Quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial

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Tutte polynomial of graphs

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where

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$T_G(1, 1) =$ number of spanning trees of $G$. 
Background and Motivation

Summary of results

Bollobás and Riordan extended the Tutte polynomial to an invariant of oriented ribbon graphs, now called the Bollobás–Riordan–Tutte (BRT) polynomial.
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Summary of results

Graphs $\leftrightarrow$ Ribbon graphs

Activity w.r.t. spanning trees $\leftrightarrow$ Activity w.r.t. quasi-trees

Spanning tree expansion $\leftrightarrow$ Quasi-tree expansion
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\begin{align*}
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\end{align*}
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- **(C-Kofman-Stoltzfus, Manturov)** Maximal genus of quasi-trees of $\mathbb{G}_D$ plus two gives an upper bound on the thickness of Khovanov homology.
Definition

An (oriented) **ribbon graph** $G$ is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface, such that its complement is a union of 2-cells.

The **genus** of $G$, $g(G)$, is the genus of the surface on which it is embedded.
Ribbon graphs

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$G$ can also be described by a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, 2, \ldots, 2n\}$ such that $\sigma_1$ is a fixed point free involution and $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \text{Identity}$. This triple gives a cell complex structure for the surface of $G$ and the orbits of $\sigma_i$ correspond to the $i$–cells for $i = 0, 1, 2$. 

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Plane graphs are genus zero ribbon graphs.
Example

\[ \sigma_0 = (1234)(56) \]

\[ \sigma_1 = (14)(25)(36) \]

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Ribbon graphs

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Example

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Quasi-trees: Motivation

A spanning tree of a graph is a spanning subgraph without any cycles.

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Example
Quasi-trees and chord diagrams

The spanning tree expansion of the Tutte polynomial is defined using activity of edges with respect to a spanning tree.

We extend Tutte’s definition of activities to edges of a ribbon graph with respect to a quasi-tree.
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We extend Tutte’s definition of activities to edges of a ribbon graph with respect to a quasi-tree.

**Proposition**

Every quasi-tree $\mathbb{Q}$ corresponds to the ordered chord diagram $C_\mathbb{Q}$ with consecutive markings in the positive direction given by the permutation:

$$\sigma(i) = \begin{cases} 
\sigma_0(i) & i \notin \mathbb{Q} \\
\sigma_2^{-1}(i) & i \in \mathbb{Q}
\end{cases}$$
Example
Fix a total order on the edges of a connected ribbon graph $G$. An edge $e$ in a quasi-tree $Q$ is **live** if the corresponding chord in $C_Q$ does not intersect any lower-ordered chords and otherwise it is **dead**.

An edge $e$ is **internal** or **external**, according to $e \in Q$ or $e \in G - Q$, respectively.
Extending Tutte’s activities to ribbon graphs

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For a genus zero ribbon graph, activities using our definition agree with those using Tutte’s definition.

Any spanning tree of a ribbon graph is also a quasi-tree of genus zero. In this case, the activities using our definition are different from those using Tutte’s definition.
Let $\mathcal{H}$ be a spanning ribbon subgraph of $G$ (i.e. contains all vertices and some edges of $G$).

- $v(\mathcal{H}) = \text{number of vertices of } \mathcal{H}$;
- $e(\mathcal{H}) = \text{number of edges of } \mathcal{H}$;
- $f(\mathcal{H}) = \text{number of faces of } \mathcal{H}$;
- $k(\mathcal{H}) = \text{number of components of } \mathcal{H}$;
- $n(\mathcal{H}) = \text{nullity of } \mathcal{H} = e(\mathcal{H}) - v(\mathcal{H}) + k(\mathcal{H})$;
- $g(\mathcal{H}) = \text{genus of } \mathcal{H} = k(\mathcal{H}) + n(\mathcal{H}) - f(\mathcal{H})$.
Bollobás–Riordan–Tutte polynomial for ribbon graphs

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Bollobás and Riordan extended the Tutte polynomial to an invariant $C(G; X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ of ribbon graphs using the following state sum over all spanning subgraphs:

$$C(G; X, Y, Z) = \sum_{\mathcal{H} \subset G} (X - 1)^{k(\mathcal{H}) - k(G)} Y^{n(\mathcal{H})} Z^{g(\mathcal{H})}$$
Properties

If $G$ is the underlying graph of $\mathbb{G}$ then

$$C(\mathbb{G}; X, Y, 1) = T_G(X, 1 + Y).$$
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If \( G \) is the underlying graph of \( \mathcal{G} \) then

\[
C(\mathcal{G}; X, Y, 1) = T_G(X, 1 + Y).
\]

Proposition (C-Kofman-Stoltzfus)

Let \( q(\mathcal{G}; t, Y) = C(\mathcal{G}; 1, Y, tY^{-2}) \). Then \( q(\mathcal{G}; t, Y) \) is a polynomial in \( t \) and \( Y \) such that

\[
q(\mathcal{G}; t, 0) = \sum_j a_j t^j
\]

where \( a_j \) is the number of quasi-trees of genus \( j \). Consequently, \( q(\mathcal{G}; 1, 0) \) equals the number of quasi-trees of \( G \).
Main Theorem

Order the edges of $G$. Let $Q$ be a quasi-tree of $G$.

- Let $ID(Q)$ be dead edges in $Q$;
- Let $IL(Q)$ be the live edges in $Q$;
- Let $EL(Q)$ be the live edges in $G - Q$;
- Let $D_Q$ be the spanning subgraph with edges in $ID(Q)$;
- Let $G_Q$ be the graph whose vertices are components of $D_Q$, and edges are in $IL(Q)$.

Theorem (C-Kofman-Stoltzfus)

For any connected ribbon graph $G$,

$$C(G) = \sum_{Q \subset G} Y^n(D_Q) Z^{g(D_Q)} (1 + Y) |EL(Q)| T_{G-Q}(X, 1 + YZ).$$
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Special cases

Planar Case

If \( g(G) = 0 \), \( G \) is a planar graph with underlying graph \( G \). Quasi-trees of \( G \) are spanning trees of \( G \), and each \( G_Q \) is a tree with \( |IL(Q)| \) edges.

So if \( Y = y - 1 \) and \( Z = 1 \), we recover \( T_G(x, y) = \sum_T x^{i(T)} y^{j(T)} \) from the Theorem.
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One-vertex case

If \( G \) has a single vertex, all edges are loops, so we get

\[
C(G) = \sum_{Q \subset G} Y^{n(D_Q)} Z^{g(D_Q)} (1 + Y)^{|EL(Q)|} (1 + YZ)^{|IL(Q)|}
\]
Idea of proof

\[ \rho : E(G) \rightarrow \{0, 1, *\} \text{ is a partial resolution of } G; \text{ *-edges are unresolved.} \]

An unresolved edge \( e \) is nugatory if \( \rho(e) = 0 \) or \( \rho(e) = 1 \) has no quasi-trees.
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\[ \rho : E(\mathcal{G}) \to \{0, 1, *\} \] is a partial resolution of \( \mathcal{G} \); *-edges are unresolved. An unresolved edge \( e \) is nugatory if \( \rho(e) = 0 \) or \( \rho(e) = 1 \) has no quasi-trees.

Binary tree \( T \) of partial resolutions of \( \mathcal{G} \) \( \leftrightarrow \) Skein resolution tree of a link diagram \( D \)
Idea of proof

\( \rho : E(\mathbb{G}) \rightarrow \{0, 1, *\} \) is a **partial resolution** of \( \mathbb{G} \); \(*\)-edges are **unresolved**. An unresolved edge \( e \) is **nugatory** if \( \rho(e) = 0 \) or \( \rho(e) = 1 \) has no quasi-trees.

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- Leaves of \( T \) ←→ Quasi-trees \( Q \subset G \)
- Unresolved edges in \( \rho \) ←→ Live edges in \( G \) wrt \( Q \)
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Unresolved edges of a leaf \( \rho \) are nugatory, and can be uniquely resolved to get \( Q \), just like crossings of the twisted unknot.
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State sum for \( T \) is the state sum for poset of all resolutions of \( G \) and this gives the required expansion.
Example

\[
C(G) = Z^2 Y^4 + 2XZYZ^3 + 4ZY^3 + X^2 Y^2 + 3XY^2 + 3XZY^2 + 4ZY^2 + 2Y^2 + 2X^2 Y + 6XY + 4Y + X^2 + 2X + 1
\]

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<th>Q</th>
<th>Activity</th>
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<td>001010</td>
<td>ℓdDdDd</td>
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<tr>
<td>001100</td>
<td>ℓdDLdd</td>
<td>X(1 + Y)</td>
</tr>
<tr>
<td>001111</td>
<td>ℓdDDDD</td>
<td>Y^2 Z(1 + Y)</td>
</tr>
<tr>
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</tr>
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<td>ℓLdlld</td>
<td>X^2 (1 + Y)</td>
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<tr>
<td>010111</td>
<td>ℓLdDDD</td>
<td>XY^2 Z(1 + Y)</td>
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<td>Y(1 + Y)(X + 1 + YZ)</td>
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### Main Theorem and Examples

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\[ Y^n(\mathcal{D}_\mathbb{Q}) \ Z_g(\mathcal{D}_\mathbb{Q}) \ (1 + Y)^{\left| EL(\mathbb{Q}) \right|} \quad T_{G_\mathbb{Q}}(X, 1 + YZ) = XY(1 + Y)(X + 1 + YZ) \]