On the Master-Equation Approach to Kinetic Theory: Linear and Nonlinear Fokker-Planck Equations

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Abstract: We discuss the relationship between kinetic equations of the Fokker-Planck type (two linear and one nonlinear) and the Kolmogorov (a.k.a. master) equations of certain N-body diffusion processes, in the context of Kac's propagation-of-chaos limit. The linear Fokker-Planck equations are well known, but here they are derived as a limit $N \to \infty$ of a simple linear diffusion equation on $3N$-C-dimensional N-velocity spheres of radius $\propto \sqrt{N}$ (where $C = 1$ or 4 depending on whether the system conserves energy only or energy and momentum). In this case, a spectral gap separating the zero eigenvalue from the positive spectrum of the Laplacian remains as $N \to \infty$, so that the exponential approach to equilibrium of the master evolution is passed on to the limiting Fokker-Planck evolution in $\mathbb{R}^3$. The nonlinear Fokker-Planck equation is known as Landau's equation in the plasma-physics literature. Its N-particle master equation, originally introduced (in the 1950s) by Balescu and Prigogine (BP), is studied here on the $3N$-4-dimensional N-velocity sphere. It is shown that the BP master equation represents a superposition of diffusion processes on certain two-dimensional submanifolds of $\mathbb{R}^{3N}$ determined by the conservation laws for two-particle collisions. The initial value problem for the BP master equation is proved to be well posed, and its solutions are shown to decay exponentially fast to equilibrium. However, the first nonzero eigenvalue of the BP operator is shown to vanish in the limit $N \to \infty$. This indicates that the exponentially fast approach to

equilibrium may not be passed from the finite-N master equation on to Landau's nonlinear kinetic equation.

Keywords: Propagation of chaos, kinetic theory, Fokker-Planck equation, Landau equation

1.   INTRODUCTION

Kinetic equations play a crucial role in the transport theory of gases and plasmas, in particular for studying the approach to equilibrium. Apart from rigorous mathematical studies of their solvability properties and the classification and description of their solutions, it is essential to establish their *validity*. The validation of a kinetic equation consists of its derivation from some deeper, deterministic microscopic model, for instance from the classical Newtonian dynamics of an isolated system of many interacting (point) particles. Clearly, a complete validation automatically involves existence and uniqueness results for the kinetic equation that is being validated, and it also involves proving some version of the second law of thermodynamics. A priori knowledge of existence and uniqueness of solutions to the kinetic equation can aid the proof of its validity, while in the absence of such a priori knowledge the successful validation would yield existence and uniqueness for the kinetic equation as a corollary. Unfortunately, validation has turned out to be a very difficult problem. Even the probably most re-known and most studied of the kinetic equations, namely Boltzmann's equation, has been validated only in a few "simple" situations (Lanford 1975; Spohn 1991; Cercignani, Illner, and Pulvirenti 1994).

   Meanwhile, inspired by the pioneering work of Kac (1956), a large body of literature has accumulated in which the deterministic N-body dynamics is replaced by an interacting stochastic Markov process that preserves, in each binary interaction, at least particle number and energy, but preferably also momentum and angular momentum, and which is designed to formally lead to the same kinetic equation that one expects from the "physical" N-body system in the infinitely many particles limit through a law of large numbers. Technically, the law of large numbers for the stochastic evolution of a family of individual system of N particles is equivalent to studying Kac's propagation of chaos limit $N \to \infty$ for the corresponding *ensemble* of individual systems, the evolution of which is being described by the Kolmogorov equation (called *master equation* in the physics literature) for the selected Markov process. Like Liouville's equation, the Kolmogorov equation is a *linear* deterministic partial differential equation for the ensemble probability density on $6N$-dimensional phase space. Unlike Liouville's equation, the Kolmogorov equation typically defines a contraction semigroup instead of a group, as does Liouville's equation. Hence, the relaxation of the ensemble density to a uniform density is now built into the ensemble evolution, and

because of Boltzmann's result that in the limit $N \to \infty$ almost every point of phase space corresponds to the Maxwellian velocity distribution, the second law of thermodynamics is a foregone conclusion. In this sense, the Kac approach is simpler than the original validation problem for kinetic equations, but it still retains some flavor of validation. From the perspective of validation, one could say that the Kac-type approach goes "half the way" toward that one would like to prove. From the perspective of the mathematical analysis of the kinetic equations itself, the Kac-type approach offers a new angle of attack to establish existence and uniqueness of the evolution and the relaxation to equilibrium in those cases where these results have not yet been obtained by direct partial-differential-equation (PDE) methods.

Yet the question whether the information obtained for a linear master (Kolmogorov) equation for finite N (such as global existence and uniqueness of solutions, as well as exponentially fast approach to equilibrium) carries on to the typically nonlinear kinetic equation that is expected to arise from the master equation in the limit $N \to \infty$ has turned out to be more subtle than originally anticipated. The current state of the art of this approach for short-range binary processes is presumably Carlen, Carvalho, and Loss (2003).

Our primary concern in this paper is the master-equation approach to certain kinetic equations that arise in the theory of systems with long-range interactions, such as Coulomb plasmas and Newtonian gravitating systems. In particular, we discuss a master equation, originally introduced (in the 1950s) by Balescu and Prigogine (BP), that leads formally to the spatially homogeneous Landau kinetic equation (Landau 1937), which plays a fundamental role in the classical transport theory of Coulomb plasmas (Balescu 1988; Hinton 1983). Considering here only the one-component case, the Landau equation for the particle density function $f(\cdot; t): \mathbb{R}^3 \to \mathbb{R}_+$ on velocity space at time $t \in \mathbb{R}_+$ has the form

$$\partial_t f(v; t) = \partial_v \cdot \int_{\mathbb{R}^3} \boldsymbol{Q_L}(v, w) \cdot (\partial_v - \partial_w)(f(v; t)f(w; t))\mathrm{d}^3 w, \qquad (1)$$

where $\boldsymbol{Q_L}(v, w)$ is the Landau collision kernel

$$\boldsymbol{Q_L}(v, w) = \partial_w^{\otimes 2}|v - w| = |v - w|^{-1}\boldsymbol{P}_{v-w}^{\perp} \qquad (2)$$

with $\boldsymbol{P}_{v-w}^{\perp}$ the projector onto the plane perpendicular to $v - w \in \mathbb{R}^3$. We remark that with the help of the so-called Rosenbluth potentials (Rosenbluth, McDonald, and Judd 1957) of f, the Landau equation can be recast as a *nonlinear* Fokker-Planck equation, the form that is better known to the astrophysics community. Formally, the Landau equation satisfies the standard conservation laws of mass, momentum, and energy; also the *H*-theorem holds. Thus, it is commonly believed that at late times the solution f evolves into the Maxwellian f_M associated, via the conservation laws, with the initial data f_0. Estimates of the relaxation time are usually obtained by linearization of the equations, but without estimates of the time it takes the dynamics to reach

the linar regime. Despite its physical importance, the mathematically rigorous confirmation of the expected behavior of the solutions to this equation is lacking. Only very recently has the spatially homogeneous Landau equation attracted some attention in the PDE literature, where it has been studied as a member of a more general family of equations with kernels

$$Q(v, w) = |v - w|^{2+\gamma} P_{v-w}^{\perp} \qquad (\gamma > -5) \qquad (3)$$

formally associated with $1/r^{\gamma-5/\gamma-1}$ force laws. However, for Coulomb and Newton interactions ($\gamma = -3$), these PDE methods have provided only weak existence results (Villani 1996, 1998), leaving the questions of uniqueness, regularity, and approach to equilibrium largely unanswered, except for initial conditions close to equilibrium (Guo 2002) or locally in time (Zhan 1994).

As regards the Balescu-Prigogine N-particle master equation for the Landau equation, we will show that its initial value problem is well posed, and that its solutions approach equilibrium exponentially fast, on the $3N$-4-dimensional N-velocity sphere of constant mass, energy, and momentum. However, in the limit $N \to \infty$, with energy and momentum scaled so that the corresponding quantities per particle are constant (the N-velocity sphere has radius $\propto \sqrt{N}$), the first nonzero eigenvalue of the BP operator is shown to vanish. This indicates that the exponentially fast approach to equilibrium described by the finite-N master equation may not be passed on to Landau's nonlinear kinetic equation. To resolve this issue requires a more detailed knowledge of the spectrum of the BP operator.

While our efforts have not yet revealed all the details of the BP operator spectrum, we have discovered that the BP master equation represents a superposition of diffusion processes on certain two-dimensional sub-manifolds of $\mathbb{R}^{3N}$ determined by the conservation laws for two-particle collisions. This has prompted us study in more detail the completely solvable cases in which the Kolmogorov equation is just the linear diffusion equation $\partial_t F = \Delta F$ on a $3N - C$-dimensional many-velocity sphere, where $C = 1$ or 4. The underlying stochastic processes are the perhaps simplest single + binary processes preserving either particle number N and total energy $\sum_{k=1}^{N} \frac{1}{2}|v_k|^2 = N_\varepsilon$ ($C = 1$), or particle number, total energy, and total momentum $\sum_{k=1}^{N} v_k = N\mathbf{u}$ ($C = 4$). The results are interesting enough to deserve being included in this paper.

Thus, we show explicity that in the limit $N \to \infty$ we obtain an essentially linear Fokker-Planck equation for the particle density function $f(\cdot; t)$: $\mathbb{R}^3 \to \mathbb{R}_+$ on velocity (= momentum) space at time $t \in \mathbb{R}_+$, which for $C = 4$ has the form

$$\partial_t f(v; t) = \partial_v \cdot \left(\partial_v f(v; t) + \frac{3}{2\varepsilon_0}(v - u)f(v; t) \right), \qquad (4)$$

for initial data $f(v; 0) \geq 0$ having mass per particle

$$\int_{\mathbb{R}^3} f(v; \ 0)\mathrm{d}^3 v = 1, \tag{5}$$

momentum per particle

$$\int_{\mathbb{R}^3} v f(v; \ 0)\mathrm{d}^3 v = \mathbf{u}, \tag{6}$$

and an energy per particle

$$\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f(v; \ 0)\mathrm{d}^3 v = \varepsilon, \tag{7}$$

which itself is a sum of the energy per particle in the center-of-mass frame, ε_0, and the energy per particle of the center-of-mass motion, $\varepsilon_{CM} = 1/2|\mathbf{u}|^2$; viz., $\varepsilon = \varepsilon_0 + 1/2|\mathbf{u}|^2$. It is easy to show that with such initial data the mass per particle $\int_{\mathbb{R}^3} f(v; t)\mathrm{d}^3 v$, the momentum per particle $\int_{\mathbb{R}^3} v f(v; t)\mathrm{d}^3 v$, and energy per particle $\int_{\mathbb{R}^3} (1/2)|v|^2 f(v; t)\mathrm{d}^3 v$ are conserved during the evolution. Moreover, for such data the solution f of (4) evolves, as $t \to \infty$, into the drifting Maxwellian f_M associated, via the conservation laws, with the initial data $f(v; 0)$, viz.,

$$f_M(v) = \left(\frac{3}{4\pi\varepsilon_0}\right)^{3/2} \exp\left(-\frac{3|v-\mathbf{u}|^2}{4\varepsilon_0}\right), \tag{8}$$

and it does so exponentially fast and with monotonically increasing relative entropy

$$S(f|f_M) = -\int_{\mathbb{R}^3} f(v; \ t) \ln \frac{f(v; \ t)}{f_M(v)} \mathrm{d}^3 v, \tag{9}$$

so that an *H*-theorem holds. The treatment without momentum conservation ($C = 1$) is similar; of course, $\mathbf{u}$ does not show in this case.

We remark that our derivation of (4) together with (5), (6), and (7) from an *isolated* system of N particles preserving energy and momentum may not be new; however, since we could not find it in the literature, this interpretation of (4) may perhaps not be so widely known. Indeed, (4) is usually associated with an Ornstein-Uhlenbeck process for a swarm of individual, independent particles, the velocities (in $\mathbb{R}^3$) of which are being thermalized through contact with a heat bath of temperature T. In this case $2\varepsilon_0$ in (4) is replaced by $3T$, and the restrictions (6) and (7) on the initial data have to be dropped.

In the remainder of the paper, we first set up the master-equation approach for isolated spatially uniform systems in general. Then, to have a simple illustration of validation à la Kac, we first discuss the diffusion equations on the $3N$-C-dimensional velocities spheres and derive the linear Fokker-Planck equation(s) in the limit $N \to \infty$. Then we turn to the BP master

equation and its putative $N \to \infty$ limit, the nonlinear Fokker-Planck equation due to Landau.

2. ENSEMBLES OF ISOLATED SYSTEMS

2.1. The Velocity Manifolds

Let $\{V_\alpha\}_{\alpha=1}^{\infty}$ denote an infinite ensemble of identically distributed random vectors taking values in $\mathbb{R}^{3N}$. Each vector $V = (v_1, \ldots, v_N) \in \mathbb{R}^{3N}$ represents a possible micro-state of an individual system of N particles with velocities $v_i = (v_{i1}, v_{i2}, v_{i3}) \in \mathbb{R}^3$. The positions of the particles are assumed to be uniformly distributed over either a periodic box or a container with reflecting boundaries and have been integrated out. The micro-state is assumed to evolve in time according to some stochastic process that conserves

$$m(V) = N \qquad \text{(mass of } V\text{)} \tag{10}$$

and

$$e(V) = \frac{1}{2} \sum_{k \in \mathbb{I}_N} |v_k|^2 \qquad \text{(energy of } V\text{)}, \tag{11}$$

where $\mathbb{I}_N = \{1, \ldots, N\}$; in the periodic box also

$$p(V) = \sum_{k \in \mathbb{I}_N} v_k \qquad \text{(momentum of } V\text{)}, \tag{12}$$

is preserved. We are only interested in ensembles of N particles systems in which all members have the same energy, or same energy and same momentum. Thus, depending on the circumstances, an ensemble consists of vectors V_α taking values either in the $3N$-1-dimensional manifold of constant energy

$$\mathbb{M}_\varepsilon^{3N-1} = \left\{ V \colon \frac{1}{2} \sum_{k \in \mathbb{I}_N} |v_k|^2 = N\varepsilon, \quad \varepsilon > 0 \right\}, \tag{13}$$

or in the $3N$-4-dimensional manifold of constant energy and momentum

$$\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4} = \left\{ V \colon \sum_{k \in \mathbb{I}_N} v_k = N\mathbf{u}, \ \frac{1}{2} \sum_{k \in \mathbb{I}_N} |v_k|^2 = N\varepsilon, \ \varepsilon > \frac{1}{2}|\mathbf{u}|^2 \right\}. \tag{14}$$

Each such manifold is invariant under the process that generates the evolution of an individual, isolated N body system, which in turn traces out a trajectory on one of these manifolds $\mathbb{M}_\varepsilon^{3N-1}$ or $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$.

    The manifold $\mathbb{M}_\varepsilon^{3N-1}$ is identical to a $3N$-1-dimensional sphere $\mathbb{S}_{\sqrt{2N\varepsilon}}^{3N-1}$ of radius $\sqrt{2N\varepsilon}$ and centered at the origin of $\mathbb{R}^{3N}$; the manifold $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ is

identical to a $3N$-4-dimensional sphere of radius $\sqrt{2N\varepsilon_0}$ and centered at $U = (u, \ldots, u)$, embedded in the $3(N-1)$-dimensional affine linear subspace of $\mathbb{R}^{3N}$ given by $U + \mathbb{L}^{3N-3}$, where $\mathbb{L}^{3N-3} \equiv \mathbb{R}^{3N} \cap \{V \in \mathbb{R}^{3N}: \sum_{k \in \mathbb{I}_N} v_k = 0\}$. In the following, when we write $\mathbb{S}^{3N-4}_{\sqrt{2N\varepsilon_0}}$, we mean $\mathbb{S}^{3N-4}_{\sqrt{2N\varepsilon_0}} \subset \mathbb{L}^{3N-3}$.

2.2. Master Equations

Any ensemble $\{V_\alpha\}_{\alpha=1}^\infty$ at time t is characterized by a probability density on either $\mathbb{M}_\varepsilon^{3N-1}$ or $\mathbb{M}_{u,\varepsilon}^{3N-4}$, for simplicity denoted $F^{(N)}(V; t)$. The time evolution of $F^{(N)}(V; t)$ is determined by a master equation on $\mathfrak{L}^2 \cap \mathfrak{L}^1(\mathbb{M}_{u,\varepsilon}^{3N-4})$ or $\mathfrak{L}^2 \cap \mathfrak{L}^1(\mathbb{M}_\varepsilon^{3N-1})$

$$\partial_t F^{(N)} = -\mathcal{L}^{(N)} F^{(N)}, \tag{15}$$

where $\mathcal{L}^{(N)}$ is a positive semidefinite operator on $\mathfrak{L}^2(\mathbb{M}_{u,\varepsilon}^{3N-4})$ or $\mathfrak{L}^2(\mathbb{M}_\varepsilon^{3N-1})$; (15) is the Kolmogorov equation adjoint to the underlying stochastic process. In general, the operator $\mathcal{L}^{(N)}$ has a nondegenerate smallest eigenvalue 0 and corresponding eigenspace consisting of the constant functions. Since all particles are of the same kind, we consider only operators $\mathcal{L}^{(N)}$ and probability densities $F^{(N)}$, which are invariant under the symmetric group S_N applied to the N components in $\mathbb{R}^3$ of V. Also, the density $F^{(N)}(V; t)$ has to satisfy the initial condition $\lim_{t \downarrow 0} F^{(N)}(V; t) = F_0^{(N)}(V)$.

3. THE DIFFUSION MASTER EQUATION

Since it is instructive to have some explicitly solvable examples, in this section we consider first the case of an isolated gas in a container, then we turn to the case of an isolated gas in a periodic box.

3.1. Gas in a Container

3.1.1. Finite N

Taking

$$\mathcal{L}^{(N)} = -\Delta_{\mathbb{M}_\varepsilon^{3N-1}}, \tag{16}$$

the master equation is then simply the diffusion equation on $\mathbb{M}_\varepsilon^{3N-1} = \mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}$,

$$\partial_t F^{(N)}(V; t) = \Delta_{\mathbb{M}_\varepsilon^{3N-1}} F^{(N)}(V; t). \tag{17}$$

Now, the spectrum and eigenfunctions of the Laplacian on a D-dimensional sphere are well-known. Since $\Delta_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}} = (1/2N\varepsilon)\Delta_{\mathbb{S}^{3N-1}}$, it will be enough to consider them on the unit sphere; the results can then be adapted to $\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}$ by simple scaling. With $D = 3N - 1$, the spectrum of $-\Delta_{\mathbb{S}^{3N-1}}$ is given by $\{\lambda^{(j)}_{\mathbb{S}^{3N-1}}\}_{j=0}^{\infty}$ with $\lambda^{(j)}_{\mathbb{S}^{3N-1}} = j(j + 3N - 2)$, and the eigenspace for $\lambda^{(j)}_{\mathbb{S}^{3N-1}}$ is spanned by the restrictions to $\mathbb{S}^{3N-1} \subset \mathbb{R}^{3N}$ of the harmonic polynomials that are homogeneous of degree j in $\mathbb{R}^{3N}$; when $j > 0$, this restriction to $\mathbb{S}^{3N-1} \subset \mathbb{R}^{3N}$ has to be nonconstant. Solutions of the diffusion master equation that are invariant under the symmetry group S_N acting on V, however, can be expanded entirely in terms of eigenfunctions having that same symmetry. The simplest such eigenfuntions are the restriction to $\mathbb{S}^{3N-1}$ of the polynomials of the form $P^{(1)}_j(V) = \sum_{k \in \mathbb{I}_N} p_j(v_k)$ where the $p_j(v)$'s are harmonic polynomials that are homogeneous of degree j in $\mathbb{R}^3$ (however, the special case of $p_2(v) = v_1^2 + v_2^2 - 2v_3^2$ simply leads to the constant function on $\mathbb{S}^{3N-1}$ and, hence, does not lead to an element of the eigenspace of $\lambda^{(2)}_{\mathbb{S}^{3N-1}}$). In the next more complicated case, the S_N-invariant eigenfunctions for $j = j_1 + j_2$ are of the form $P^{(2)}_{j_1+j_2}(V) = \sum_{k \in \mathbb{I}_N} \sum_{l \in \mathbb{I}^{(k)}_{N-1}} p_{j_1}(v_k)p_{j_2}(v_l)$, restricted to $\mathbb{S}^{3N-1}$; etc.

Thus, the S_N-symmetric solutions of equation (17) on $\mathbb{M}^{3N-1}_\varepsilon = \mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}$ are given by a generalized Fourier series

$$F^{(N)}(V; t) = \left|\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}\right|^{-1} + \sum_{j \in \mathbb{N}} \sum_{\ell \in \mathbb{D}_j} F^{(N)}_{j,\ell} G^{(N)}_{j,\ell}(V) \exp\left(-\frac{j(j + 3N - 2)}{2N\varepsilon} t\right),$$

(18)

where $\{G^{(N)}_{j,\ell}(V), \ell \in \mathbb{D}_j\}$ are the eigenfunctions of $-\Delta_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}}$ spanning the S_N-symmetric eigen-subspace for $\lambda^{(j)}_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}}$ for $j \in \mathbb{N}$, with $\mathbb{D}_j \subset \mathbb{N}$ the set of indices labeling the degeneracy of the jth eigenvalue, and the $F^{(N)}_{j,\ell}$ are the expansion coefficients. (Although the eigenfunctions are quite explicitly computable, we here refrain from listing them; we shall only work with some simple eigenfunctions for the purpose of illustration.) Evidently, the ensemble probability density function on $\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}$ evolves exponentially fast into the uniformly spread-out probability density $\left|\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}\right|^{-1}$, which is the eigenfunction for $\lambda^{(0)}_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}} = 0$.

3.1.2.   The Limit $N \to \infty$

To discuss the limit $N \to \infty$ for the time-evolution of the ensemble, we consider the time-evolution of the hierarchy of n-velocity marginal distributions $F^{(n|N)}(v_1, \ldots, v_n; t)$ with domains $\{(v_1, \ldots, v_n): \sum_{k=1}^{n} |v_k|^2 \leq 2N\varepsilon\} \subset \mathbb{R}^{3n}$, which obtains by integrating (18) over the available domains

$\mathbb{S}^{3(N-n)-1}_{\sqrt{2N\varepsilon}-\sum_{k=1}^{n}|v_k|^2}$ of the remaining $N-n$ velocities variables; thus

$$
\begin{aligned}
F^{(n|N)}(v_1, \ldots, v_n;\ t) &= F^{(n|N)}_{\text{stat}}(v_1, \ldots, v_n) \\
&\quad + \sum_{j\in\mathbb{N}}\sum_{\ell\in\mathbb{D}_j} F^{(N)}_{j,\ell}\, g^{(n|N)}_{j,\ell}(v_1, \ldots, v_n) e^{\frac{-j(j+3N-2)}{2N\varepsilon}t},
\end{aligned} \qquad (19)
$$

where $F^{(n|N)}_{\text{stat}}(v_1,\ldots,v_n)$ and $g^{(n|N)}_{j,\ell}(v_1,\ldots,v_n)$ are the corresponding integrals of $\left|\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}\right|^{-1}$ and $G^{(N)}_{j,\ell}(V)$ over $\mathbb{S}^{3(N-n)-1}_{\sqrt{2N\varepsilon}-\sum_{k=1}^{n}|v_k|^2}$.

The n-velocity marginal of $\left|\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}\right|^{-1}$ is

$$
\begin{aligned}
F^{(n|N)}_{\text{stat}}(v_1, \ldots, v_n) &= \frac{|\mathbb{S}^{3(N-n)-1}|}{|\mathbb{S}^{3N-1}|}(2N\varepsilon)^{-3n/2} \\
&\quad \times \left(1 - \frac{1}{2N\varepsilon}\sum_{k=1}^{n}|v_k|^2\right)^{[3(N-n)-1]/2}
\end{aligned} \qquad (20)
$$

and converges in the limit $N\to\infty$ to the n-velocity Maxwellian

$$
f_M^{\otimes n}(v_1, \ldots, v_n) = \prod_{k=1}^{n}\left(\frac{3}{4\pi\varepsilon}\right)^{3/2}\exp\left(-\frac{3}{4\varepsilon}|v_k|^2\right) \qquad (21)
$$

with domain $\mathbb{R}^{3n}$. Hence, one obtains the same equilibrium statistical mechanics from $\left|\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}\right|^{-1}$, viewed as a probability density on $\mathbb{M}^{3N-1}_\varepsilon$, as one obtains from the conventional Boltzmann-Gibbs micro-canonical equilibrium ensemble $Z^{-1}\delta(1/2\sum_{k\in\mathbb{I}_N}|v_k|^2 - N\varepsilon)$, viewed as a stationary probability "density" on $\mathbb{R}^{3N}$ for the velocities of the perfect classical gas of N particles at energy N. Note that for finite N, the results differ slightly.

As for the partially integrated S_N-symmetric eigenfunctions of $-\Delta_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}}$, suitably normalized they converge pointwise to a compatible family of S_n-symmetric function on $\mathbb{R}^{3N}$,

$$
\lim_{N\to\infty} g^{(n|N)}_{j,\ell}(v_1, \ldots, v_n) = g^{(n)}_{j,\ell}(v_1, \ldots, v_n). \qquad (22)
$$

A detailed calculation, to be presented elsewhere, shows that the limit functions $g^{(n)}_{j,\ell}(v_1,\ldots,v_n)$ are identically zero unless ℓ belongs to a certain subset $\tilde{\mathbb{D}}_j \subset \mathbb{D}_j$. If $\ell \in \tilde{\mathbb{D}}_j$, each $g^{(n)}_{j,\ell}(v_1,\ldots,v_n)$ turns out to be, for all n, one of the well-known (Risken 1996) eigenfunctions of the Fokker-Planck equation in $\mathbb{R}^{3n}$ (see (26) here below). Each eigenfunction is given by the n-velocity Maxwellian (21) multiplied by a (symmetrized) product of Hermite polynomials, one in each component of the n velocities $v_1,\ldots,v_n$, of total degree j.

Regarding the spectrum of $-\Delta_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}}$, it is readily seen that in the limit $N \to \infty$ we have

$$\lim_{N\to\infty}\left\{\lambda^{(j)}_{\mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}}\right\}^{\infty}_{j=0} = \left\{\frac{3j}{2\varepsilon}\right\}^{\infty}_{j=0}. \tag{23}$$

Thus, the whole spectrum remains discrete and, as is well known (Risken 1996), coincides with the spectrum of the harmonic quantum oscillator (after a scaling and a shift). In particular, there is a spectral gap separating the origin from the rest of the spectrum.

Finally, if one chooses a sequence $\{F^{(N)}_{j,\ell}\}$ that converges to an appropriate limit sequence $\{F_{j,\ell}\}$, and if the initial n-velocity marginal in the limit $N \to \infty$ is given by

$$f^{(n)}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\ 0) = f^{\otimes n}_M(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n) + \sum_{j\in\mathbb{N}}\sum_{\ell\in\tilde{\mathbb{D}}_j} F_{j,\ell}g^{(n)}_{j,\ell}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n), \tag{24}$$

then its subsequent evolution is given by

$$f^{(n)}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\ t) = f^{\otimes n}_M(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)$$

$$+ \sum_{j\in\mathbb{N}}\sum_{\ell\in\tilde{\mathbb{D}}_j} F_{j,\ell}g^{(n)}_{j,\ell}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)e^{-(3j/2\varepsilon)t} \tag{25}$$

and describes an exponentially fast approach to equilibrium in the infinite system.

Coming now to the evolution equations for the $f^{(n)}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;t)$, we here only state the final result, which is a special case of the more general one presented in the next subsection. In the limit $N \to \infty$, the nth marginal evolution equation for the diffusion master equation on $\mathbb{M}^{3N-1}_\varepsilon = \mathbb{S}^{3N-1}_{\sqrt{2N\varepsilon}}$, equation (17), becomes the (essentially linear) Fokker-Planck equation in $\mathbb{R}^{3n}$,

$$\partial_t f^{(n)} = \sum_{i\in\mathbb{I}_n}\frac{\partial}{\partial\boldsymbol{v}_i}\cdot\left(\frac{\partial f^{(n)}}{\partial\boldsymbol{v}_i} + \frac{3}{2\varepsilon}\boldsymbol{v}_i f^{(n)}\right); \tag{26}$$

in particular, for $n = 1$ we recover (4) for $f \equiv f^{(1)}$, with $\boldsymbol{u} = \boldsymbol{0}$ and $\varepsilon_0 = \varepsilon$, and together with (5) and (7). The momentum constraint (6) on the initial data is here immaterial.

It thus appears that we have derived (4) as a kinetic equation. Appearances are, however, misleading. At this point (4) does not yet have the status of a kinetic equation; notice that Kac's concept of propagation of chaos has not entered the picture! In fact, $f^{(n)}$ in (26) may still in general be a convex linear ensemble superposition of extremal states, which are products of n one-particle functions representing the velocity density function of an actual member of the infinite ensemble. In other words, (26) for $n = 1, 2, \ldots$ defines a "Fokker-Planck hierarchy" for a general statistical superposition of initial conditions, analogous to the well-known Boltzman

hierarchy that arises in the kinetic theory of dilute gases (Spohn 1991; Cercignani, Illner, and Pulvirenti 1994). In this simple case, however, the nth linear equation in the hierarchy (26) is decoupled from the equation for the $n + 1$-th marginal. The upshot is that the first equation of the hierarchy is decoupled from $f^{(2)}$ and therefore already a closed equation for $f^{(1)}$. Since it is *essentially linear* (we say "essentially," for the parameter ε is coupled with the initial data), a linear superposition of different solutions (corresponding to a statistical ensemble of initial data for f with same energy) is again a solution. Hence, if initially

$$f^{(n)}(v_1, \ldots, v_n; \ 0) = \langle f_0^{\otimes n}(v_1, \ldots, v_n) \rangle, \tag{27}$$

where $\langle \cdot \rangle$ is the Hewitt-Savage (1955) ensemble demcomposition measure on the space of initial velocity density functions of *individual* physical systems, where each $f_0(v) \geq 0$ satisfies (5) and (7), then at later times

$$f^{(n)}(v_1, \ldots, v_n; \ t) = \langle f^{\otimes n}(v_1, \ldots, v_n; \ t) \rangle, \tag{28}$$

where $f(v; t)$ solves (4) with $u = 0$ and $\varepsilon_0 = \varepsilon$, and together with (5) and (7). Note that the Hewitt-Savage measure is of course invariant under the evolution. This finally establishes the status of (4) (together with its constraints) as a kinetic equation valid for (almost) every individual member of the limiting ensemble.

We remark that a product structure for $f^{(n)}(v_1, \ldots, v_n; 0)$ imposes interesting relations on the expansion coefficients, but we have no space to enter their discussion here.

3.2. Gas in a Periodic Box

A periodic box is physically unrealistic, but it provides a simple example of a situation in which momentum conservation has to be taken into account, too. In large parts the discussion of the previous subsection carries over to this situation.

3.2.1. Finite N

The evolution of the ensemble of finite N systems is now described by the diffusion equation on $\mathbb{M}_{u,\varepsilon}^{3N-4}$,

$$\partial_t F^{(N)}(V; \ t) = \Delta_{\mathbb{M}_{u,\varepsilon}^{3N-4}} F^{(N)}(V; \ t). \tag{29}$$

The spectrum of the Laplacian is now given by $\{\lambda_{\mathbb{S}_{\sqrt{2N\varepsilon_0}}^{3N-4}}^{(j)}\}_{j=0}^{\infty}$ with $\lambda_{\mathbb{S}_{\sqrt{2N\varepsilon_0}}^{3N-4}}^{(j)} = j(j + 3N - 5)$, and the eigenspace for $\lambda_{\mathbb{S}_{\sqrt{2N\varepsilon_0}}^{3N-4}}^{(j)}$ is spanned by the restrictions to $\mathbb{S}_{\sqrt{2N\varepsilon_0}}^{3N-4} \subset \mathbb{L}^{3N-3}$ of the harmonic polynomials that are homogeneous of degree j in $\mathbb{L}^{3N-3}$; when $j > 0$ this restriction to $\mathbb{S}_{\sqrt{2N\varepsilon_0}}^{3N-4} \subset \mathbb{L}^{3N-3}$

has to be nonconstant. The computation of the eigenfunctions is again straightforward, but the embedding of $\mathbb{S}^{3N-4}_{\sqrt{2N\varepsilon_0}} \subset \mathbb{L}^{3N-3}$ causes a minor inconvenience because a rotation of all velocity variables is involved to get back to the physical velocity variables. We shall skip the details here, which will be presented elsewhere, and now turn directly to the derivation of the hierarchy of linear evolution equations for the marginal densities that obtains from (29).

It is convenient to express the Laplacian on the right-hand side in (29) in terms of the projection operator from $\mathbb{R}^{3N}$ to the tangent space to the embedded manifold $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$. The relevant formula is discussed in the appendix, equation (71). In order to apply (71) to (29), we introduce an orthogonal basis for the orthogonal complement in $\mathbb{R}^{3N}$ of the tangent space to $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$ at $V \in \mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$. If $\mathbf{u} = \mathbf{0}$, such a basis is simply provided by the set of vectors $\{V, E_1, E_2, E_3\}$, with $E_\sigma = (e_\sigma, \ldots, e_\sigma)$, where the e_σ, $\sigma = 1, 2, 3$, are the standard unit vectors in $\mathbb{R}^3$. If $\mathbf{u} \neq \mathbf{0}$, the E_σ are mutually orthogonal but not orthogonal to $V \in \mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$; however, they are orthogonal to $W \equiv V - U$. Indeed,

$$V - U = \left(\mathbf{I}_{3N} - \frac{1}{N}\sum_{\sigma=1}^{3} E_\sigma \otimes E_\sigma\right) \cdot V. \tag{30}$$

The vectors $\{W, E_1, E_2, E_3\}$ form the desired orthogonal basis for the orthogonal complement in $\mathbb{R}^{3N}$ of the tangent space to $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$ at $V \in \mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$. The magnitudes of the vectors E_σ and W are $|E_\sigma| = \sqrt{N}$ and $|W| = \sqrt{2N\varepsilon_0}$, respectively, where we recall that $\varepsilon_0 = \varepsilon - 1/2|\mathbf{u}|^2$. Then, (29) becomes

$$\partial_t F^{(N)} = \frac{\partial}{\partial V} \cdot \left[\left(\mathbf{I}_{3N} - \frac{1}{N}\sum_{\sigma=1}^{3} E_\sigma \otimes E_\sigma - \frac{1}{2N\varepsilon_0} W \otimes W\right)\frac{\partial F^{(N)}}{\partial V}\right]. \tag{31}$$

In order to obtain an equation for the nth marginal of $F^{(N)}(V; t)$, which will be denoted by $F^{(n|N)}(v_1, \ldots, v_n; t)$, we integrate (31) over $(v_{n+1}, \ldots, v_N) \in \mathbb{R}^{3N-3n}$. Clearly,

$$\int dv_{n+1} \cdots dv_N \frac{\partial}{\partial V} \cdot \left(\frac{\partial F^{(N)}}{\partial V}\right) = \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial v_i} \cdot \frac{\partial F^{(n|N)}}{\partial v_i}. \tag{32}$$

Also,

$$\int dv_{n+1} \cdots dv_N \frac{\partial}{\partial V} \cdot \left(\sum_{\sigma=1}^{3} E_\sigma \otimes E_\sigma \frac{\partial F^{(N)}}{\partial V}\right) = \sum_{k=1}^{3}\sum_{i,j \in \mathbb{I}_n} \frac{\partial^2 F^{(n|N)}}{\partial v_{ik}\partial v_{jk}}. \tag{33}$$

Finally,

$$
\int d\boldsymbol{v}_{n+1} \cdots d\boldsymbol{v}_N \, \frac{\partial}{\partial \boldsymbol{V}} \cdot \left(W \otimes W \frac{\partial F^{(N)}}{\partial \boldsymbol{V}} \right)
$$

$$
= \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left(\boldsymbol{w}_i \int d\boldsymbol{v}_{n+1} \cdots d\boldsymbol{v}_N \sum_{j \in \mathbb{I}_N} \boldsymbol{w}_j \cdot \frac{\partial F^{(N)}}{\partial \boldsymbol{v}_j} \right)
$$

$$
= \sum_{i,j \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left(\boldsymbol{w}_i \boldsymbol{w}_j \cdot \frac{\partial F^{(n|N)}}{\partial \boldsymbol{v}_j} \right)
$$

$$
+ (N-n) \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left(\boldsymbol{w}_i \int d\boldsymbol{v}_{n+1} \cdots d\boldsymbol{v}_N \boldsymbol{w}_N \cdot \frac{\partial F^{(N)}}{\partial \boldsymbol{v}_N} \right)
$$

$$
= \sum_{i,j \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left(\boldsymbol{w}_i \boldsymbol{w}_j \cdot \frac{\partial F^{(n|N)}}{\partial \boldsymbol{v}_j} \right)
$$

$$
- 3(N-n) \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot (\boldsymbol{w}_i F^{(n|N)}) \tag{34}
$$

where $\boldsymbol{w}_i \equiv \boldsymbol{v}_i - \mathbf{u}$ and the permutation symmetry of $F^{(N)}$ was used. Putting everything together, if $F^{(N)}$ satisfies the diffusion equation on $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$, it follows that the nth marginal $F^{(n|N)}$ satisfies

$$
\partial_t F^{(n|N)} = \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \frac{\partial F^{(n|N)}}{\partial \boldsymbol{v}_i} - \frac{1}{N} \sum_{k=1}^{3} \sum_{i,j \in \mathbb{I}_n} \frac{\partial^2 F^{(n|N)}}{\partial v_{ik} \partial v_{jk}}
$$

$$
- \frac{1}{2N\varepsilon_0} \sum_{i,j \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left((\boldsymbol{v}_i - \mathbf{u})(\boldsymbol{v}_j - \mathbf{u}) \cdot \frac{\partial F^{(n|N)}}{\partial \boldsymbol{v}_j} \right)
$$

$$
+ \frac{3(N-n)}{2\varepsilon_0 N} \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot ((\boldsymbol{v}_i - \mathbf{u}) F^{(n|N)}). \tag{35}
$$

3.2.2.   The Limit $N \to \infty$

The spectrum of $-\Delta_{\mathbb{S}^{3N-4}_{\sqrt{2N\varepsilon_0}}}$ in the limit $N \to \infty$ is given by

$$
\lim_{N \to \infty} \{\lambda^{(j)}_{\mathbb{S}^{3N-4}_{\sqrt{2N\varepsilon_0}}}\}_{j=0}^{\infty} = \left\{ \frac{3j}{2\varepsilon_0} \right\}_{j=0}^{\infty}, \tag{36}
$$

which up to the replacement of ε by ε_0 agrees with (23). Thus, the whole spectrum is once again discrete, and, in particular, there is a spectral gap separating the origin from the rest of the spectrum.

As a result, once again the velocity densities $f^{(n)}$ approach equilibrium exponentially fast. The evolution equation for $f^{(n)}$ that obtains in the limit $N \to \infty$ from (35) is the (essentially linear) Fokker-Planck equation in $\mathbb{R}^{3n}$,

$$\partial_t f^{(n)} = \sum_{i \in \mathbb{I}_n} \frac{\partial}{\partial v_i} \cdot \left(\frac{\partial f^{(n)}}{\partial v_i} + \frac{3}{2\varepsilon_0} (v_i - \mathbf{u}) f^{(n)} \right). \tag{37}$$

In particular, for $n = 1$ we recover (4) with $f \equiv f^{(1)}$, together with the constraints on the initial data (5), (6), and (7). The last step to establish the status of (4) (together with its constraints) as a kinetic equation involves once again the Hewitt-Savage decomposition, which implements Kac's concept of propagation of chaos for (37).

4. THE BALESCU-PRIGOGINE MASTER EQUATION

After having established that the linear Fokker-Planck Equation (4) together with the constraints on the initial data can be derived as a kinetic equation from (29), the diffusion equation on the energy-momentum foliation of $\mathbb{R}^{3N}$, we now study the more complicated diffusion process on the foliation $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ associated with the (nonlinear) Landau equation, equation (1). At least at the formal level, the Kolmogorov equation for the diffusion process in question is given by the *BP master equation* (Prigogine and Balescu 1959) for the time evolution of $F^{(N)}$, which can be written once again as

$$\partial_t F^{(N)} = -\mathcal{L}^{(N)} F^{(N)}, \tag{38}$$

where now

$$\mathcal{L}^{(N)} = \frac{1}{N-1} \sum_{k \in \mathbb{I}_N} \sum_{l \in \mathbb{I}_{N-1}^{(k)}} \mathcal{L}_{v_k, v_l} \tag{39}$$

with

$$\mathcal{L}_{v,w} = -\frac{1}{2} (\partial_v - \partial_w) \cdot (|v - w|^{-1} P_{v-w}^{\perp} \cdot (\partial_v - \partial_w)) \tag{40}$$

and $\mathbb{I}_{N-1}^{(k)} = \mathbb{I}_N \backslash \{k\}$; the density $F^{(N)}$ has to satisfy the initial condition $\lim_{t \downarrow 0} F^{(N)}(V; t) = F_0^{(N)}(V)$. A Fourier-transformed version of (38) was constructed by Prigogine and Balescu (1957, 1959), but it does not seem to have received much attention ever since. Balescu and Prigogine already pointed out that Landau's equation (1) can be extracted from (38) by contraction onto the first marginal of $F^{(N)}(V; t)$. The formal argument runs as follows: (38) is equivalent to a hierarchy of evolution equations for all the marginals of $F^{(N)}(V; t)$, which are denoted by $F^{(n|N)}(v_1, \ldots, v_n; t)$ with $n = 1, \ldots, N$.

Of course, $F^{(N|N)} \equiv F^{(N)}$. The time evolution of $F^{(n|N)}(v_1, \ldots, v_n; t)$ is given by the corresponding contration of the BP master equation onto n variables,

$$\partial_t F^{(n|N)} = -\frac{n-1}{N-1} \mathcal{L}^{(n)} F^{(n|N)} - \frac{N-n}{N-1} \sum_{k=1}^{n} \int \mathcal{L}_{v_k, v_{n+1}} F^{(n+1|N)} \mathrm{d}^3 v_{n+1}. \quad (41)$$

Introducing the shorthands v for v_1 and w for v_2, the evolution equation for the first marginal takes the form

$$\partial_t F^{(1|N)}(v; \ t) = \partial_v \cdot \int_{\mathbb{R}^3} |v - w|^{-1} P^{\perp}_{v-w} \cdot (\partial_v - \partial_w) F^{(2|N)}(v, w; \ t) \mathrm{d}^3 w. \quad (42)$$

Clearly, if we had $F^{(2|N)}(v, w; t) = F^{(1|N)}(v; t)F^{(1|N)}(w; t)$, then (42) would be a closed equation for $F^{(1|N)}$. However, since (38) with (39) and (40) does not preserve a product structure of $F^{(N)}$ for any finite N, $F^{(2|N)}$ cannot remain a product of the first marginals even if that is the case initially. It was Kac's insight that if one lets $N \to \infty$ and assumes that $\lim_{N \uparrow \infty} F_0^{(N)}(V) = \otimes_{k=1}^{\infty} f_0(v_k)$, then one ought to be able to show that the product structure persists in time, viz., $\lim_{N \uparrow \infty} F^{(N)}(V; t) = \otimes_{k=1}^{\infty} f(v_k; t)$ for all $t > 0$, for which Kac coined the phrase "the propagation of molecular chaos." Kac proved propagation of chaos for his caricature of the Maxwellian gas, and Balescu and Prigogine surmised that propagation of chaos will hold also for their master equation when $N \to \infty$, in which limit equation (42) then becomes the Landau equation (1). We remark that, for initial conditions that do not necessarily factorize, in the $N \to \infty$ limit equation (41) leads (formally) to the *Landau hierarchy*

$$\partial_t f^{(n)}(v_1, \ldots, v_n; \ t) = -\sum_{k=1}^{n} \int_{\mathbb{R}^3} \mathcal{L}_{v_k, v_{n+1}} f^{(n+1)}(v_1, \ldots, v_{n+1}; \ t) \mathrm{d}^3 v_{n+1}. \quad (43)$$

A rigorous justification of propagation of chaos for the BP equation is an interesting open problem, on which we recently made some progress. Here we report on our results:

1. The BP diffusion operator has an interesting geometric interpretation: it is a weighted average of the Laplacians associated with a certain family of submanifolds of $\mathbb{M}^{3N-4}_{\mathbf{u}, \varepsilon}$. These submanifolds are determined by the conservation laws for binary collisions between particles, as will be made clear below.
2. For each finite N, the BP master equation is well posed in $\mathfrak{L}^2 \cap \mathfrak{L}^1$ ($\mathbb{M}^{3N-4}_{\mathbf{u}, \varepsilon}$) and displays exponential decay to equilibrium.
3. As $N \to \infty$, the first nonzero eigenvalue converges to zero.

Our results show on the one hand that the finite-N BP equation is more similar to the diffusion equation than meets the eye, yet on the other hand,

the limit $N \to \infty$ is markedly different. In particular, our results suggest that the spectral gap vanishes.

4.1. Geometric Aspects of the BP Master Equation

Clearly (38) is a linear parabolic PDE in $3N$ variables, and it is easily verified that $\mathcal{L}^{(N)}$, like $\Delta_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}}$, annihilates the constant function $m(V)$, the linear 3-vector polynomial $p(V)$, and the quadratic polynominal $e(V)$, ensuring the conservation of mass, energy, and momentum. Hence, also in this case the evolution of the ensemble factors into independent evolutions on the invariant manifolds $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$.

Now, for $1 \leq l < k \leq N$ consider the family of $N(N-1)/2$ two-dimensional manifolds

$$\mathbb{B}_{kl}^2 = \{V: \ v_k + v_l = \alpha_{kl} \quad |v_k - v_l|^2 = \beta_{kl}^2 \quad v_i = \gamma_{kl}^{(i)}, \ i \neq k, l\} \quad (44)$$

where α_{kl}, β_{kl}, and $\gamma_{kl}^{(i)}$ are $3N - 2$ arbitrary constants. Clearly, each point in $\mathbb{R}^{3N}$ determines one such family of manifolds, and each manifold corresponds to the conservation laws associated with the "collision" between particles k and l. In fact, the conservation of $|v_k - v_l|^2$ is equivalent to the conservation of the two-particle energy $|v_k|^2 + |v_l|^2$, as long as the two-particle momentum $v_k + v_l$ is also constant; thus, if $V \in \mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$, it follows that the manifolds $\mathbb{B}_{kl}^2$ determined by V are submanifolds of $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ for all k, l.

Theorem 4.1 Let $\mathcal{L}_{v,w}$ be operator in (40) and $F^{(N)}$ a probability density on $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$. Then

$$\mathcal{L}_{v_k, v_l} F^{(N)} = -|v_k - v_l|^{-1} \Delta_{\mathbb{B}_{kl}^2} F^{(N)} \quad (45)$$

where $\Delta_{\mathbb{B}_{kl}^2}$ is the Laplace-Beltrami operator on $\mathbb{B}_{kl}^2$.

Proof

We first observe that the factor $|v_k - v_l|^{-1}$ can be moved out of the differential operator in (40), since $(\partial_{v_l} - \partial_{v_k})|v_k - v_l|^{-1}$ yields terms parallel to $v_k - v_l$, which are annihilated by the projector $P_{v_k-v_l}^\perp$. Hence,

$$\mathcal{L}_{v_k, v_l} = -\frac{1}{2}|v_k - v_l|^{-1}(\partial_{v_l} - \partial_{v_k}) \cdot (P_{v_k-v_l}^\perp \cdot (\partial_{v_k} - \partial_{v_l})). \quad (46)$$

Next, we calculate $\Delta_{\mathbb{B}_{kl}^2}$. At any point $V \in \mathbb{R}^{3N}$ the manifold $\mathbb{B}_{kl}^2$ has the $3N - 2$

normal unit vectors $\boldsymbol{M}_\sigma$, $\boldsymbol{E}$, and $\boldsymbol{W}_{i,\sigma}$ where

$$
\boldsymbol{M}_\sigma \equiv \frac{1}{\sqrt{2}}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
e_\sigma \\
0 \\
\vdots \\
0 \\
e_\sigma \\
0 \\
\vdots \\
0
\end{pmatrix}
\qquad
\boldsymbol{E} \equiv \frac{1}{\sqrt{2}|v_k - v_l|}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
v_k - v_l \qquad (k \text{ entry}) \\
0 \\
\vdots \\
0 \\
-v_k + v_l \qquad (l \text{ entry}) \\
0 \\
\vdots \\
0
\end{pmatrix}
\tag{47}
$$

and the e_σ, $\sigma = 1, 2, 3$, are again the standard unit vectors in $\mathbb{R}^3$; $\boldsymbol{W}_{i,\sigma}$ is the standard unit vector in $\mathbb{R}^{3N}$ with e_σ as the ith 3-block, $i \neq k$, l, and zeroes everywhere else. Since all these vectors are mutually orthogonal, the projector on the tangent space to $\mathbb{B}_{kl}^2$ is

$$
\mathrm{P}_{\mathbb{B}_{kl}^2} = \mathbf{I}_{3N} - \boldsymbol{E} \otimes \boldsymbol{E} - \sum_\sigma \boldsymbol{M}_\sigma \otimes \boldsymbol{M}_\sigma - \sum_{i,\sigma} \boldsymbol{W}_{i,\sigma} \otimes \boldsymbol{W}_{i,\sigma}
$$

$$
= \frac{1}{2}
\begin{pmatrix}
& \vdots & & \vdots & \\
\cdots & P_{v_k-v_l}^\perp & \cdots & -P_{v_k-v_l}^\perp & \cdots \\
& \vdots & & \vdots & \\
\cdots & -P_{v_k-v_l}^\perp & \cdots & P_{v_k-v_l}^\perp & \cdots \\
& \vdots & & \vdots & \\
& k & & l &
\end{pmatrix}
\begin{matrix} \\ k \\ \\ l \\ \\ \end{matrix}
\tag{48}
$$

where all the unmarked enteries are zero and, of course,

$$
P_{v_k-v_l}^\perp = \mathbf{I}_3 - \frac{v_k - v_l}{|v_k - v_l|} \otimes \frac{v_k - v_l}{|v_k - v_l|}.
\tag{49}
$$

It follows immediately from (71) that

$$
\Delta_{\mathbb{B}_{kl}^2} = \frac{1}{2}(\partial_{v_l} - \partial_{v_k}) \cdot (P_{v_k-v_l}^\perp \cdot (\partial_{v_k} - \partial_{v_l})),
\tag{50}
$$

which, together with (46), gives (45).

Hence, the BP master equation can be written as

$$
\partial_t F^{(N)} = -\frac{1}{N-1} \sum_{k \in \mathbb{I}_N} \sum_{l \in \mathbb{I}_{N-1}^{(k)}} |v_k - v_l|^{-1} \Delta_{\mathbb{B}_{kl}^2} F^{(N)}.
\tag{51}
$$

It is instructive to compare (51), which leads formally to the Landau equation, to the diffusion equation on $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$, (29), which leads to the linear Fokker-Planck equation. In the Landau case the diffusion does not take place isotropically over the manifold of constant energy and momentum, but on the collection of the submanifolds $\mathbb{B}_{kl}^2$ determined by the two-particle conservation laws. To be precise, the BP operator in (51) is a sort of weighted average of the Laplacians on all the $\mathbb{B}_{kl}^2$, where the kl-Laplacian has weight ("diffusivity") $|v_k - v_l|^{-1}$. These diffusivities are constant quantities on the corresponding manifolds, but they can take arbitrarily small values in certain regions of $\mathbb{R}^{3N}$ as N increases. Loosely speaking, the ellipticity of the BP equation degenerates as $N \to \infty$. This is bound to affect the rate of decay to equilibrium of the solutions in the same limit.

4.2. Well-Posedness of the BP Master Equation and Decay to Equilibrium

We now identify the evolution of $F^{(N)}(V; t)$ with the motion of a point ψ_t in the Hilbert space $\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$. We define the Sobolev-type space $\mathfrak{H}$ as the closure of $\mathfrak{C}^\infty(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$ w.r.t. the norm $\| \cdot \|_{\mathfrak{H}}$, given by

$$\|\psi\|_{\mathcal{H}}^2 \overset{\text{def}}{=} Q^{(N)}(\psi, \psi) + \|\psi\|_{\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})}^2, \tag{52}$$

where $Q^{(N)}$ is the manifestly symmetric positive semidefinite quadratic form associated with the operator $\mathcal{L}^{(N)}$ defined on $\mathfrak{C}^\infty(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$:

$$Q^{(N)}(\psi, \phi) = \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} \psi \mathcal{L}^{(N)} \phi \mathrm{d}\tau \tag{53}$$

for (ψ, ϕ) in $\mathfrak{C}^\infty(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})^{\times 2}$. The form closure of $Q^{(N)}(\psi, \phi)$ in $\mathfrak{H} \times \mathfrak{H}$, which is also denoted by $Q^{(N)}(\psi, \phi)$, defines a unique self-adjoint, positive semidefinite operator with dense domain $\tilde{\mathfrak{H}} \subset \mathfrak{H}$, the Friedrichs extension of $\mathcal{L}^{(N)}$, also denoted by $\mathcal{L}^{(N)}$. We recall that the Sobolev-type space $\mathfrak{H}$ coincides with the domain of the square root of the Friedrichs extension of $\mathcal{L}^{(N)}$, i.e., the operator $[\mathcal{L}^{(N)}]^{1/2}$; see Reed and Simon (1980) for general background material. Thus $\mathcal{L}^{(N)}$ is a densely defined, unbounded operator on the Hilbert space $\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$.

It is easily seen that the kernel space of $\mathrm{Ker}(\mathcal{L}^{(N)}) \equiv \mathcal{N}_0$ is one-dimensional, consisting of the constant functions. Hence, $\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$ decomposes as $\mathcal{N}_0 \oplus \mathfrak{L}^{2,+}$, where $\mathfrak{L}^{2,+}$ is the orthogonal complement of $\mathcal{N}_0$ in $\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$, i.e., the subspace of $\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})$ on which $\mathcal{L}^{(N)}$ is strictly positive. Also, $\mathfrak{H}$ decomposes as $\mathcal{N}_0 \oplus \mathfrak{n}^+$; we remark that $\mathfrak{n}^+ \subset \mathcal{N}^{2,+}$ can be equivalently defined as the closure of $\{\psi \in \mathfrak{C}^\infty(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}): \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} \psi \mathrm{d}\tau = 0\}$

w.r.t. the norm $\| \cdot \|_{\mathfrak{H}^+}$, where

$$\|\psi\|^2_{\mathfrak{H}^+} \overset{\text{def}}{=} Q^{(N)}(\psi, \psi). \tag{54}$$

Since $\mathcal{L}^{(N)}$ is self-adjoint and strictly positive on $\mathfrak{H}^+ \cap \bar{\mathfrak{H}}$, it follows immediately that the operator $\mathcal{L}^{(N)}$ is the generator of the contraction semi-group $e^{-t\mathcal{L}^{(N)}}$ on $\mathfrak{L}^{2,+}$ (Reed and Simon 1975). This implies that $\mathcal{L}^{(N)}$ is also the generator of a strongly continuous semi-group on all of $\mathfrak{L}^2(\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon})$, also denoted by $e^{-t\mathcal{L}^{(N)}}$. Precisely, $e^{-t\mathcal{L}^{(N)}}$ acts isometrically on the invariant subspace $\mathcal{N}_0$ and strictly contracting on its orthogonal complement $\mathfrak{L}^{2,+}$; hence $e^{-t\mathcal{L}^{(N)}}$ is also positivity preserving. Thus, if $\psi_0^{(N)} \in \mathfrak{L}^2(\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon})$, then

$$\psi_t^{(N)} \overset{\text{def}}{=} e^{-t\mathcal{L}^{(N)}} \psi_0^{(N)} \tag{55}$$

solves (38) uniquely for Cauchy data $\lim_{t \downarrow 0} \psi^{(N)} = \psi_0^{(N)}$, and the initial value problem for the BP master equation is well posed. The evolution of an initial density $\psi_0^{(N)} \in \mathfrak{L}^1_{+,1} \cap \mathfrak{L}^2(\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon})$ (i.e., $\psi_0^{(N)} > 0$ and $\int_{\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}} \psi_0^{(N)} d\tau = 1$), actually takes place in the intersection of the positive cone of $\mathfrak{L}^2(\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon})$ with the affine Hilbert space

$$\mathfrak{U} = \psi_\infty^{(N)} + \mathfrak{L}^{2,+}, \tag{56}$$

where $\psi_\infty^{(N)} \overset{\text{def}}{=} \|\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}\|^{-1}$ is the Hilbert space vector of the (analog of the) micro-canonical equilibrium ensemble.

The spectrum of $\mathcal{L}^{(N)}$ can be studied with the standard techniques developed for weak solutions of linear inhomogeneous PDE in divergence form (Gilbarg and Trudinger 1998) extended to operators on compact manifolds without boundary (Hebey 1996) (here $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$). Some care must be taken due to the fact that the ellipticity is *not uniform*, since the coefficients are unbounded above; on the other hand, the ellipticity condition is satisfied uniformly from below. First of all, since the bilinear form $Q^{(N)}$ is clearly positive and bounded on $\mathfrak{H}^+$, by the Lax-Milgram theorem it follows that for $\lambda < 0$ the operator $\mathcal{L}^{(N)}_\lambda \equiv \mathcal{L}^{(N)} + \lambda \mathcal{I}$ determines a bijective mapping from $\mathfrak{H}^+$ to $(\mathfrak{H}^+)^*$ (the dual of $\mathfrak{H}^+$). Next, we introduce a compact embedding $\mathcal{E}$ from $\mathfrak{H}^+$ to $(\mathfrak{H}^+)^*$ such that $(\mathcal{E}u)(v) = \int_{\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}} uv \, d\tau$ for all $v \in \mathfrak{H}^+$. In order to define the operator $\mathcal{E}$, we first observe that points on $\mathbb{M}^{3N-4}_{\mathbf{u},\varepsilon}$ satisfy

$$|v_k - v_l|^2 = |v_k|^2 + |v_l|^2 - 2v_k \cdot v_l \leq 2(|v_k|^2 + |v_l|^2) \leq 4N\varepsilon \tag{57}$$

so that

$$Q^{(N)}(\psi, \psi) \geq \frac{1}{2\sqrt{2N\varepsilon}(N-1)} \sum_{k \in \mathbb{I}_N} \sum_{l \in \mathbb{I}^{(k)}_{N-1}} \hat{Q}^{(N)}_{k,l}(\psi, \psi) \tag{58}$$

where

$$\hat{Q}_{k,l}^{(N)}(\psi, \phi) = \frac{1}{2} \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} (\partial_{v_k} - \partial_{v_l})\psi \cdot \mathbf{P}_{v_k - v_l}^{\perp} \cdot (\partial_{v_k} - \partial_{v_l})\phi \mathrm{d}\tau. \quad (59)$$

Since $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ is a compact manifold, it is easy to see that there is some constant C_N such that

$$\sum_{k \in \mathbb{I}_N} \sum_{l \in \mathbb{I}_{N-1}^{(k)}} \hat{Q}_{k,l}^{(N)}(\psi, \psi) \geq C_N \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} |\nabla\psi|^2 \mathrm{d}\tau \quad (60)$$

where $\nabla\psi$ is the covariant derivative of ψ on $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ e.g. $|\nabla\psi|^2 = g^{ij}\partial_i\psi\partial_j\psi$, g^{ij} being the metric tensor and $\partial_j\psi$ the derivative with respect to the jth coordinate. Combined with (58), this gives

$$\|\psi\|_{\mathfrak{H}^+}^2 \geq \frac{C_N}{2\sqrt{2N\varepsilon}(N-1)} \|\psi\|_{\dot{\mathfrak{W}}_+^{1,2}}^2 \quad (61)$$

where $\dot{\mathfrak{W}}_+^{1,2}$ is the closure of $\{\psi \in \mathfrak{C}^\infty(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}): \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}}\psi\mathrm{d}\tau = 0\}$ w.r.t the norm

$$\|\psi\|_{\dot{\mathfrak{W}}_+^{1,2}}^2 \overset{\text{def}}{=} \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} |\nabla\psi|^2 \mathrm{d}\tau. \quad (62)$$

Equation (61) implies that $\mathfrak{H}^+$ is continuously embedded in $\dot{\mathfrak{W}}_+^{1,2}$ which in turn is compactly embedded in $\mathfrak{L}^{2,+}$ by the Sobolev embedding theorem and the Rellich-Kondrasov theorem, both of which hold on a compact manifold (Hebey 1996). Finally, $\mathfrak{L}^{2,+}$ is continuously embedded in $(\mathfrak{H}^+)^*$ (via the Riesz Representation Theorem), and the compact embedding $\mathcal{E}$ if $\mathfrak{H}^+$ into $(\mathfrak{H}^+)^*$ is obtained as the composition $\mathfrak{H}^+ \to \dot{\mathfrak{W}}_+^{1,2} \to \mathfrak{L}^{2,+} \to (\mathfrak{H}^+)^*$.

Finally, one obtains the standard Fredholm alternative (Gilbarg and Trudinger 1998) by rewriting the equation $\mathcal{L}_\lambda^{(N)}\psi = \sigma$ as

$$\psi + (\lambda - \lambda_0)\mathscr{G}_{\lambda_0}^{(N)}\mathcal{E}\psi = \mathscr{G}_{\lambda_0}^{(N)}\sigma \quad (63)$$

where $\mathscr{G}_{\lambda_0}^{(N)}(\lambda_0 < 0)$ is the (continuous) inverse of $\mathcal{L}_{\lambda_0}^{(N)}$. Then, since $\mathcal{E}$ is compact, $-(\lambda - \lambda_0)\mathscr{G}_{\lambda_0}^{(N)}\mathcal{E}$ is also compact from $\mathfrak{H}^+$ to $\mathfrak{H}^+$, and the standard Riesz-Schauder theory of compact operators in a Hilbert space leads to the conclusion that $\mathcal{L}^{(N)}$ has a purely discrete spectrum and that each eigenvalue has a finite-dimensional eigenspace. Since the spectrum of $\mathcal{L}^{(N)}$ is discrete, we have a spectral gap between $\mathcal{L}_0^{(N)} = 0$ and the smallest nonzero eigenvalue $\lambda_1^{(N)} > 0$ of $\mathcal{L}^{(N)}$, and we conclude that the equilbrium ensemble is approached exponentially in time

$$\|\psi_t^{(N)} - \psi_\infty^N\|_{\mathfrak{L}^2} = \|e^{-t\mathcal{L}^{(N)}}(\psi_0^{(N)} - \psi_\infty^N)\|_{\mathfrak{L}^2} \leq e^{-t\lambda_1^{(N)}}\|\psi_0^{(N)} - \psi_\infty^N\|_{\mathfrak{L}^2}. \quad (64)$$

4.3.   The Limit $N \to \infty$

The question whether the spectral gap remains finite in the limit $N \to \infty$, which was recently answered affirmatively in the context of the Kac model and other models related to the Boltzmann equation (Carlen, Carvalho and Loss 2003), and also by us in the previous section for the diffusion master equation, has presumably a negative answer for the BP master equation, at least for the Coulomb case studied here. Indeed, we now show that the smallest positive eigenvalue with permutation symmetric eigenfunction vanishes in the limit $\lim_{N \to \infty} \lambda_1^{(N)} = 0$.

Consider

$$\lambda_1^{(N)} \overset{\text{def}}{=} \inf_{\psi \in \Sigma_N^+} (\psi, \mathcal{L}^{(N)}\psi)_{\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})} \tag{65}$$

where (using permutation symmetry)

$$(\psi, \mathcal{L}^{(N)}\psi)_{\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})} = \frac{N}{2} \int_{\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}} (\partial_{v_2} - \partial_{v_1})\psi \cdot \frac{P_{v_2-v_1}^{\perp}}{|v_2 - v_1|} \cdot (\partial_{v_2} - \partial_{v_1})\psi d\tau. \tag{66}$$

Here, $\Sigma_N^+ = \{\psi \in \mathfrak{H}_S^+ : \|\psi\|_{\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})} = 1\}$ and $\mathfrak{H}_S^+$ is the permutation symmetric subspace of $\mathfrak{H}^+$. An upper bound on $\lambda_1^{(N)}$ is obtained by selecting a suitable trial function $\widehat{\psi} \in \Sigma_N^+$. We use

$$\widehat{\psi} = A\left(\sum_{i=1}^{N} g(v_i) - C\right) \tag{67}$$

with $g(v) = v_1^2/2$. In order to satisfy the condition $\widehat{\psi} \in \Sigma_N^+$ one has to choose

$$C = \frac{N}{3} \qquad A = \frac{3}{2N}\sqrt{\frac{3N-1}{|\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}|}}. \tag{68}$$

Then, the quantity $\left(\widehat{\psi}, \mathcal{L}^{(N)}\widehat{\psi}\right)_{\mathfrak{L}^2(\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4})}$ can be calculated exactly. In the "standard" case[1] $\mathbf{u} = 0$, $\varepsilon = 1$, a tedious exercise leads to the estimate

$$\lambda_1^{(N)} < (N > 1)\frac{9}{5\sqrt{\pi}}\frac{1}{\sqrt{3N-4}}. \tag{69}$$

Thus, $\mathcal{L}_1^{(N)} \to 0$ as $N \to \infty$.

Estimate (69) is not enough to conclude rigorously that the BP equation has a vanishing spectral gap as $N \to \infty$ (because we have not determined the asymptotics for *all* the eigenvalues). However, in view of our previous remarks on the fact that the ellipticity of the equation degenerates as $N \to \infty$, it seems reasonable to conjecture that this may well be the case. In

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[1]In general, the evolution of $F_t^{(N)}$ on $\mathbb{M}_{\mathbf{u},\varepsilon}^{3N-4}$ can always be obtained from the evolution on $\mathbb{M}_{0,1}^{3N-4}$ via the simple transformation $V \to U + \varepsilon_0 V$.

turn, a vanishing spectral gap for the BP equation would lend support to the conjecture (Villani 1998) that the Landau equation itself possesses solutions that decay to equilibrium slower than exponentially.

We end on the remark that this result can be easily generalized to any BP master equation corresponding to a Landau kinetic equation with "soft" potentials (for $\gamma < 7/3$ in (3)). Conversely, it is not difficult to prove that for BP master equations with "hard" potentials ($\gamma \geq 7/3$), the spectral gap remains finite in the limit $N \to \infty$.

5. APPENDIX

We here recall some basic facts about the Laplacian on a Riemannian manifold. Let $\mathbb{M}$ be a n-dimensional Riemannian manifold with metric g_{ij}, and let f be a functions on $\mathbb{M}$, $f \in \mathcal{C}^\infty(\mathbb{M})$. The familiar expression for the Laplace-Beltrami operator acting on f in local coordinates is

$$\Delta f = g^{-1/2} \partial_{x_i}(g^{1/2} g^{ij} \partial_{x_j} f) \tag{70}$$

where $g \equiv \det(g_{ij})$. Now, let us take $\mathbb{M}$ to be embedded in the Euclidean space $\mathbb{R}^m$, $m > n$, and let g_{ij} be the metric induced by the standard metric in $\mathbb{R}^m$. Let the function f be defined and differentiable over some suitable subdomain of $\mathbb{R}^m$ containing $\mathbb{M}$. We associate to any point $x \in \mathbb{M}$ a set of orthonormal basis vectors $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ such that $e_1, \ldots, e_n$ span the tangent plane $T_x\mathbb{M}$, whereas $e_{n+1}, \ldots, e_m$ span the orthogonal complement $T_x\mathbb{M}^\perp$ in $\mathbb{R}^m$. The expression for Δf is coordinate-independent, and we are free to choose local coordinates $(x_1, \ldots, x_n)$ such that the coordinate curves are tangent to the $e_1, \ldots, e_n$. In such coordinates $g_{ij} = \delta_{ij}$, $g = 1$ and $\partial_{x_j} f = (\nabla f)_j$, i.e., the component along e_j of the gradient ∇f of f in $\mathbb{R}^m$. Then (70) becomes

$$\Delta f = (P_\mathbb{M} \nabla) \cdot [P_\mathbb{M} \nabla f] = \nabla \cdot [P_\mathbb{M} \nabla f] \tag{71}$$

where $P_\mathbb{M}$ is the orthogonal projection from $\mathbb{R}^m$ to $T_x\mathbb{M}$. Note that (71) is coordinate-independent in $\mathbb{R}^m$ and preserves permutation summetry with respect to the Cartesian coordinates of $\mathbb{R}^m$. This is important when dealing with probability densities for N-particle systems, which are constrained by the dynamical conservation laws to evolve on certain lower-dimensional manifolds and at the same time must be permutation-symmetric.

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