Quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial

Abhijit Champanerkar

College of Staten Island, CUNY

Joint work with Ilya Kofman & Neal Stoltzfus

KNOTS IN WASHINGTON XXVII
3rd Japan-USA Workshop in Knot Theory
January 9-11, 2009
George Washington University, Washington, DC

Tutte polynomial of graphs

Let G be a connected graph. A spanning tree T of G is a connected, spanning subgraph with no cycles.

The Tutte polynomial is defined by the spanning tree expansion:

$$T_G(x,y) = \sum_{T \subseteq G} x^{i(T)}\, y^{j(T)}$$

where

- $i(T) =$ number of internally active edges for spanning tree T;
- $j(T) =$ number of externally active edges for spanning tree T.

$T_G(1,1) =$ number of spanning trees of G.

Summary of results

Bollobás and Riordan extended the Tutte polynomial to an invariant of oriented ribbon graphs, now called the Bollobás–Riordan–Tutte (BRT) polynomial.

Graphs	$\longleftrightarrow$	Ribbon graphs
Spanning trees	$\longleftrightarrow$	Quasi-trees
Tutte polynomial	$\longleftrightarrow$	BRT polynomial
Activity w.r.t. spanning trees	$\longleftrightarrow$	Activity w.r.t. quasi-trees
Spanning tree expansion of the Tutte polynomial	$\longleftrightarrow$	Quasi-tree expansion of the BRT polynomial

Motivation

- (Dasbach-Futer-Kalfagianni-Lin-Stoltzfus) Associated to every link diagram D a ribbon graph $\mathbb{G}_D$ and obtained the Jones polynomial as a specialization of the Bollobás–Riordan–Tutte polynomial.

- (C-Kofman-Stoltzfus) Quasi-trees of $\mathbb{G}_D$ correspond to spanning trees of the Tait graph (checkerboard graph) of D and generate a complex whose homology is the Khovanov homology of D.

- (C-Kofman-Stoltzfus, Manturov) Maximal genus of quasi-trees of $\mathbb{G}_D$ plus two gives an upper bound on the thickness of Khovanov homology.

Ribbon graphs

Definition

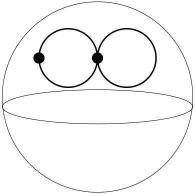
An (oriented) ribbon graph $\mathbb{G}$ is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface, such that its complement is a union of 2-cells.

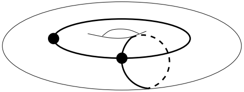
The genus of $\mathbb{G}$, $g(\mathbb{G})$, is the genus of the surface on which it is embedded.

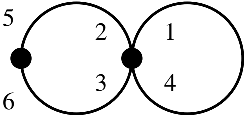
$\mathbb{G}$ can also be described by a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, 2, \ldots, 2n\}$ such that σ_1 is a fixed point free involution and $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \mathrm{Identity}$. This triple gives a cell complex structure for the surface of $\mathbb{G}$ and the orbits of σ_i correspond to the i–cells for $i = 0, 1, 2$.

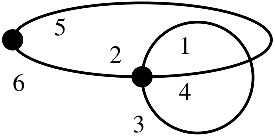
Plane graphs are genus zero ribbon graphs.

Example









$$\sigma_0 = (1234)(56)$$
$$\sigma_1 = (14)(25)(36)$$
$$\sigma_2 = (246)(35)$$

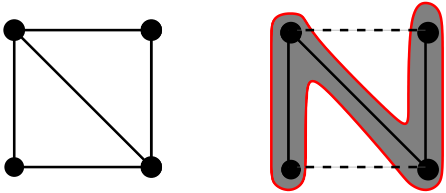
$$\sigma_0 = (1234)(56)$$
$$\sigma_1 = (13)(26)(45)$$
$$\sigma_2 = (152364)$$

Quasi-trees: Motivation

A spanning tree of a graph is a spanning subgraph without any cycles.

For a plane graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example



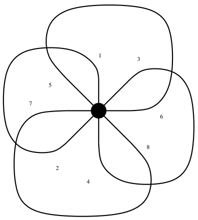
Quasi-trees: Definition

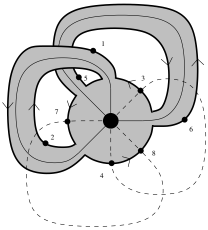
Definition

A quasi-tree of a ribbon graph is a spanning ribbon subgraph with one face.

- The quasi-trees of a plane graph are spanning trees.
- The genus zero quasi-trees of a ribbon graph are its spanning trees.

Example





Quasi-trees and chord diagrams

The spanning tree expansion of the Tutte polynomial is defined using activity of edges with respect to a spanning tree.

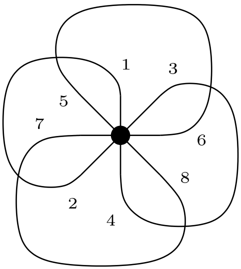
We extend Tutte's definition of activities to edges of a ribbon graph with respect to a quasi-tree.

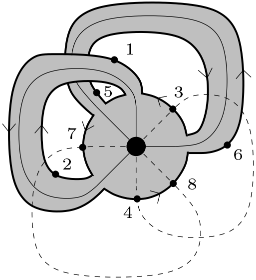
Proposition

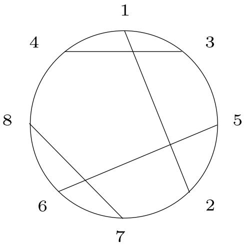
Every quasi-tree $\mathbb{Q}$ corresponds to the ordered chord diagram $C_{\mathbb{Q}}$ with consecutive markings in the positive direction given by the permutation:

$$\sigma(i) = \begin{cases} \sigma_0(i) & i \notin \mathbb{Q} \\ \sigma_2^{-1}(i) & i \in \mathbb{Q} \end{cases}$$

Example







Extending Tutte's activites to ribbon graphs

Definition

Fix a total order on the edges of a connected ribbon graph G. An edge e in a quasi-tree $\mathbb{Q}$ is live if the corresponding chord in $C_{\mathbb{Q}}$ does not intersect any lower-ordered chords and otherwise it is dead.

An edge e is internal or external, according to $e \in \mathbb{Q}$ or $e \in G - \mathbb{Q}$, respectively.

- For a genus zero ribbon graph, activities using our definition agree with those using Tutte's definition.
- Any spanning tree of a ribbon graph is also a quasi-tree of genus zero. In this case, the activities using our definition are different from those using Tutte's definition.

Bollobás–Riordan–Tutte polynomial for ribbon graphs

Let $\mathbb{H}$ be a spanning ribbon subgraph of $\mathbb{G}$ (i.e. contains all vertices and some edges of $\mathbb{G}$).

- $v(\mathbb{H})$ = number of vertices of $\mathbb{H}$;
- $e(\mathbb{H})$ = number of edges of $\mathbb{H}$;
- $f(\mathbb{H})$ = number of faces of $\mathbb{H}$;
- $k(\mathbb{H})$ = number of components of $\mathbb{H}$;
- $n(\mathbb{H})$ = nullity of $\mathbb{H} = e(\mathbb{H}) - v(\mathbb{H}) + k(\mathbb{H})$;
- $g(\mathbb{H})$ = genus of $\mathbb{H} = k(\mathbb{H}) + n(\mathbb{H}) - f(\mathbb{H})$.

Bollobás and Riordan extended the Tutte polynomial to an invariant $C(\mathbb{G}; X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ of ribbon graphs using the following state sum over all spanning subgraphs:

$$C(\mathbb{G}; X, Y, Z) = \sum_{\mathbb{H} \subset \mathbb{G}} (X-1)^{k(\mathbb{H}) - k(G)}\, Y^{n(\mathbb{H})}\, Z^{g(\mathbb{H})}$$

Properties

If G is the underlying graph of $\mathbb{G}$ then

$$C(\mathbb{G}; X, Y, 1) = T_G(X, 1 + Y).$$

Proposition (C-Kofman-Stoltzfus)

Let $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2})$. Then $q(\mathbb{G}; t, Y)$ is a polynomial in t and Y such that

$$q(\mathbb{G}; t, 0) = \sum_j a_j t^j$$

where a_j is the number of quasi-trees of genus j. Consequently, $q(\mathbb{G}; 1, 0)$ equals the number of quasi-trees of G.

Main Theorem

Order the edges of $\mathbb{G}$. Let $\mathbb{Q}$ be a quasi-tree of $\mathbb{G}$.

- Let $ID(\mathbb{Q})$ be dead edges in $\mathbb{Q}$;
- Let $IL(\mathbb{Q})$ be the live edges in $\mathbb{Q}$;
- Let $EL(\mathbb{Q})$ be the live edges in $\mathbb{G} - \mathbb{Q}$;
- Let $\mathcal{D}_{\mathbb{Q}}$ be the spanning subgraph with edges in $ID(\mathbb{Q})$;
- Let $\mathcal{G}_{\mathbb{Q}}$ be the graph whose vertices are components of $\mathcal{D}_{\mathbb{Q}}$, and edges are in $IL(\mathbb{Q})$.

Theorem (C-Kofman-Stoltzfus)

For any connected ribbon graph $\mathbb{G}$,

$$C(\mathbb{G}) = \sum_{\mathbb{Q} \subset G} Y^{n(\mathcal{D}_{\mathbb{Q}})}\, Z^{g(\mathcal{D}_{\mathbb{Q}})}\, (1+Y)^{|EL(\mathbb{Q})|}\, T_{\mathcal{G}_{\mathbb{Q}}}(X, 1+YZ)$$

Special cases

Planar Case

If $g(\mathbb{G}) = 0$, $\mathbb{G}$ is a planar graph with underlying graph G. Quasi-trees of $\mathbb{G}$ are spanning trees of G, and each $\mathcal{G}_{\mathbb{Q}}$ is a tree with $|IL(\mathbb{Q})|$ edges.

So if $Y = y - 1$ and $Z = 1$, we recover $T_G(x, y) = \sum_T x^{i(T)}\, y^{j(T)}$ from the Theorem.

One-vertex case

If $\mathbb{G}$ has a single vertex, all edges are loops, so we get

$$C(\mathbb{G}) = \sum_{\mathbb{Q} \subset \mathbb{G}} Y^{n(\mathcal{D}_{\mathbb{Q}})}\, Z^{g(\mathcal{D}_{\mathbb{Q}})}\, (1 + Y)^{|EL(\mathbb{Q})|}\, (1 + YZ)^{|IL(\mathbb{Q})|}$$

Idea of proof

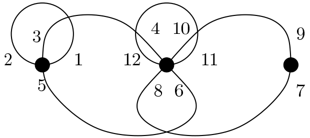
$\rho : E(\mathbb{G}) \to \{0, 1, *\}$ is a partial resolution of $\mathbb{G}$; $*$–edges are unresolved.
An unresolved edge e is nugatory if $\rho(e) = 0$ or $\rho(e) = 1$ has no quasi-trees.

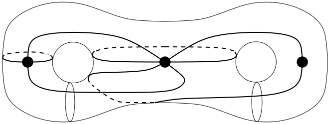
Binary tree $\mathcal{T}$ of partial resolutions of $\mathbb{G}$	$\longleftrightarrow$	Skein resolution tree of a link diagram D
Leaves of $\mathcal{T}$	$\longleftrightarrow$	Quasi-trees $\mathbb{Q} \subset \mathbb{G}$
Unresolved edges in ρ	$\longleftrightarrow$	Live edges in G wrt $\mathbb{Q}$

Unresolved edges of a leaf ρ are nugatory, and can be uniquely resolved to get $\mathbb{Q}$, just like crossings of the twisted unknot.

State sum for $\mathcal{T}$ is the state sum for poset of all resolutions of $\mathbb{G}$ and this gives the required expansion.

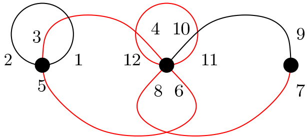
Example

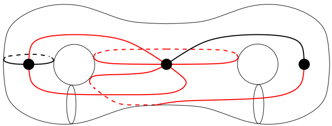




$\mathbb{Q}$	Activity	Weight
001010	$\ell dDdDd$	$(1+Y)$
001100	$\ell dDLdd$	$X(1+Y)$
001111	$\ell dDDDD$	$Y^2Z(1+Y)$
010010	$\ell LddDd$	$X(1+Y)$
010100	$\ell LdLdd$	$X^2(1+Y)$
010111	$\ell LdDDD$	$XY^2Z(1+Y)$
011011	$\ell LLdDD$	$Y(1+Y)(X+1+YZ)$
011101	$\ell LLLdD$	$XY(1+Y)(X+1+YZ)$
011110	$\ell LLDDd$	$Y(1+Y)(X+1+YZ)$
111010	$LDDdDd$	$Y(1+YZ)$
111100	$LDDLdd$	$XY(1+YZ)$
111111	$LDDDDD$	$Y^3Z(1+YZ)$

$$C(\mathbb{G}) = Z^2Y^4 + 2XZY^3 + 4ZY^3 + X^2Y^2 + 3XY^2 + 3XZY^2 + 4ZY^2 + 2Y^2 + 2X^2Y + 6XY + 4Y + X^2 + 2X + 1$$





$\mathbb{Q}$	Activity	Weight
001010	$\ell dDdDd$	$(1+Y)$
001100	$\ell dDLdd$	$X(1+Y)$
001111	$\ell dDDDD$	$Y^2Z(1+Y)$
010010	$\ell LddDd$	$X(1+Y)$
010100	$\ell LdLdd$	$X^2(1+Y)$
010111	$\ell LdDDD$	$XY^2Z(1+Y)$
011011	$\ell LLdDD$	$Y(1+Y)(X+1+YZ)$
011101	$\ell LLLdD$	$XY(1+Y)(X+1+YZ)$
011110	$\ell LLDDd$	$Y(1+Y)(X+1+YZ)$
111010	$LDDdDd$	$Y(1+YZ)$
111100	$LDDLdd$	$XY(1+YZ)$
111111	$LDDDDD$	$Y^3Z(1+YZ)$

$$Y^{n(\mathcal{D}_{\mathbb{Q}})}\ Z^{g(\mathcal{D}_{\mathbb{Q}})}\ (1+Y)^{|EL(\mathbb{Q})|}\ T_{\mathcal{G}_{\mathbb{Q}}}(X, 1+YZ) = XY(1+Y)(X+1+YZ)$$