RCA_0}$, we have that $\emptyset' \subseteq \mathbb{N}$ exists).

To prove that $\mathsf{ABW_0}$ implies the existence of the halting set $\emptyset'$, let $A(X)$ be the arithmetic predicate which says that X is of the form $X = \sigma 0^\infty$, where $\sigma \in 2^{<\mathbb{N}}$, and $\sigma = \emptyset' \restriction |\sigma|$. Now, since $\emptyset' \subseteq \mathbb{N}$ is an infinite/coinfinite set, it follows that $A(X)$ does not have finitely many solutions. Therefore, by $\mathsf{ABW_0}$, $A(X)$ must have an accumulation point. Now, by $\mathsf{I}\Sigma^1_1$, for every $n \in \mathbb{N}$ there are at most n-many solutions of $A(X)$ such that $X \restriction n \neq \emptyset' \restriction n$). Therefore, we have that $\emptyset'$ is the *unique* accumulation point of $\{X : A(X)\}$, and by $\mathsf{ABW_0}$, $\emptyset'$ exists.

Note that we only used the hypothesis $\mathsf{I}\Sigma^1_1$ in the second paragraph to show that $\mathsf{ABW_0}$ implies $\mathsf{ACA_0}$. Therefore, (1) of Lemma 2.2 is valid over $\mathsf{RCA_0}$. $\square$

Proof. Proof of Theorem 2.1 We reason in $\mathsf{RCA_0} + \mathsf{I\Sigma^1_1}$. Note that, by Lemma 2.2 above, if $\mathsf{S} \in \{\mathsf{\Sigma^1_1} - \mathsf{AC_0}, \mathsf{SL_0}, \mathsf{ABW_0}, \mathsf{weak} - \mathsf{\Sigma^1_1} - \mathsf{AC_0}\}$, then S implies $\mathsf{ACA_0}$, and therefore we will assume that $\mathsf{ACA_0}$ holds throughout the proof of Theorem 2.1. We prove the four implications in order. Before we begin the proof of Theorem 2.1, we require the following elementary observations, definitions, and notation.

Now, let

$$V = \{X \in 2^{\mathbb{N}} : (\exists^{\infty} n)[X(n) = 1]\};$$

we will refer to $V \subset 2^{\mathbb{N}}$ as the set of *irrational numbers*, and the complement of V (in $2^{\mathbb{N}}$) as the set of *rational numbers*. Note that there is a natural computable homeomorphism $Z : V \to \mathbb{N}^{\mathbb{N}}$, such that $Z(X)(n),\ X \in V,\ n \in \mathbb{N}$, is equal to the number of 0s between the $(n-1)^{th}$ and n^{th} 1s appearing in $X \in V$. Note also that for every $X \in V$, $Z(X) \equiv_T X$.

To prove (1), first assume $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$. Now, let $A(X)$ be an arithmetic predicate with a single free set variable X. Furthermore, let $X_0 \in 2^{\mathbb{N}}$ be such that for every open neighborhood $U \subseteq 2^{\mathbb{N}}$ of X_0, there exist $X_1, X_2 \in U,\ X_1 \neq X_2$, such that both $A(X_1)$ and $A(X_2)$ hold. We shall use $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$ to construct a set $Y \in 2^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $A(Y^{[k]})$ holds and $\lim_{k\to\infty} Y^{[k]} = X_0$, thus proving that $\mathsf{\Sigma^1_1} - \mathsf{AC_0} \to \mathsf{SL_0}$.

Let $B(X, n),\ X \in 2^{\mathbb{N}}, n \in \mathbb{N}$, be the arithmetic predicate with a single free set variable X, and a single free number variable n, such that

$$B(X, n) \equiv\ A(X) \wedge (X{\restriction}n = X_0{\restriction}n).$$

For any $X \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, $B(X, n)$ says that $A(X)$ holds and $A{\restriction}n = X_0{\restriction}n$. Now, by hypothesis (above), we know that for every $n \in \mathbb{N}$ there exists a $Y \in 2^{\mathbb{N}}$ such that $B(Y)$. Therefore, by $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$, there exists a set $Y \in 2^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have that $B(Y^{[k]}, k)$ holds. By definition of $B(X, n)$, this implies that for all $k \in \mathbb{N}$ we have that $Y^{[k]}{\restriction}k = X_0{\restriction}k$ (and so $\lim_{k\to\infty} Y^{[k]} = X_0$) and $A(Y^{[k]})$ holds.

To prove (2), assume $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$ (recall that $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$ implies $\mathsf{ACA_0}$), and let $A(X)$ be a bounded arithmetic predicate with a single free set variable X. We need to show that if for every $n \in \mathbb{N}$, A has at least n-many solutions, then A has an accumulation point $X_0 \in 2^{\mathbb{N}}$ (note that X_0 need not satisfy A). Define an arithmetic predicate $B(X, n)$ with a single free set variable X and a single free number variable n, such that

$$B(X, n) \equiv (X = \bigoplus_{i=0}^{n} X^i) \wedge (\forall i \leq n)[A(X^i)] \wedge (\forall i, j \leq n)[i \neq j \to X^i \neq X^j].$$

For a fixed $n \in \mathbb{N}$, $B(X, n)$ says that $X \in 2^{\mathbb{N}}$ is the join of n-many distinct solutions to $A(X)$. We shall apply $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$ to the predicate $B(X)$ in order to prove that $\mathsf{ABW_0}$ holds for the predicate $A(X)$. To do this, we first need to prove that, for every $n \in \mathbb{N}$, there exists $X_n \in 2^{\mathbb{N}}$ such that $B(X_n, n)$ holds. This follows from $\mathsf{I\Sigma^1_1}$ and our assumption that for all $n \in \mathbb{N}$, $A(X)$ has at least n-many solutions. Hence, by $\mathsf{\Sigma^1_1} - \mathsf{AC_0}$, we have that there exists a set

$$X = \bigoplus_{n\in\mathbb{N}} X_n = \bigoplus_{n\in\mathbb{N}} \oplus_{i=0}^{n} X_n^i$$

such that $B(X_n, n)$ holds for every $n \in \mathbb{N}$. Note that, by our definition of $B(X, n)$, X_n^i satisfies $A(X_n^i)$ for all $n \in \mathbb{N},\ 0 \leq i \leq n$. Moreover, if for every $n \in \mathbb{N}$ we define

$$Y_n = \bigoplus_{\substack{0\leq m\leq n \\ 0\leq i\leq m}} X_m^i,$$

then Y_n contains at least n-many distinct columns because $X_n^i,\ 0 \le i \le n$, are columns of Y_n. Using this fact, and arithmetic comprehension (i.e. $\mathsf{ACA_0}$) relative to X, we can construct a set $Y \in 2^{\mathbb{N}}$ such that Y is of the form

$$Y = \bigoplus_{n\in\mathbb{N}} Y_n$$

and for every $n \in \mathbb{N}$ we have that $A(Y_n)$ holds and $Y_0, Y_1, \ldots, Y_n, \ldots \in 2^{\mathbb{N}}$ are mutually distinct.

Now, using $\mathsf{ACA_0}$ relative to Y, we may construct a tree $T \subseteq 2^{<\mathbb{N}}$ via the following definition:

$$T = \{\sigma \in 2^{<\mathbb{N}} : (\exists^\infty n \in \mathbb{N})[\sigma = Y_n\restriction|\sigma|]\}.$$

$T \subseteq 2^{<\mathbb{N}}$ consists of all nodes $\sigma \in 2^{<\mathbb{N}}$ that are initial segments of Y_n for infinitely many $n \in \mathbb{N}$. We now wish to show that T contains infinitely many nodes, then, since we are assuming $\mathsf{ACA_0}$, we can conclude that there exists an infinite path $Z \in 2^{\mathbb{N}}$ through T. Lastly, we will show that Z is in fact an accumulation point of the set $\{X \in 2^{\mathbb{N}} : A(X)\}$. Let $n \in \mathbb{N}$ be given. We shall show that T contains a node of length n. Note that there are 2^n-many binary strings of length n, and by $\mathsf{ACA_0}$ [4] it follows that for some $\sigma \in 2^{<\mathbb{N}},\ |\sigma| = n$, there exist infinitely many $m \in \mathbb{N}$ such that $\sigma = Y_m\restriction n$. By definition of T, we have that $\sigma \in T$. Hence, T is infinite. Now, using $\mathsf{ACA_0}$ with the parameter T, we can construct an infinite path $Z \in 2^{\mathbb{N}}$ through T. We now claim that Z is an accumulation point of the set $\{X \in 2^{\mathbb{N}} : A(X)\}$. Let $n \in \mathbb{N}$ be given. We must construct a real $X \in 2^{\mathbb{N}}$ such that $Z\restriction n = X\restriction n$, $Z \ne X$, and $A(X)$ holds. Since $Z\restriction n \in T$, (by definition of T) we know that there are infinitely many $m \in \mathbb{N}$ such that $Y_m\restriction n = Z\restriction n$ and $A(Y_m)$. Now, since $Y_0, Y_1, Y_2, \ldots$ are mutually distinct sets, we can find some $Y_m,\ m \in \mathbb{N}$, such that $Y_m \ne Z$, $Y_m\restriction n = Z\restriction n$, and $A(Y_m)$ holds. This proves that $Z \in 2^{\mathbb{N}}$ is indeed an accumulation point for the set $\{X \in 2^{\mathbb{N}} : A(X)\}$.

To prove (3), assume that $\mathsf{SL_0}$ holds, and suppose that $A(X, n)$ is an arithmetic predicate with one free set variable X and one free number variable n such that for every $m \in \mathbb{N}$ there exists $X \in 2^{\mathbb{N}}$ such that $A(X, m)$ holds. We need to construct a set $Y = \oplus_{n\in\mathbb{N}} Y_n$ such that, for every $n \in \mathbb{N}$, we have $A(Y_n, n)$. Without loss of generality, assume that $A(X, 0)$ has a solution of the form $0X,\ X \in 2^{\mathbb{N}}$, and define an arithmetic predicate $B(X),\ X \in 2^{\mathbb{N}}$, as follows:

$$B(X) \equiv (\exists n \in \mathbb{N})(\forall m \in \mathbb{N})[(m < n \to X^{[m]} = 0^\infty) \wedge (m > n \to X^{[m]} = 10^\infty) \wedge$$
$$(m = n \to (X^{[n]} = \bigoplus_{i=0}^{n} X_i^n) \wedge (X^{[n]}(0) = 0) \wedge (\forall i \le n)[A(X_i^n, i)])].$$

$B(X)$ says that there exists a number $n \in \mathbb{N}$ such that every row of X, $X^{[m]},\ m \in \mathbb{N}$, is equal to 0^∞ if $m < n$ or 10^∞ if $m > n$, except for possibly the single row, $X^{[n]}$, which is the join of sets $X_0^n, X_1^n, \ldots, X_n^n$ such that for all $0 \le i \le n$, X_i^n satisfies $A(X_i^n, i)$. If $A(X, 0)$ has no solution of the form $0X,\ X \in 2^{\mathbb{N}}$, then replace $X^{[n]}(0) = 0$ in the definition of $B(X)$ with $X^{[n]}(0) = 1$. The rest of the proof would change only slightly in this case. We assume that after having read the rest of our proof below, the reader could supply the proof of Theorem 2.1 (3) in the case where $A(X, 0)$ has no solution of

[4] $\mathsf{ACA_0}$ implies $\mathsf{B\Sigma_2}$, which is equivalent to the infinite pigeonhole principle. The infinite pigeonhole principle states that if $X \subseteq \mathbb{N}$ is infinite and we have that $X = \cup_{i=0}^n X_i$, then there is an $i \in \{0, 1, 2, \ldots, n\}$ such that X_i is infinite.

the form $0X$, $X \in 2^{\mathbb{N}}$ (and therefore we will neglect to consider this case in our proof below).

Let $n \in \mathbb{N}$ be given, and (by $\mathsf{I}\Sigma^1_1$) let $X^n_0, X^n_1, \ldots, X^n_n \in 2^{\mathbb{N}}$ be such that $A(X^n_i, i)$ holds for all $0 \leq i \leq n$. By definition of $B(X)$, the set $X_n \in 2^{\mathbb{N}}$, $n \in \mathbb{N}$, defined by $X^{[k]}_n = 0^\infty$ if $k < n$, $X^{[k]}_n = 10^\infty$ if $k > n$, and $X^{[n]}_n = \oplus^n_{i=0} X^n_i$, satisfies $B(X_n)$. Since, by assumption, there exists a number $m \in \mathbb{N}$ such that $A(X, m) \to X \neq 0^\infty$, then it follows that every neighborhood of 0^∞ contains at least two solutions to $B(X)$. Therefore, by SL_0, 0^∞ is the limit of some sequence of solutions to $B(X)$. In other words, there exists $Y = \oplus_{n\in\mathbb{N}} Y_n \in 2^{\mathbb{N}}$ such that $\lim_{n\to\infty} Y_n = 0^\infty$ and for all $n \in \mathbb{N}$, $B(Y_n)$ holds.

Notice that Y must (uniformly) code solutions to $A(X, n)$, for all $n \in \mathbb{N}$. To see this, for all $n \in \mathbb{N}$ (via ACA_0) let $f(n) \in \mathbb{N}$ be the largest natural number such that that for all $m \leq f(n)$ we have that $Y^{[m]}_n(0) = 0$. By our construction of $B(X)$ above, it follows that, for every $n \in \mathbb{N}$, $f(n)$ exists and $X = Y^{[f(n)]}_n$ satisfies $A(X^{[k]}, k)$, $0 \leq k \leq f(n)$. In other words, $Y^{[f(n)]}_n$ codes solutions to $A(X, k)$ for $k = 0, 1, \ldots, f(n)$. Also, note that since $\lim_{n\to\infty} Y_n = 0^\infty$, it follows that $\lim_{n\to\infty} f(n) = \infty$.

Now, from $Y = \oplus_{n\in\mathbb{N}} Y_n$ and $f(n)$, we shall uniformly compute a set $X = \oplus_{n\in\mathbb{N}} X_n$ such that for every $n \in \mathbb{N}$, $A(X_n, n)$ holds. To compute $X^{[n]}$ (using the fact that $\lim_n f(n) = \infty$) find the smallest argument $m \in \mathbb{N}$ such that $f(m) > n$, and set $X^{[n]}$ equal to $(Y^{[f(m)]}_m)^{[n]}$ (i.e. the n^{th} column of the $f(m)^{th}$ column of Y_m). By our construction of $B(X)$ above, we have that $A(X^{[n]}, n)$ holds, as required.

To prove implication (4), assume ABW_0, and suppose that $A(X, n)$ is an arithmetic predicate with a single free set variable X and a single free number variable n such that for every $n \in \mathbb{N}$ there exists a *unique* $X \in \mathbb{N}^{\mathbb{N}}$ such that $A(X, n)$ holds. Recall our definitions of $V \subseteq 2^{\mathbb{N}}$ and $Z : V \to \mathbb{N}^{\mathbb{N}}$ given at the beginning of this proof. Via ABW_0, we shall construct the set $Y \in 2^{\mathbb{N}}$ such that $Y = \oplus_{n\in\mathbb{N}} X_n$ and $A(Z(X_n), n)$ (it is not difficult to check that, over RCA_0, constructing such a $Y \in 2^{\mathbb{N}}$ is equivalent to proving that weak choice holds for $A(X, n)$). To construct $Y \in 2^{\mathbb{N}}$, consider the bounded arithmetic predicate $B(X)$ defined by

$$B(X) \equiv (\exists n_0 \in \mathbb{N})[(\forall n \leq n_0)[A(Z(X^{[n]}), n)] \wedge (\forall n > n_0)[X^{[n]} = 0^\infty]].$$

$B(X)$ says that there exists a number $n_0 \in \mathbb{N}$ such that if $n \leq n_0$ then $X^{[n]} \in 2^{\mathbb{N}}$ satisfies $A(Z(X^{[n]}), n)$, and if $n > n_0$ then $X^{[n]} = 0^\infty$. By $\mathsf{I}\Sigma^1_1$, and the fact that $0^\infty \notin V$, it follows that for all $k \in \mathbb{N}$, $B(X)$ has at least k-many solutions.

Now, we can apply ABW_0 to conclude that there exists $Y \in 2^{\mathbb{N}}$ such that Y is an accumulation point for the set $\{X \in 2^{\mathbb{N}} : B(X)\}$. Note that, for any given number $k \in \mathbb{N}$, Y is also an accumulation point for the class $\{X \in 2^{\mathbb{N}} : B(X)\} \cap \{X \in V : (\forall i \leq k)[A(Z(X^{[k]}), k)]\}$, since this class differs from $\{X \in 2^{\mathbb{N}} : B(X)\}$ by at most k-many elements. We now show that $Y = \oplus_{n\in\mathbb{N}} X_n$, where X_n, $n \in \mathbb{N}$, is such that $A(Z(X_n), n)$ holds. It suffices to show that, for every $n \in \mathbb{N}$ and $k \leq n$, we have that $Y^{[k]} \restriction n = X \restriction n$, where $X \in 2^{\mathbb{N}}$ is the unique solution to $A(Z(X), k)$. Suppose, for a contradiction, that this were not the case. In other words, suppose that for some $n \in \mathbb{N}$ there is some $k \in \mathbb{N}$, $k \leq n$, such that $Y^{[k]} \restriction n \neq X_0 \restriction n$, $X_0 \in V$, and $A(Z(X_0), k)$ holds. Then, since $Y \in 2^{\mathbb{N}}$ is an accumulation point for the class $\{X \in 2^{\mathbb{N}} : B(X)\} \cap \{X \in V : (\forall i \leq k)[A(Z(X^{[i]}), i)]\}$, there exists $W \in \{X \in 2^{\mathbb{N}} : B(X)\} \cap \{X \in V : (\forall i \leq k)[A(Z(X^{[i]}), i)]\}$ such that $Y^{[k]} \restriction n = W^{[k]} \restriction n \neq X_0 \restriction n$ and $B(W)$ holds. Now, by definition of $X_0, W \in V$, we have that $A(Z(X_0), k)$ holds, $A(Z(W^{[k]}), k)$ holds, but $Z(X_0) \neq Z(W^{[k]})$, and therefore we have contradicted the fact that for every $m \in \mathbb{N}$ there is a *unique* set $X \in \mathbb{N}^{\mathbb{N}}$ such that $A(X, m)$ holds. We now conclude that $Y \in 2^{\mathbb{N}}$, $Y = \oplus_{n \in \mathbb{N}} X_n$, $A(Z(X_n), n)$, as required.

At this point we wish to note that we did not use our hypothesis of $\mathsf{I}\Sigma^1_1$ in the proofs of (1) and (3) above, therefore, we have that $\mathsf{SL}_0 \leftrightarrow \Sigma^1_1 - \mathsf{AC}_0$ holds over RCA_0. □

3. $\Delta^1_1 - \mathsf{CA}_0$ does not imply ABW_0

In this section we prove the following theorem.

Theorem 3.1. *There is an ω-model of $\Delta^1_1 - \mathsf{CA}_0$ that is not a model of ABW_0. Hence, $\Delta^1_1 - \mathsf{CA}_0$ does not imply ABW_0.*

In [Ste78], Steel constructs an ω-model $M_\infty \cap \mathcal{P}(\omega)$ that satisfies $\Delta^1_1 - \mathsf{CA}_0$, but not $\Sigma^1_1 - \mathsf{AC}_0$. In [Mon], Montalbán shows that Steel's model also satisfies $\Pi^1_1 - \mathsf{SEP}_0$. We shall provide a subtle modification of Steel's construction that produces a model M_∞ satisfying $\Delta^1_1 - \mathsf{CA}_0 + \neg\mathsf{ABW}_0$. The rest of this section closely follows Montalbán's treatment of the matter in Section 2 of [Mon].

3.1. **Constructing M_∞.** To construct M_∞, we first construct a generic object

$$G = \langle T^G, \{\alpha^G_i : i \in \omega\}, h^G \rangle,$$

where $T^G \subseteq \omega^{<\omega}$ is a tree, $\{\alpha_i : i \in \omega\}$ is a set of paths through T^G, and $h^G : T^G \to \omega_1^{CK} \cup \{\infty\}$ is the well-founded rank function for T^G; i.e. $h^G(s) = |s|_{T^G}$. The function h^G ensures that $[T^G] \cap M_\infty = \{\alpha_i : i \in \omega\}$, and helps to prove certain properties of the forcing notion.

Throughout this section F will denote a finite subset of ω, i.e. $F \subset_f \omega$. For all such F, we let M_F be the class of sets that can be defined by a computable infinitary formula relative to the parameters T^G and α_i, $i \in F$. In other words,

$$M_F = \{X \subseteq \omega : (\exists \mu < \omega_1^{CK})[X \in \Sigma^0_\mu(T^G, \alpha^G_i : i \in F)]\}.$$

This can also be expressed as follows

$$M_F = \mathcal{P}(\omega) \cap L_{\omega_1^{CK}}(\{T^G\} \cup \{\alpha_i : i \in F\}),$$

where $L_{\omega_1^{CK}}(\{T^G\} \cup \{\alpha_i : i \in F\})$ is the class of Gödel constructible sets up to level ω_1^{CK}, starting from $\{T^G\} \cup \{\alpha^G_i : i \in F\}$ [Ste78, page 57]. Lemma 3.8 says that, for every $F \subset_f \omega$, we have that that $M_F =$HYP$(T \oplus \bigoplus_{i \in F} \alpha_i)$. It follows that, for every $F \subset_f \omega$, M_F is closed under hyperarithmetic reductions.

We now define our desired model M_∞ as follows:

$$M_\infty = \bigcup_{F \subset_f \omega} M_F.$$

As in [Ste78, Mon], we will show that for every $F \subset_f \omega$, the set of paths through T^G in M_F is equal to $\{\alpha_i : i \in F\}$. It follows that the set of paths through T^G in M_∞ is $\{\alpha_i : i \in \omega\}$ (Lemma 3.6), and from this fact we shall be able to deduce that $M_\infty \nvDash \mathsf{ABW}_0$ (Corollary 3.7).

For any given $\mu < \omega_1^{CK}$ and $F \subset_f \omega$, we define $M_{\mu,F}$, $M_{\mu,\infty}$, $H_{F,\mu}$, and $S_{\mu,F,e}$ exactly as in Section 2 of [Mon]. $M_{\mu,F}$ is the class of Σ^0_ν-definable sets in the parameters T and α_i, $i \in F$, where ν ranges over all ordinals less than μ. $M_{\mu,\infty} = \cup_{F \subset_f \omega} M_{\mu,F}$. $H_{F,1}$ is defined as the join of T^G, α_i, $i \in F$; $S_{\mu,F,e}$ is the e-th c.e. set relative to $H_{F,\mu}$; and for $\mu > 1$, $H_{F,\mu}$ is the join of $S_{\nu,F,e}$ such that $\nu < \mu, e \in \omega$. By our definitions above, and some well-known facts about infinitary formulas and definability, it follows that the sets $S_{\mu,F,e}$ belong to $\Sigma^0_\mu(T^G, \alpha_i : i \in F)$ and that $M_{\mu,F} = \{S_{\nu,F,e} : e \in \omega, \nu < \mu\}$.

3.2. **Our Forcing Conditions.** Our forcing conditions are motivated by those of Steel [Ste78] and Montalbán [Mon]. One major difference, however, is the introduction of a new tagging function $g : T^p_\infty \to \{0,1\}$ that tags the set of nodes $T^p_\infty = \{\sigma \in T^p : h^p(\sigma) = \infty\}$ (see below for more details). The function g ensures that we do not add too many paths to the generic tree $T^G = \cup_{p\in G} T^p$, which allows us to conclude that $[T^G]$ has no accumulation points in M_∞, and therefore M_∞ does not satisfy $\mathsf{ABW_0}$ (see Lemma 3.6 and Corollary 3.7).

Our forcing conditions are quadruplets $\langle T^p, f^p, h^p, g^p\rangle$ such that

(1) $T^p \subset \omega^{<\omega}$ is a finite nonempty tree.
(2) $f^p : \omega \to T^p$ is such that $\mathrm{dom}(f^p) \subset_f \omega$.
(3) $h^p : T^p \to \omega_1^{CK} \cup \{\infty\}$ so that
  (a) $(\forall \sigma, \tau \in T^p)[\sigma \subset \tau \Rightarrow h^p(\sigma) > h^p(\tau)]$.
  (b) $(\forall \sigma \in T^p)[((\exists i \in \omega)\sigma \subseteq f^p(i)) \Rightarrow h^p(\sigma) = \infty]$.
  (c) $h^p(\emptyset) = \infty$.
(4) $g^p : T^p_\infty \to \{0,1\}$, where $T^p_\infty = \{\sigma \in T^p : h^p(\sigma) = \infty\}$, is such that if $T^p = \{\emptyset\}$ then $g^p(\emptyset) = 1$.

By fiat, $\infty > \infty$ and $\infty > \omega_1^{CK}$. From now on, let P denote the set of our forcing conditions.

For $p, q \in P$, we define $p \leq q$ if and only if

(5) $T^q \subseteq T^p$.
(6) (a) $\mathrm{dom}(f^q) \subseteq \mathrm{dom}(f^p)$.
  (b) $(\forall i \in \mathrm{dom}(f^q))[f^q(i) \subseteq f^p(i)]$.
  (c) $(\forall i \in \mathrm{dom}(f^q))(\nexists \sigma \in T^q)[f^q(i) \subset \sigma \subseteq f^p(i)]$.
(7) $h^q = h^p \restriction T^q$.
(8) (a) If $g^q(\sigma) = 1$, but $g^p(\sigma) = 0$, then there exists some $\tau \supset \sigma$, $\tau \in T^p$, such that $\sigma \in T^q$ is the longest initial segment of τ on T^q, and $g^p(\tau) = 1$.
  (b) For all $\sigma \in T^p \setminus T^q$ such that $g^p(\sigma) = 1$, there exists $\tau_\sigma \in T^q$, $\tau_\sigma \subset \sigma$ such that $g^q(\tau_\sigma) = 1$ and $g^p(\tau_\sigma) = 0$.
(9) For every $i \in \mathrm{dom}(f^p) \setminus \mathrm{dom}(f^q)$, if $\sigma_i \in T^q$ is the longest initial segment of $f^p(i)$ on T^q, then we have that $g^q(\sigma_i) = 1$.

For any given $p \in P$, T^p, h^p, and f^p play the same role here as they did in [Mon, Ste78]. The most significant difference between our conditions and those of [Mon, Ste78] is the introduction of our function $g^p : T^p_\infty \to \{0,1\}$ that tags the nodes of $T^p_\infty = \{\sigma \in T^p : h^p(\sigma) = \infty\}$. More precisely, g^p acts as a lock on $\sigma \in T^p_\infty$. The lock is open when $g^p(\sigma) = 1$, and the lock is closed when $g^p(\sigma) = 0$. Condition (9) says that if $p \leq q$ wishes to add new paths to our model by extending the domain of $f^q \subset f^p$, then the longest initial segment of all such paths in T^q must be unlocked. The primary goal of the lock $g^p(\sigma)$ is to restrict the creation of new paths, so that we can prove that $[T] \subset \omega^\omega$ has no accumulation points in M_∞, and hence M_∞ does not satisfy $\mathsf{ABW_0}$ (Lemma 3.6 and Corollary 3.7). Our second requirements on g is given in condition (8) above. Roughly speaking, condition (8a) above says that whenever we lock a node $\sigma \in T^q$, there must be some node $\tau \supset \sigma$, $\tau \in T^p \setminus T^q$, that is unlocked. This property will be used to prove that M_∞ satisfies $\mathsf{\Delta^1_1 - CA_0}$ (Lemma 3.13). Condition (8b) is a converse to (8a), and says that in order to unlock a node $\tau \in T^p$, we *must* lock some initial segment of τ. We will use condition (8b) in the proof of Corollary 3.7 below, which says that $\mathsf{ABW_0}$ does not hold in our model M_∞.

Let $\boldsymbol{P} = \langle P, \leq \rangle$, and let $\boldsymbol{G} \subset \boldsymbol{P}$ be a sufficiently $\boldsymbol{P}$-generic filter. More specifically, let $\boldsymbol{G}$ be generic enough to force every Σ-over-$\mathcal{L}_F$ formula, which we will define later in this section. Define $T^G = \cup_{p \in \boldsymbol{G}} T^p$, $\alpha_i^G = \cup_{p \in \boldsymbol{G}} f^p(i)$, $i \in \omega$, and $h^G = \cup_{p \in \boldsymbol{G}} h^p$. By definition of $M_\infty = \cup_{F \subset_f \omega} M_F$, we will show that for all $i \in \omega$, we have that $[T^G] \cap M_\infty = \{\alpha_i^G : i \in \omega\}$. From this it will follow that M_∞ does not satisfy $\mathsf{ABW_0}$.

3.3. **The Forcing Language.** The forcing language $\mathcal{L}_\infty$, as well as the languages $\mathcal{L}_F,\ F \subset_f \omega$, are identical to those defined by Montalbán in Section 2.3 of [Mon]. For a complete description of Montalbán's languages, we refer the reader to [Mon]. Here we will give a brief overview of our languages, which we denote by $\mathcal{L}_\infty$ and $\mathcal{L}_F,\ F \subset_f \omega$.

The languages $\mathcal{L}_F,\ F \subset_f \omega$, consist of the symbols $\in, =, +, \times, , \leq$; constants for natural numbers; number variables; unranked set variables $X_H, Y_H, \ldots,\ H \subseteq F$; ranked set variables $X_H^\nu, Y_H^\nu, \ldots,\ H \subseteq F$; the usual logical connectives; the usual quantifiers for both number an set variables; the symbols $\boldsymbol{T}, \boldsymbol{\alpha}_i, \boldsymbol{S}_{\nu,F,e}, \boldsymbol{H}_{\nu,F}$: $i \in F \subset_f \omega, e \in \omega, \nu < \omega_1^{CK}$; and the elements of the sets of constants $C_\lambda^F,\ \lambda < \omega_1^{CK}$, that name elements of M_F.

The symbols of $\mathcal{L}_\infty$ include those in $\cup_{F \subset_f \omega} \mathcal{L}_F$, but also includes both ranked and unranked set variables of the form X^ν and X, respectively. A variable $X \in \mathcal{L}_\infty$ is *F-restricted* if it is subscripted H for some $H \subseteq F$. A formula of $\mathcal{L}_\infty$ is F-restricted if and only if all of its bounded variables are F-restricted.

The semantics of the languages $\mathcal{L}_\infty, \mathcal{L}_F,\ F \subset_f \omega$, are straightforward. Simply remember that $\boldsymbol{T}$ denotes $T^G \subseteq \omega^{<\omega}$; $\boldsymbol{\alpha}_i$ denotes $\alpha_i^G \in \omega^\omega$; $\boldsymbol{S}_{\nu,F,e}$ denotes $S_{\nu,F,e}$; $\boldsymbol{H}_{\nu,F}$ denotes $H_{\nu,F}$; X_F^ν ranges over $M_{\nu,F}$; X^ν ranges over $M_{\nu,\infty}$; X_F ranges over M_F; and X ranges over M_∞.

We say that a formula of $\mathcal{L}_\infty$ is *ranked* if all of its bounded variables are ranked. If ψ is a formula of $\mathcal{L}_\infty$, then $o(\psi)$ denotes the least upper bound of $\{\nu : \nu \text{ is the superscript of a quantified variable in } \psi\} \cup \{\nu + 1 : \text{some constant of the form } \boldsymbol{S}_{\nu,F,e} \text{ or } \boldsymbol{H}_{\nu,F} \text{ appears in } \psi\}$. For every constant $c = C_\lambda^F$ above, we let $o(c) = o(\emptyset \in C)$. Also, for all $\psi \in \mathcal{L}_\infty$, define

$$\text{rk}(\psi) = \omega_1^{CK} \cdot u(\psi) + \omega^2 \cdot o(\psi) + \omega \cdot r(\psi) + n(\psi),$$

where $u(\psi)$ is the number of unranked quantifiers in ψ, $r(\psi)$ is the number of ranked quantifiers in ψ, and $n(\psi)$ is the number of connectives in ψ.

3.4. **The Forcing Relation.** The definition of the forcing relation is standard; for further details consult [Mon]. One can show (by transfinite induction) that if $p \in \boldsymbol{P}$ and $\psi \in \mathcal{L}_\infty$, then $p \Vdash \psi$ if and only if whenever G is a sufficiently generic filter such that $p \in G$ and $\mathcal{M}_\infty$ is the model obtained from G, then $\mathcal{M}_\infty \vDash \psi$.

3.5. **Retaggings in $\boldsymbol{P}$.** We now introduce the notion of a *retagging*. This notion will play a significant role throughout the rest of Section 3, and we will define other notions of retaggings later on. For now, our definition of retagging is similar to that of Steel [Ste78] and Montalbán [Mon], and, as a result, both the statements and proofs of the results in this section are similar to those found in Section 2.5 of [Mon].

Definition 3.2. Let $p, p^* \in \boldsymbol{P}$, $F \subset_f \omega$, and $\mu \in \omega_1^{CK}$ be given. Then p^* is a *$\mu - F-$absolute retagging* of p, and we write $Ret(\mu, F, p, p^*)$, if the following three conditions are satisfied:

(1) $T^p = T^{p^*}$, $F \subseteq \text{dom}(f^p)$, $f^p \restriction F = f^{p^*} \restriction F$.
(2) $(\forall \sigma \in T^p)[h^p(\sigma) < \mu \Rightarrow h^{p^*}(\sigma) = h^p(\sigma)]$.
(3) $(\forall \sigma \in T^p)[h^p(\sigma) \geq \mu \Rightarrow h^{p^*}(\sigma) \geq \mu]$.

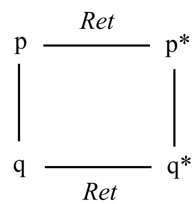
It can be shown that, for a fixed subset $F \subset_f \omega$ and ordinal $\mu < \omega_1^{CK}$, we have that $Ret(\mu, F, \cdot, \cdot)$ is an equivalence relation on $\boldsymbol{P}$. The intuition behind the following lemma is that $Ret(\mu, F, p, p^*)$ holds only if $p, p^* \in \boldsymbol{P}$ are indistinguishable inside $M_{\mu,F}$.

Lemma 3.3. [Mon, Lemma 2.4] *Let ψ be a ranked formula in $\mathcal{L}_F$, and let $p, p^* \in \boldsymbol{P}$. Then,*

$$Ret(\omega \cdot \mathrm{rk}(\psi), F, p, p^*) \quad \Rightarrow \quad (p \Vdash \psi \Leftrightarrow p^* \Vdash \psi).$$

The following lemma is crucial to the proof of Lemma 3.3. It is also crucial to the proof of Theorem 3.1.

Lemma 3.4. [Mon, Lemma 2.5] *Let p^* be an $\omega \cdot \beta - F -$absolute retagging of $p \in \boldsymbol{P}$ and suppose that $\gamma < \beta$ and $q \le p$. Then, there exists $q^* \le p^*$ such that $Ret(\omega \cdot \gamma, F, q, q^*)$.*



Proof. We construct $q^* \in \boldsymbol{P}$ as follows. First, set $T^{q^*} = T^q$. Secondly, set $f^{q^*}(i) = f^q(i)$, for all $i \in F$, and $f^{q^*}(i) = f^{p^*}(i)$, for all $i \in$dom$(f^{p^*}) \setminus F$. Next, let $h^{q^*} \supseteq h^{p^*}$ be such that $h^{q^*}(\sigma) = \infty$ for all $\sigma \in T^{q^*} \setminus T^{p^*}$, $\sigma \subseteq f^{q^*}(i)$, $i \in F$. Also, define $h^{q^*}(\sigma) = h^q(\sigma)$ whenever $h^q(\sigma) < \omega \cdot \gamma$, and set $h^{q^*}(\sigma) = \omega \cdot \gamma + |\sigma|_Q$, for all $\sigma \in Q$, where $Q \subseteq T^{q^*}$ is the set of nodes in T^{q^*} on which h^{q^*} is currently undefined. Finally, let $g^{q^*} \supseteq g^{p^*}$ be such that $g^{q^*}(\sigma) = 0$, for all $\sigma \in T^{q^*}_\infty \setminus T^{p^*}_\infty$.

One can check that $q^* \in \boldsymbol{P}$, $q^* \le p^*$, and $Ret(\omega \cdot \gamma, F, q, q^*)$. □

The proof of Lemma 3.3 (above) is exactly the same as that of [Mon, Lemma 2.4].

Define $P_\beta = \{p \in \boldsymbol{P} : \mathrm{ran}(h^p) \subseteq \beta \cup \{\infty\}\}$. By Lemma 3.3, it follows that if $\psi = \neg\varphi$ is of rank β and $p \in P_{\omega\cdot\beta}$, then $p \Vdash \neg\varphi$ if for every $q \in P_{\omega\cdot\beta}$, $q \le p$, we have that $q \nVdash \varphi$. From this, we can conclude the following corollary (by transfinite induction on β).

Corollary 3.5. [Mon, Corollary 2.6] *For a formula ψ of rank β, $0^{(\beta)}$ can decide whether or not $p \Vdash \psi$ uniformly in ψ, p, and β.*

The first major application of Lemmas 3.3 and 3.4 is in the proof the following lemma, a corollary of which says that our generic model $\mathcal{M}_\infty$ does not satisfy $\mathsf{ABW_0}$.

Lemma 3.6. [Mon, Lemma 2.7] *For every $F \subset_f \omega$, we have that*

$$M_F \cap [T^G] = \{\alpha_i^G : i \in F\}.$$

Proof. Suppose, for a contradiction, that $S = S_{\nu,F,e} \in \mathcal{M}_F$ is a path through T^G that is different from α_i^G, for $i \in F$. Then there exists $\sigma \subset S$, $|\sigma| > 1$, such that σ is not an initial segment of α_i^G, $i \in F$. Now, let $p \in G$ be such that dom$(f^p) \supseteq F$, $\sigma \in T^p$, and

$$p \Vdash \boldsymbol{S} \in [\boldsymbol{T}] \;\&\; \boldsymbol{\sigma} \subseteq \boldsymbol{S} \;\&\; \forall i \in F(\boldsymbol{\sigma} \nsubseteq \boldsymbol{\alpha}_i).$$

Now, let β be greater than ω times the rank of ψ, and large enough so that $p \in P_\beta$. Since S is a path in T^G, it follows that $h^G(\sigma) = \infty$, and thus $h^p(\sigma) = \infty$. We will construct a $p^* \in \boldsymbol{P}$ such that $Ret(\beta, F, p, p^*)$ and $h^{p^*}(\sigma) \in \omega_1^{CK}$. To define p^*, all we need to do is change the values of $h^p(\tau)$, $\tau \supseteq \sigma$, to ordinals in ω_1^{CK} that are greater than β.

Now, by Lemma 3.3, we have that

$$p^* \Vdash \boldsymbol{S} \in [\boldsymbol{T}] \ \& \ \sigma \subseteq \boldsymbol{S} \ \& \ \forall i \in F(\boldsymbol{\sigma} \nsubseteq \boldsymbol{\alpha}_i),$$

a contradiction since σ is in the well-founded part of T^{G^*}, for any generic filter G^* extending p^*. □

The following corollary says that our generic model $\mathcal{M}_\infty$ does not satisfy $\mathsf{ABW_0}$. The reason for this, informally speaking, is that the set

$$\{X : X \in [T^G]\}$$

has no accumulation point in M_∞.

Before we state and prove the corollary, however, we require some elementary definitions. Recall (from the proof of Theorem 2.1) that

$$V = \{X \in 2^\omega : (\exists^\infty n)[X(n) = 1]\};$$

we will refer to $V \subset 2^\omega$ as the set of *irrational numbers*, and the complement of V (in 2^ω) as the set of *rational numbers*. Note that there is a natural computable homeomorphism $Z : V \to \omega^\omega$, such that $Z(X)(n),\ X \in V,\ n \in \omega$, is equal to the number of 0s between the n^{th} and $(n+1)^{th}$ 1s appearing in $X \in V$. Note also that for every $X \in V$, $Z(X) \equiv_T X$.

Corollary 3.7.

$$\mathcal{M}_\infty \nvDash \mathsf{ABW_0}.$$

Proof. Let $A(X)$ be the bounded arithmetic formula of a single free set variable X such that

$$A(X) \equiv \ (X \in V) \wedge (Z(X) \in [T^G]).$$

First, note that $A(X)$ has infinitely many solutions in M_∞ of the form $Z^{-1}(\alpha_i^G),\ i \in \omega$. Next, we will show that $\{X : A(X)\}$ has no accumulation point in M_∞.

The first step in proving that $\{X : A(X)\}$ has no accumulation point in M_∞ is to show that such an accumulation point cannot be rational. To see why this is the case, note that by condition (8b) and the genericity of $G \subseteq \boldsymbol{P}$, it follows that for any given node $\sigma \in T^G$ there exists a number $n_\sigma \in \omega$ such that for every $p \in G$ and every $\tau \in T^G$ such that $\tau \supseteq \sigma k,\ k \geq n_\sigma$, we have that $g^p(\tau) = 0$ whenever it is defined. Therefore, if $X \in 2^\omega$ is an accumulation point of $\{X \in M_\infty : A(X)\}$, then X must be irrational (i.e. $Z(X) \in \omega^\omega$ is defined).

Now, since $[T^G] \subseteq \omega^\omega$ and $Z^{-1}([T^G]) \subseteq 2^\omega$ are *closed* sets, by the previous paragraph we have that every accumulation point of $\{X : A(X)\}$ must live inside $Z^{-1}([T^G])$ (but not necessarily inside $M_\infty \cap Z^{-1}([T^G])$). Furthermore, by Lemma 3.6, to show that $\{X : A(X)\}$ has no accumulation point in M_∞, it suffices to show that for every $i \in \omega$, $Z^{-1}(\alpha_i^G) \in 2^\omega$ is not an accumulation point of $\{X \in M_\infty : A(X)\}$. We prove this by contradiction.

Suppose, for a contradiction, that there exists some $i \in \omega$ such that $Z^{-1}(\alpha_i^G) \in 2^\omega$ is an accumulation point of the set $\{X : A(X)\}$, and let $p \in \boldsymbol{P}$ be such that $i \in$dom(f^p). Now, by genericity of $\boldsymbol{G} \subseteq \boldsymbol{P}$, let $q \leq p,\ q \in \boldsymbol{P}$, be such that all $\sigma \in T^q_\infty$ such that $g^q(\sigma) = 1$ are incomparable with $f^q(i) \in T^q_\infty$. By our condition (9) in Section 3.2, it follows that $Z^{-1}(\alpha_i^G) \in 2^\omega$ is not an accumulation point for the set $\{X : A(X)\}$, because $[T^G] \cap [f^q(i)] \cap M_\infty = \{\alpha_i^G\}$, and thus α_i^G is isolated in $[T^G]$, from the point of view of M_∞. □

Following [Ste78, Mon], for all $F \subset_f \omega$, we say that a formula $\psi \in \mathcal{L}_\infty$ is Σ-over-$\mathcal{L}_F$ if it is built up from ranked, F-restricted formulas using $\wedge$,$\forall n$, and $\exists X$. For any formula $\psi \in \mathcal{L}_\infty$ and $\mu < \omega_1^{CK}$, ψ^μ is the result of replacing "X" by "X^μ", for every unranked variable X. Note that if $F \subset_f \omega$, ψ is Σ-over-$\mathcal{L}_F$, $\mu < \omega_1^{CK}$, and $\mu > o(\boldsymbol{d})$ for any constant $\boldsymbol{d}$ appearing in ψ, then we have that $\psi^\mu \Rightarrow \psi$.

The proof of the following lemma is very similar to the one given in [Mon].

Lemma 3.8. [Mon, Lemma 2.9]

(1) *Let $p \in \boldsymbol{P}$, $p \Vdash \psi$, where $\psi \in \mathcal{L}_\infty$ is Σ-over-$\mathcal{L}_F$, and $F \subset_f \omega$. Then $(\exists \mu < \omega_1^{CK})(\forall \rho)[\mu \leq \rho < \omega_1^{CK} \Rightarrow p \Vdash \psi^\rho]$.*
(2) *$M_F \models \Sigma_1^1 - \mathsf{AC}_0$, and hence M_F is hyperarithmetically closed. Moreover, $M_F = HYP(T \oplus \bigoplus_{i \in F} \alpha_i^G)$.*

3.6. **Automorphisms of $\boldsymbol{P}$.** Let $\pi : \omega \to \omega$ be an automorphism (i.e. permutation) of ω. Then π induces an automorphism $\hat{\pi}$ of $\boldsymbol{P}$ as follows: $T^{\hat{\pi}(p)} = T^p, h^{\hat{\pi}(p)} = h^p, g^{\hat{\pi}(p)} = g^p$, and $f^{\hat{\pi}(p)}(\pi(i)) = f^p(i)$, $i \in \omega$. Automorphisms of the form $\hat{\pi}$, $\pi : \omega \to \omega$ a permutation, play a significant role throughout the rest of this section. Given $\varphi \in \mathcal{L}_\infty$, let $\pi(\psi)$ be the formula obtained from ψ by replacing $\boldsymbol{\alpha}_i$ by $\boldsymbol{\alpha}_{\pi(i)}$ for every $i \in \omega$.

The following lemma is proved by induction on the rank of $\psi \in \mathcal{L}_\infty$.

Lemma 3.9. [Mon, Lemma 2.10] *Let π be a permutation of ω, let $p \in \boldsymbol{P}$, and let $\psi \in \mathcal{L}_\infty$. Then*

$$p \Vdash \psi \Leftrightarrow \hat{\pi}(p) \Vdash \pi(\psi).$$

Remark: As in [Mon], we shall mainly use automorphisms (of $\boldsymbol{P}$) of the form $\hat{\pi}$ in the following way. Let $F \subset_f \omega$, and let $K \subset_f \omega$ be such that $F \cap K = \emptyset$. Also, suppose that $p \in \boldsymbol{P}$ has dom(f^p)$\subseteq F$, that ψ has constants in $\mathcal{L}_F$, and that $r \in \boldsymbol{P}$ is such that $r \leq p$ and $r \Vdash \psi$. Now, via an automorphism of $\boldsymbol{P}$, we can replace r by a condition $\hat{\pi}(r)$ such that $\hat{\pi}(r) \leq p, \hat{\pi}(r) \Vdash \psi$, and dom($f^{\hat{\pi}}(r)) \cap K$. In other words, we could replace r by $\hat{\pi}(r)$ if necessary to guarantee that dom(r)$\cap K = \emptyset$ is disjoint from $F \subset_f \omega$.

We now introduce a stronger notion of retagging, which we shall use throughout the rest of this section.

Definition 3.10. Let $p, p^* \in \boldsymbol{P}$, $F \subset_f \omega$, and $\mu \in \omega_1^{CK}$ be given. Then p^* is a *good $\mu - F -$absolute retagging of* p, and we write $\underline{Ret}(\mu, F, p, p^*)$, if the following conditions are satisfied:

(1) $T^p = T^{p^*}$, and $f^p \restriction F = f^{p^*} \restriction F$.
(2) $(\forall \sigma \in T^p)[h^p(\sigma) < \mu \Rightarrow h^{p^*}(\sigma) = h^p(\sigma)]$.
(3) $(\forall \sigma \in T^p)[h^p(\sigma) \geq \mu \Rightarrow h^{p^*} \geq \mu]$.
(4) $T^p_\infty \subseteq T^{p^*}_\infty$.
(5) $(\forall \sigma \in T^p_\infty)[g^p(\sigma) = 1 \Rightarrow g^{p^*}(\sigma) = 1]$.

We also define $\underline{Ret}_F(\mu, F, p, p^*)$ exactly as $\underline{Ret}(\mu, F, p, p^*)$, except that we also require $F \subseteq$dom(f^p) in condition (1). Note that $\underline{Ret}$ and $\underline{Ret}_F$ are not equivalence relations (because of condition (5)).

We now prove some retagging lemmas concerning $\underline{Ret}$ and $\underline{Ret}_F$. These lemmas are similar in spirit to Lemmas 3.3 and 3.4 above (or [Mon, Lemma 2.4, Lemma 2.5]).

Lemma 3.11. *Let ψ be a ranked formula in $\mathcal{L}_F$, and let $p, p^* \in \boldsymbol{P}$. Then,*

$$\underline{Ret}_F(\omega \cdot \mathrm{rk}(\psi), F, p, p^*) \quad \Rightarrow \quad (p \Vdash \psi \Leftrightarrow p^* \Vdash \psi).$$

Note that Lemma 3.3 is is a particular case of Lemma 3.11. It depends heavily upon Corollary 3.13 and Lemma 3.14 below. First, however, we prove Lemma 3.12, which is a stronger version of Corollary 3.13. Lemma 3.12 will play a major role in the proof of Lemma 3.15 below.

Lemma 3.12. *Suppose that* $F \subset_f \omega$, $\underline{Ret}(\omega \cdot \beta, F, p, p^*)$, *and that* $\gamma < \beta$ *and* $q \leq p$. *Then, there exists* $q^* \leq p^*$ *such that* $\underline{Ret}(\omega \cdot \gamma, F, q, q^*)$.

Proof. We construct $q^* \in \boldsymbol{P}$ as follows. First, define $T^{q^*} = T^q$. Secondly, set $f^{q^*}(i) = f^q(i)$, for all $i \in F \cap \mathrm{dom}(f^q)$, and $f^{q^*}(i) = f^{p^*}(i)$, for all $i \in \mathrm{dom}(f^{p^*}) \setminus F$. Next, we define $h^{q^*} : T^{q^*} \to \omega_1^{CK} \cup \{\infty\}$ as follows.

(a) Let $h^{q^*}(\sigma) = h^{p^*}(\sigma)$, for all $\sigma \in T^{p^*}$.
(b) Let $h^{q^*}(\sigma) = \infty$, for all $\sigma \in T^{q^*} \setminus T^{p^*}$ such that $h^q(\sigma) = \infty$.
(c) Let $h^{q^*}(\sigma) = h^q(\sigma)$, whenever $\sigma \in T^{q^*} \setminus T^{p^*}$ is such that $h^q(\sigma) < \omega \cdot \gamma$.
(d) Let $h^{q^*}(\sigma) = \omega \cdot \gamma + |\sigma|_Q$ for all $\sigma \in Q$, where $Q \subseteq T^{q^*} \setminus T^{p^*}$ is the set of nodes in $T^{q^*} \setminus T^{p^*}$ not covered by cases (a)-(c) above.

Lastly, we define $g^{q^*} : T^{q^*}_\infty \to \{0, 1\}$ as follows.

(i) For all $\sigma \in T^{p^*}$, let $g^{q^*}(\sigma) = 0$, if either $g^{p^*}(\sigma) = 0$, or else $g^p(\sigma) = 1$ and $g^q(\sigma) = 0$. Let $g^{q^*}(\sigma)$, $\sigma \in T^{p^*}$, be equal to 1 otherwise.
(ii) Let $g^{q^*}(\sigma) = 1$, for all $\sigma \in T^{q^*} \setminus T^{p^*}$ such that $g^q(\sigma) = 1$.
(iii) Let $g^{q^*}(\sigma) = 0$, for all $\sigma \in T^{q^*} \setminus T^{p^*}$ such that $g^q(\sigma) = 0$.

One can verify that, by the construction of q^*, we have that $q^* \in \boldsymbol{P}$, $q^* \leq p^*$, and $\underline{Ret}(\omega \cdot \gamma, F, q, q^*)$, as required. □

The statement of the following corollary is the same as that of the previous lemma, except that we replace $\underline{Ret}$ with $\underline{Ret}_F$. Moreover, it follows immediately from Lemma 3.12.

Corollary 3.13. *Suppose that* $F \subset_f \omega$, $\underline{Ret}_F(\omega \cdot \beta, F, p, p^*)$, *and that* $\gamma < \beta$ *and* $q \leq p$. *Then, there exists* $q^* \leq p^*$ *such that* $\underline{Ret}_F(\omega \cdot \gamma, F, q, q^*)$.

The next lemma is the counterpart to Lemma 3.12 above, except that it can only be verified with $\underline{Ret}_F$ in place of $\underline{Ret}$.

Lemma 3.14. *Suppose that* $F \subset_f \omega$, $\underline{Ret}_F(\omega \cdot \beta, F, p, p^*)$, *and that* $\gamma < \beta$ *and* $q^* \leq p^*$. *Then, there exists* $q \leq p$ *such that* $\underline{Ret}_F(\omega \cdot \gamma, F, q, q^*)$.

Proof. We construct $q \in \boldsymbol{P}$ as follows. First, define $T^q = T^{q^*}$. Secondly, set $f^q(i) = f^{q^*}(i)$, for all $i \in F$, and $f^q(i) = f^p(i)$, for all $i \in \mathrm{dom}(f^p) \setminus F$. Next, we define $h^q : T^q \to \omega_1^{CK} \cup \{\infty\}$ as follows.

(a) $h^q(\sigma) = h^p(\sigma)$, for all $\sigma \in T^p$.
(b) $h^q(\sigma) = \infty$, for all $\sigma \in T^q \setminus T^p$ such that $h^{q^*}(\sigma) = \infty$ and $h^p(\tau_\sigma) = \infty$, where $\tau_\sigma \in T^p$ is the longest initial segment of σ on T^p.
(c) $h^q(\sigma) = h^{q^*}(\sigma)$, whenever $\sigma \in T^q \setminus T^p$ and $h^{q^*}(\sigma) < \omega \cdot \gamma$.
(d) $h^q(\sigma) = \omega \cdot \gamma + |\sigma|_Q$ for all $\sigma \in Q$, where $Q \subseteq T^q \setminus T^p$ is the set of nodes in $T^q \setminus T^p$ not covered by cases (a)-(c) above.

Lastly, we define $g^q : T^q_\infty \to \{0, 1\}$ as follows.

(i) For all $\sigma \in T^p$, let $g^q(\sigma) = 0$, if either $g^p(\sigma) = 0$, or else $g^{p^*}(\sigma) = 1$ and $g^{q^*}(\sigma) = 0$. Let $g^q(\sigma)$, $\sigma \in T^p$, be equal to 1 otherwise.
(ii) Let $g^q(\sigma) = 1$, for all $\sigma \in T^q \setminus T^p$ such that $g^{q^*}(\sigma) = 1$ and there is some $\tau \subset \sigma$, $\tau \in T^p$, such that $g^p(\tau) = 1$ but $g^q(\tau) = 0$ via case (i) above.

(iii) Let $g^{q^*}(\sigma) = 0$, for all $\sigma \in T^{q^*} \setminus T^{p^*}$ not covered by case (ii) above.

One can verify that, by condition (8a) in Section 3.2 and our construction of q above, we have that $q \in \boldsymbol{P}$, $q \leq p$, and $\underline{Ret}_F(\omega \cdot \gamma, F, q, q^*)$, as required. □

The following key lemma (Lemma 3.15) will play a major role in the proof of Theorem 3.17 (below), which says that M_∞ satisfies $\Delta^1_1 - \mathsf{CA}_0$. Its statement is similar to that of [Mon, Lemma 2.12], but our proof depends heavily on the previous two (new) lemmas and corollary.

Lemma 3.15. *Let $F \subset_f \omega$ and $\psi \in \mathcal{L}_\infty$ be a Σ-over-$\mathcal{L}_F$ sentence. Suppose also that $\nu = \mathrm{rk}(\psi^\mu)$, where $\mu < \omega_1^{CK}$. Then,*

$$\underline{Ret}_F(\omega\nu + \omega^2, F, p, p^*) \ \&\ \mathrm{dom}(f^p) = F \qquad \Rightarrow \qquad (p \Vdash \psi^\mu \Rightarrow p^* \Vdash \psi^\mu).$$

(Note that ψ^μ is not necessarily in $\mathcal{L}_F$, because it may have quantifiers of the form $\exists X^\mu$.)

Proof. The proof is by induction on the number $k \in \omega$ of steps needed to build ψ from ranked, F-restricted formulas; we will show that the formula holds with "$\omega\nu + \omega 2k$" replacing "$\omega\nu + \omega^2$". The case $k = 0$ follows directly from Lemma 3.3 above. All of the cases follow easily from the induction hypothesis, except when ψ is of the form $\exists X \varphi(X)$. In this case we must show that $(\forall q^* \leq p^*)(\exists r^* \leq q^*)(\exists \boldsymbol{S} \in C_\mu)[r^* \Vdash \varphi^\mu(\boldsymbol{S})]$.

Now, let $q^* \leq p^*$ be given. By Lemma 3.14, there exists $q \leq p$ such that $\underline{Ret}_F(\omega\nu + \omega(2k+1), F, q, q^*)$. Moreover, the proof of Lemma 3.14 produces such a $q \leq p$ with $\mathrm{dom}(f^q) = F$. Since $p \Vdash \psi^\mu$, there exists $r \leq q$ and $\boldsymbol{S} \in C_\mu$ such that $r \Vdash \varphi^\mu(\boldsymbol{S})$. Choose $H \subset_f \omega$ such that $\boldsymbol{S} \in C_\mu^{F \cup H}$, $\mathrm{dom}(f^r) = F \cup H$, and $F \cap H = \emptyset$. Using an automorphism of $\boldsymbol{P}$ if necessary, we can assume without loss of generality that $H \cap \mathrm{dom}(f^{q^*}) = \emptyset$. Then we have that $\underline{Ret}(\omega\nu + \omega(2k+1), F \cup H, q, q^*)$, and by Lemma 3.12, there exists $r^* \leq q^*$ such that $\underline{Ret}_{F \cup H}(\omega\nu + \omega 2k, F \cup H, r, r^*)$. Finally, we can apply the induction hypothesis to conclude that $r^* \Vdash \varphi^\mu(\boldsymbol{S})$. □

3.7. $\mathcal{M}_\infty$ **satisfies** $\Delta^1_1 - \mathsf{CA}_0$**.** Before we can prove that M_∞ satisfies $\Delta^1_1 - \mathsf{CA}_0$, we require the following definition.

Definition 3.16. Suppose that $T' \subseteq_f T^G$ and $g : T' \to \omega_1^{CK} \cup \{\infty\}$. We say that g is *ν-good* if

$$(\forall \sigma \in T')[(h^G(\sigma) < \nu \Rightarrow g(\sigma) = h^G(\sigma)) \ \&\ (h^g(\sigma) \geq \nu \Rightarrow g(\sigma) \geq \nu)].$$

Note that deciding whether or not g is ν-good is hyperarithmetic in g, T^G, and ν, since it requires at most $(\nu + \omega)$-many Turing jumps of T^G.

We are now ready to prove that M_∞ satisfies $\Delta^1_1 - \mathsf{CA}_0$.

Theorem 3.17.

$$M_\infty \models \Delta^1_1 - \mathsf{CA}_0.$$

Proof. The proof is similar to that of [Mon, Lemma 2.14], with a few modifications.

Let $\varphi(n), \psi(n)$ be Σ-over-$\mathcal{L}_F$ with only n free, $F \subset_f \omega$, and such that

$$M_\infty \models (\forall n)[\psi(n) \Leftrightarrow \neg\varphi(n)].$$

We need to show that there exists $D \in M_\infty$ such that

$$M_\infty \models (\forall n)[\psi(n) \Leftrightarrow n \in D].$$

Let $p \in G$ be such that $p \Vdash (\forall n)[\psi(n) \Leftrightarrow \neg\varphi(n)]$. By enlarging $F \subset_f \omega$ (if necessary) and taking an extension $p' \leq p$, we can assume without any loss of generality that $\mathrm{dom}(f^p) = F$. By Lemma 3.8, there exists $\mu < \omega_1^{CK}$ such that

$$p \Vdash (\forall n)[\psi^\mu(n) \vee \varphi^\mu(n)]$$

and $\mu > o(\boldsymbol{S})$, for any constant $\boldsymbol{S}$ occurring in either ψ or φ. Fix $\nu < \omega_1^{CK}$ such that $p \in P_\nu$ and for all $n \in \omega$ we have that $\mathrm{rk}(\varphi^\mu(\boldsymbol{n}) \Leftrightarrow \neg\psi^\mu(\boldsymbol{n})) < \nu$. We are now ready to define the set $D \in M_\infty,\ D \subseteq \omega$ described above.

Let $d \in D$ if and only if there exists $q \in P_{\omega\nu+\omega^2+\omega 2},\ q \leq p$, such that

(1) $q \Vdash \psi^\mu(\boldsymbol{d})$;
(2) $T^q \subset T^G$;
(3) h^q is $\omega\nu + \omega^2 + \omega 2$-good;
(4) $(\forall i \in F)[f^q(i)$ is the longest initial segment of α_i^G on $T^q]$;

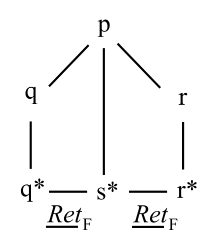
We will show that, for every $d \in \omega$, we have that $d \in D$ if and only if $\neg\varphi(d)$ holds.

Note that $D \subseteq \omega$ is hyperarithmetic in $T \oplus \bigoplus_{i \in F} \alpha_i^G$. Furthermore, since $M_F = HYP(T \oplus \bigoplus_{i \in F} \alpha_i^G)$, we have that $D \in M_F \subseteq M_\infty$.

Now, assume that $d \in \omega$ is such that $\neg\varphi(d)$. We will show that $d \in D$. Since φ is Σ-over-$\mathcal{L}_F$, by definition of μ, $\neg\varphi^\mu(d)$ holds. Let $q \in G$, $q \leq p$, be such that $q \Vdash \neg\varphi^\mu(\boldsymbol{d})$, and hence $q \Vdash \psi^\mu(\boldsymbol{d})$. By construction, q satisfies conditions (1)-(4) above, but q may not be in $P_{\omega\nu+\omega^2+\omega 2}$. To fix this issue, define $q^* \in \boldsymbol{P}$ as follows. Begin by setting $T^q = T^{q^*}$, $f^q = f^{q^*} \restriction F$, and $h^{q^*} \supseteq h^p$. Then, for all $\sigma \in T^{q^*} \setminus T^p$, define $h^{q^*}(\sigma) = \infty$, whenever $h^q(\sigma) \geq \omega\nu + \omega^2 + \omega 2$, and $h^{q^*}(\sigma) = h^q(\sigma)$ otherwise. Finally, set $g^{q^*}(\sigma) = 1$ for all $\sigma \in T_\infty^{q^*}$ such that $\sigma \in T_\infty^q$ and $g^q(\sigma) = 1$, and set $g^{q^*}(\sigma) = 0$ otherwise. By our construction of q^*, it follows easily that $q^* \in P_{\omega\nu+\omega^2+\omega 2}$, $q^* \leq p$, and that q^* satisfies (2)-(4) above. To see that q^* also satisfies condition (1), note that $\underline{Ret}_F(\omega\nu + \omega^2 + \omega 2, F, q, q^*)$. Hence, by definition of $q \leq p$ and Lemma 3.15 above, it follows that q^* also satisfies (1). Hence, q^* witnesses that $d \in D$.

Now, assume that $d \in \omega$ is such that $\varphi(d)$ holds. We will show that $d \notin D$. Let $r \leq p$, $r \in G$, be such that $r \Vdash \varphi(\boldsymbol{d})$, and thus $r \Vdash \varphi^\mu(\boldsymbol{d}) \wedge \neg\psi^\mu(\boldsymbol{d})$. Now, suppose for a contradiction that $d \in D$ and $q \in \boldsymbol{P}$ witnesses it. Via an automorphism of $\boldsymbol{P}$, we may assume without loss of generality that $\mathrm{dom}(f^q)= F \cup H$, $H \subset_f \omega$, where $F \cap H = \emptyset$ and $H \cap \mathrm{dom}(f^r) = \emptyset$. Let $F_{q^*} \subset_f \omega$ denote $\mathrm{dom}(f^{q^*})$, and let $F_{r^*} \subset_f \omega$ denote $\mathrm{dom}(f^{r^*})$. Next, we will define $q^* \leq q$, $r^* \leq r$, and $s^* \leq p$, such that

(i) $\underline{Ret}_{F_{q^*}}(\omega\nu + \omega^2 + \omega, F_{q^*}, q^*, s^*)$.
(ii) $\underline{Ret}_{F_{r^*}}(\omega\nu + \omega^2 + \omega, F_{r^*}, r^*, s^*)$.



Now, since $r \Vdash \varphi^\mu(\boldsymbol{d})$ and $q \Vdash \psi^\mu(\boldsymbol{d})$, then by Lemma 3.15, it will follow that $s^* \Vdash \varphi^\mu(\boldsymbol{d}) \wedge \psi^\mu(\boldsymbol{d})$. However, since $s^* \leq p$ and $p \Vdash (\forall n)[\psi^\mu(n) \Leftrightarrow \neg\varphi^\mu(n)]$, we have a contradiction. All that is left to do is construct q^*, r^*, and s^*.

We now construct a tree $T \supseteq T^q \cup T^r,\ T \subseteq 2^{<\omega}$, as follows. First, let $A \subseteq \omega^{<\omega}$ be the set of $\rho \in T_\infty^q \cup T_\infty^r$ such that either $g^q(\rho) = 1$ or $g^r(\rho) = 1$. Next, for every $\rho \in A$, let $\tau_\rho \in \omega^{<\omega}$ be any node such that $\tau_\rho \supset \rho$, $|\tau_\rho| = |\rho| + 1$, and $\tau_\rho \notin T^q \cup T^r$. Now, let

$$T = T^q \cup T^r \cup \{\tau_\rho : \rho \in A\};$$

note that, by its construction, $T \subseteq \omega^{<\omega}$ is a tree. Let $R = \{\tau_\rho : \rho \in A\} = T \setminus (T^q \cup T^r)$.

Define q^* as follows.

(1) $T^{q^*} = T$;
(2) (i) $f^{q^*}(i) = f^q(i)$, for $i \in H$.
  (ii) $f^{q^*}(i) = \alpha_i^G \restriction n$, where $n \in \omega$ is the largest number such that $\alpha_i^G \restriction n \in T^{q^*}$, and $i \in F$.
(3) (a) $h^{q^*}(\tau) = h^q(\tau)$, for all $\tau \in T^q$.
  (b) $h^{q^*}(\tau) = h^r(\tau)$, for all $\tau \in T^r \setminus T^q$ such that $h^r(\tau) < \omega\nu + \omega^2 + \omega$.
  (c) $h^{q^*}(\tau) = \infty$, if $(\exists i)[\tau \subseteq f^{q^*}(i)]$.
  (d) $h^{q^*}(\tau) = \infty$, if $\tau \in R$, and $h^q(\sigma_\tau) = \infty$, where σ_τ is the longest initial segment of τ on T^q.
  (e) $h^{q^*}(\tau) = \omega\nu + \omega^2 + \omega + |\tau|_Q$, for all $\tau \in Q$, where $Q = \{\tau \in T : \tau \text{ is not covered by cases (a), (b), or (c) above}\}$.
(4) $g^{q^*}(\sigma) = 1$ for all $\sigma \in R$ such that $g^q(\tau) = 1$, for some $\tau \subset \sigma$, $\tau \in T^q$; set $g^{q^*}(\sigma) = 0$ otherwise.

By the constuction of q^*, it follows that $q^* \in \boldsymbol{P}$, h^{q^*} is $(\omega\nu + \omega^2 + \omega)-$good, and $q^* \leq q$.

We define r^* as follows.

(1) $T^{r^*} = T$;
(2) (i) $f^{r^*}(i) = f^r(i)$, for $i \in \mathrm{dom}(f^r) \setminus F$.
  (ii) $f^{r^*}(i) = \alpha_i^G \restriction n$, where n is the largest number such that $\alpha_i^G \restriction n \in T$, for each $i \in F$.
(3) (a) $h^{r^*}(\tau) = \infty$, for all $\tau \in R$ such that $h^r(\sigma_\tau) = \infty$, where $\sigma_\tau \in T^r$ is the longest initial segment of τ on T^r.
  (b) $h^{r^*}(\tau) = h^G(\tau)$, for all $\tau \in T$ not covered by case (a).
(4) $g^{r^*}(\sigma) = 1$ for all $\sigma \in R$ such that $g^r(\tau) = 1$, for some $\tau \subset \sigma$, $\tau \in T^r$; set $g^{r^*}(\sigma) = 0$ otherwise.

It is not difficult to check that $r^* \in \boldsymbol{P}$, $r^* \leq r$, and h^{r^*} is $\omega\nu + \omega^2 + \omega-$good.

Lastly, we construct $s^* \leq p$ as follows.

(1) $T^{s^*} = T$.
(2) (i) $f^{s^*}(i) = f^{q^*}(i) = f^{r^*}(i)$, for all $i \in F$.
  (ii) $f^{s^*}(i) = f^{q^*}(i)$, for all $i \in \mathrm{dom}(f^{q^*}) \setminus F$.
  (iii) $f^{s^*}(i) = f^{r^*}(i)$, for all $i \in \mathrm{dom}(f^{r^*}) \setminus F$.
(3) (a) $h^{s^*}(\tau) = h^p(\tau)$, for all $\tau \in T^p$.
  (b) $h^{s^*}(\tau) = h^{q^*}(\tau) = h^{r^*}(\tau) = h^G(\tau)$, for all $\tau \in T^{s^*} \setminus T^p$ such that $h^G(\tau) < \omega\nu + \omega^2 + \omega$.
  (c) $h^{s^*}(\tau) = \infty$, for all $\tau \in T^{s^*} \setminus T^p$ such that either $h^{q^*}(\tau) = \infty$ or $h^{r^*}(\tau) = \infty$.
  (d) $h^{s^*}(\tau) = \omega\nu + \omega^2 + \omega + |\tau|_Q$, for all $\tau \in Q$, where $Q = \{\tau \in T^{s^*} : \tau \text{ is not covered by cases (a), (b), or (c) above}\}$.
(4) $g^{s^*}(\sigma) = 1$, for all $\sigma \in R$; set $g^{s^*}(\sigma) = 0$ otherwise.

It follows from the constructions of q^*, r^*, s^* above that $q^*, r^*, s^* \in \boldsymbol{P}$, $q^* \leq q$, $r^* \leq r$, $s^* \leq p$, and that conditions (i)-(ii) above are satisfied, as required. □

This completes the proof of Theorem 3.1, and marks the end of this section.

4. $\mathsf{ABW_0}$ does not imply $\mathsf{INDEC_0}$

Our main goal in this section is to prove the following theorem.

Theorem 4.1. *There is an ω-model $\mathcal{N}$ that satisfies $\mathsf{ABW_0}$, but does not satisfy $\mathsf{INDEC_0}$. Therefore, $\mathsf{ABW_0}$ does not imply $\mathsf{INDEC_0}$.*

To prove Theorem 4.1, we will show that van Wesep's model $\mathcal{N}$ in [Nee, Section 3],[vW], is indeed a model of $\mathsf{ABW_0}$. Neeman [Nee] has already shown that $\mathcal{N}$ is a model of $\neg\mathsf{INDEC_0}$. The key to showing that $\mathcal{N}$ satisfies $\mathsf{ABW_0}$ is to prove a modified version of [vW, Lemma 1.5, Sublemma 1], which Neeman describes in [Nee, Lemma 3.6].

Van Wesep's forcing conditions are similar to the ones that we defined in the previous section, with one main difference. Rather than tagging with ordinals, [vW] tags his trees with elements of a fixed computable linear order with no infinite hyperarithmetic descending sequences. Fix a nonstandard initial segment of such a (computable) linear ordering (for example, one could use the Harrison linear ordering [Har68]), and call it γ ([vW] refers to it as I_a).

Our forcing conditions are similar to those of [Nee, vW]; they are triplets $\langle T^p, f^p, h^p\rangle$ such that

(1) $T^p \subset \omega^{<\omega}$ is a finite tree.
(2) $f^p : \omega \to T^p$ is such that $\mathrm{dom}(f^p) \subset_f \omega$.
(3) $h^p : T^p \to \gamma \cup \{\infty\}$ so that
  (a) $(\forall \sigma, \tau \in T^p)[\sigma \subset \tau \Rightarrow h^p(\sigma) >_\gamma h^p(\tau)]$,
  (b) $(\forall \sigma \in T^p)[((\exists i)\sigma \subseteq f^p(i)) \Rightarrow h^p(\sigma) = \infty]$, and
  (c) $h^p(\emptyset) = \infty$.

By fiat, $\infty > \infty$ and $\infty > \omega_1^{CK}$. Let P denote the set of the above forcing conditions. We note that in [vW], instead of tagging nodes of T^p with ∞, the author simply leaves these nodes untagged, and so h^p is only defined on the set of nodes in T^p that are not extended by $f^p(i)$, $i \in \mathrm{dom}(f^p) \subset_f \omega$.

For $p, q \in P$, [vW] defines $p \leq q$ if and only if

(4) $T^q \subseteq T^p$,
(5) (a) $\mathrm{dom}(f^q) \subseteq \mathrm{dom}(f^p)$,
  (b) $(\forall i \in \mathrm{dom}(f^q))[f^q(i) \subseteq f^p(i)]$,
(6) $h^q = h^p \restriction T^q$.

Note that we have eliminated conditions (5c) and (5d) from Section 3.2. Our proof that $\mathcal{N} \models \mathsf{ABW_0}$ will depend upon this fact (Lemma 4.5 below). Let $\mathbb{P} = \langle P, \leq\rangle$, and G be any sufficiently $\mathbb{P}$-generic filter.

Another difference between the construction of van Wesep's model, $\mathcal{N}$, in [Nee, vW] and our construction of $\mathcal{M}_\infty$ in Section 3 is that $\mathcal{N}$ is of the form

$$N = \bigcup_{\substack{F \subset_f \omega \\ F \subseteq S^*}} M_F^G,$$

where M_F^G is defined analogously to our notion of M_F in Section 3, *but* $S^* \subseteq \omega$ is such that

$$(\forall i, j \in \omega)[(f_i^G(0) = f_j^G(0)) \Rightarrow (i \in S^* \Leftrightarrow j \in S^*)], \tag{1}$$

and $f_i^G = \cup_{p\in G} f^p(i),\ i \in \omega$ (analogous to the definition of α_i^G in Section 3). Note that (1) above is a consequence of the (more complicated) definition of S^* given in [vW].

In [vW], van Wesep uses the definition of S^* and the construction of $\mathcal{N}$ to show that $\mathcal{N} \not\models \Delta^1_1 - \mathsf{CA}_0$, while, in [Nee], Neeman uses these facts to conclude that $\mathcal{N} \not\models \mathsf{INDEC}_0$.

For every finite $F \subset_f \omega$, we construct M^G_F analogously to our construction of M_F in Section 3. The construction of M^G_F given by [Nee, vW] is different than, but equivalent to our construction of M^G_F. In particular, [Nee, vW] construct M^G_F in $L_{\omega_1^{CK}}(T, f^G_i : i \in F)$, while our construction lives in $\mathcal{P}(\omega)$. However, since the elements T^G and $f^G_i,\ i \in \omega$, are generic, it follows that ω_1^{CK} relative to $T, f^G_i,\ i \in F$, is equal to ω_1^{CK} (relative to $\emptyset$), and we therefore have that

$$M^G_F = \mathcal{P}(\omega) \cap L_{\omega_1^{CK}}(\{T^G\} \cup \{f^G_i : i \in F\}),$$

where M^G_F is constructed analogously to M_F in Section 3 of this article. Furthermore, for every $F \subset_f \omega$, we have $[T^G] \cap M^G_F = \{f^G_i : i \in F\}$, $M^G_F =$HYP$(T^G, f^G_i : i \in F)$, and $\mathcal{M}_F \models \Sigma^1_1 - \mathsf{AC}_0$ (see [Nee, vW] for more details).

Van Wesep [vW, Definition 1.2] defines retaggings (or *absolute reducts*, as he calls them) exactly as we do in Definition 3.2 above. Using the genericity of $G \subseteq \mathbb{P}$ and retaggings, one can prove the following theorem [vW, Lemma 1.5, Sublemma 2], which we will use in our proof of Theorem 4.6 below to show that $\mathcal{N} \models \mathsf{ABW}_0$. The main idea behind the proof of [vW, Lemma 1.5, Sublemma 2] is similar to that of Theorem 3.17 above, or [Mon, Lemma 2.14]. We give only but a sketch of the proof; for more details we ask the reader to consult [vW, Lemma 1.5, Sublemma 2].

Lemma 4.2. [vW, Lemma 1.5, Sublemma 2] *For any $I, J \subset_f \omega$, we have that*

$$M^G_I \cap M^G_J = M^G_{I\cap J}.$$

Proof: (Sketch). By our construction of $M^G_F,\ F \subset_f \omega$, above, it is clear that $M^G_{I\cap J} \subseteq M^G_I \cap M^G_J$. Therefore, it suffices to prove that $M^G_I \cap M^G_J \subseteq M^G_{I\cap J}$. In other words, we will show that if $X \in \omega^\omega$ satisfies $X \in M^G_I$ and $X \in M^G_J$, then we also have that $X \in M^G_{I\cap J}$.

Let $X \in M^G_I \cap M^G_J$. Let $F = I \cap J$, and $H, G \subset_f \omega$ be such that $F \cap H = F \cap G = \emptyset$, $I = F \cup H$, $J = F \cup G$. Let $\nu < \omega_1^{CK}$ be such that there exist $e_0, e_1 \in \omega$ such that

$$X = S_{\nu,I,e_0} = S_{\nu,J,e_1},$$

(where $S_{\mu,A,z},\ \mu < \omega_1^{CK},\ A \subset_f \omega,\ z \in \omega$, is as defined in Section 3.1). The proof rests on the following proposition, which we state, but will not prove (for a proof, consult [vW, Lemma 1.5, Sublemma 2]). As we have already remarked, the main idea behind the proof of the following proposition can be found in the proof of Theorem 3.17 above, or [Mon, Lemma 2.14].

Claim 4.3. *Let $p \in \mathbb{P}$ be such that*

$$p \Vdash \boldsymbol{X} = \boldsymbol{S}_{\nu,I,e_0} = \boldsymbol{S}_{\nu,J,e_1}.$$

Then, for all $n \in \omega$, we have that $n \in X$ if and only if there exists $q \leq p$ such that

(1) $q \Vdash \boldsymbol{n} \in \boldsymbol{S}_{\nu,I,e_0}$.
(2) $T^q \subseteq T^G$.
(3) h^q *is $\omega\nu + \omega$-good for T^G (see Definition 3.16 above).*
(4) *For all $i \in F$, we have that $f^q(i)$ is the longest initial segment of $f^G_i \in [T^G] \subseteq \omega^\omega$ on T^q.*

The proposition implies that X is hyperarithmetic in $T^G \oplus_{i\in F} f^G_i$, and hence $X \in M^G_F = M^G_{I\cap J}$, as required. □

We now state [vW, Lemma 1.5, Sublemma 1], the main idea of which can be found in [Nee, Lemma 3.6].

Lemma 4.4 ([vW], Lemma 1.5, Sublemma 1)**.** *Suppose that $F \subseteq I \subset_f \omega$, $Z_0 \in M_F^G$, $X_0 \in M_I^G$, and that $B(X, Z)$ is an arithmetical predicate with with only the free variables shown, such that $B(X_0, Z_0)$ holds. Then, there exists $J \subset_f \omega$, $X_1 \in M_J^G$, such that $I \cap J = F$ and $B(X_1, Z_0)$ holds. Moreover, for every $j \in J$ there exists $i \in I$ such that f_i^G and f_j^G have a nontrivial initial segment in common. Hence, by (1) above, $I \subseteq S^* \Rightarrow J \subseteq S^*$.*

Next, we will prove a modified version of [vW, Lemma 1.5, Sublemma 1] to show that, indeed, $\mathcal{N} \vDash \mathsf{ABW_0}$. The main idea of our proof is derived from that of van Wesep. Van Wesep's proof depends upon automorphisms of $\mathbb{P}$, which we defined in Section 3.6. Note that, via the same proof, Lemma 3.9 also holds for $\mathbb{P}$ (in place of $\boldsymbol{P}$).

Lemma 4.5. *Suppose that $F \subseteq I \subset_f \omega$, $Z_0 \in M_F^G$, $X_0 \in M_I^G$, and $B(X, Z)$ is an arithmetical predicate with only the free variables shown, such that $B(X_0, Z_0)$ holds. Then, for any given $k \in \omega$, there exists $J_k \subset_f \omega$, $X_k \in M_{J_k}^G$, such that $I \cap J_k = F$, $B(X_k, Z_0)$ holds,* ***and***

$$X_0 \restriction k = X_k \restriction k.$$

Moreover, for every $j \in J_k$ there exists $i \in I$ such that $f^G(i)$ and $f^G(j)$ have a nontrivial initial segment in common. Therefore, by (1) above, $I \subseteq S^* \Rightarrow J_k \subseteq S^*$.

Proof. The proof of Lemma 4.5 resembles that of [vW, Lemma 1.5, Sublemma 1].

Let $I = F \cup H_1$, $H_1 \cap F = \emptyset$, and $k \in \omega$ be given. Suppose that $Z_0 = S_{\nu_0,F,e_0}$ and $X_0 = S_{\nu_1,I,e_1}$, $\nu_0, \nu_1 < \omega_1^{CK}$, $F \subseteq I \subset_f \omega$, $e_0, e_1 \in \omega$. Now, let $\sigma_k \in \omega^{<\omega}$ be the initial segment of $X_0 \in \omega^\omega$ of length k, and $p \in G$ be such that

$$p \Vdash B(\boldsymbol{S}_{\nu_1,I,e_1}, \boldsymbol{S}_{\nu_0,F,e_0}) \wedge (\boldsymbol{\sigma}_k \subset \boldsymbol{S}_{\nu_1,I,e_1}).$$

Finally, let $H_2 \subset_f \omega$ be such that $H_2 \cap (F \cup H_1) = \emptyset$, and $H_0 = H_1 \cup H_2$ satisfies $\text{dom}(f^p) \subseteq F \cup H_0$.

As in [vW, Lemma 1.5, Sublemma 1], we call $q \in \mathbb{P}$ a *doublet* if there exists $n \in \omega$ such that $\text{dom}(f^q) = 2n$ and for all $m < n$ we have that $f^q(m) = f^q(m+n)$. Call n the period of the doublet, and note that for any $n \in \omega$, the set of doublets of period $> n$ is dense in $\mathbb{P}$. Let $n_1 \in \omega$ be such that $n_1 > \max\{F \cup H_0\}$, and let $q \in G$ be such that $q \leq p$ and q is a doublet with period $n > n_1$, $n \in \omega$. Let $L_i = \{l \in \omega : (\exists j \in H_i)[l = j + n]\}$, for $i = 0, 1, 2$, and note that, by the construction of L_i ($i = 0, 1, 2$) we have that $L_0 = L_1 \cup L_2$ and $L_0 \cap (F \cup H_0) = \emptyset$.

Now, let $r \geq q$ be identical to $p \in \mathbb{P}$, except that

(i) $\text{dom}(f^r) \subseteq F \cup L_0$.

(ii) $i \in \text{dom}(f^r) \Leftrightarrow [(i \in F \cap \text{dom}(f^p)) \vee (i - n \in H_0 \cap \text{dom}(f^p))]$.

(iii) $f^r(i) = f^p(i)$, for all $i \in F \cap \text{dom}(f^p)$; and $f^r(i+n) = f^p(i)$, for all $i \in H_0$.

In constructing $r \geq q$ above, we have simply translated the part of the domain of f^p that is $H_0 \subseteq \text{dom}(f^p)$ to the right by n and onto $L_0 \subseteq \text{dom}(f^r)$. Furthermore, since $r \geq q$ and $q \in G$, we have that $r \in G$.

Note that, by our construction of $r \geq q$, there is an automorphism of $\mathbb{P}$ taking p to r, and so, via a suitably modified version of Lemma 3.9 (i.e. the automorphism lemma), we have that

$$r \Vdash B(\boldsymbol{S}_{\nu_1,F\cup L_1,e_1}, \boldsymbol{S}_{\nu_0,F,e_0}) \wedge (\boldsymbol{\sigma}_k \subset \boldsymbol{S}_{\nu_1,F\cup L_1,e_1}).$$

Therefore, if we set

$$X_k = S_{\nu_1,J_k,e_1} \in \omega^\omega \text{ and } J_k = F \cup L_1 \subset_f \omega,$$

then it follows that $X_k \in M^G_{J_k}$ satisfies the conclusion of Lemma 4.5. $\square$

We are now ready to show that van Wesep's $\mathcal{N}$ is indeed a model of $\mathsf{ABW_0}$.

Theorem 4.6.

$$\mathcal{N} \vDash \mathsf{ABW_0}.$$

Proof. Let $A(X)$ be a bounded arithmetic predicate with parameters from $N = \cup_{F \subset_f S^*} M^G_F$, and such that $A(X)$ has infinitely many solutions $X \in N$. By definition of N, there exists $F \subset_f \omega$ such that $A(X)$ is an arithmetic predicate with parameters from M^G_F.

There are two cases to consider. The first case says that every solution of $A(X)$ lies in M^G_F. In this case we use the fact that $M^G_F =$HYP$(T^G, f^p_i : i \in F)$ to conclude that $\mathcal{M}_F \vDash \mathsf{\Sigma^1_1 - AC_0}$, and thus, by Theorem 2.1, $\mathcal{M}_F \vDash \mathsf{ABW_0}$. Therefore, there exists $X \in M^G_F \subseteq N$ such that X is an accumulation point for the set $\{Z \in 2^\omega : A(Z)\} \cap N$, from which it follows that $\mathcal{N} \vDash \mathsf{ABW_0}$.

The second case says that there exists $X \in N$ such that $X \notin M^G_F$ and $A(X)$ holds. In this case it follows from Lemmas 4.2 and 4.5 above that for every $k \in \omega$ there exists $Y \in N$ such that $Y \neq X$, but $Y \upharpoonright n = X \upharpoonright n$. Hence, $X \in N$ is an accumulation point for the set $\{Z \in 2^\omega : A(Z)\} \cap N$, and therefore we have that $\mathcal{N} \vDash \mathsf{ABW_0}$. $\square$

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Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, CANADA