Quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial

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Tutte polynomial of graphs

Let $G$ be a connected graph. A spanning tree $T$ of $G$ is a connected, spanning subgraph with no cycles.

The Tutte polynomial is defined by the spanning tree expansion:

$$T_G(x, y) = \sum_{T \subseteq G} x^{i(T)} y^{j(T)}$$

where

- $i(T) =$ number of internally active edges for spanning tree $T$;
- $j(T) =$ number of externally active edges for spanning tree $T$.

$T_G(1, 1) =$ number of spanning trees of $G$. 
Bollobás and Riordan extended the Tutte polynomial to an invariant of oriented ribbon graphs, now called the Bollobás–Riordan–Tutte (BRT) polynomial.

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Motivation

- **(Dasbach-Futer-Kalfagianni-Lin-Stoltzfus)** Associated to every link diagram $D$ a ribbon graph $G_D$ and obtained the Jones polynomial as a specialization of the Bollobás–Riordan–Tutte polynomial.

- **(C-Kofman-Stoltzfus)** Quasi-trees of $G_D$ correspond to spanning trees of the Tait graph (checkerboard graph) of $D$ and generate a complex whose homology is the Khovanov homology of $D$.

- **(C-Kofman-Stoltzfus, Manturov)** Maximal genus of quasi-trees of $G_D$ plus two gives an upper bound on the thickness of Khovanov homology.
Ribbon graphs

**Definition**

An (oriented) **ribbon graph** \( G \) is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface, such that its complement is a union of 2-cells.

The **genus** of \( G \), \( g(G) \), is the genus of the surface on which it is embedded.

\( G \) can also be described by a triple of permutations \((\sigma_0, \sigma_1, \sigma_2)\) of the set \(\{1, 2, \ldots, 2n\}\) such that \(\sigma_1\) is a fixed point free involution and \(\sigma_0 \circ \sigma_1 \circ \sigma_2 = \text{Identity}\). This triple gives a cell complex structure for the surface of \( G \) and the orbits of \(\sigma_i\) correspond to the \(i\)–cells for \(i = 0, 1, 2\).

Plane graphs are **genus zero** ribbon graphs.
Example

\[ \sigma_0 = (1234)(56) \]
\[ \sigma_1 = (14)(25)(36) \]
\[ \sigma_2 = (246)(35) \]
Quasi-trees: Motivation

A spanning tree of a graph is a spanning subgraph without any cycles.

For a plane graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example
**Quasi-trees: Definition**

**Definition**

A **quasi-tree** of a ribbon graph is a spanning ribbon subgraph with one face.

- The quasi-trees of a plane graph are spanning trees.
- The genus zero quasi-trees of a ribbon graph are its spanning trees.

**Example**
Quasi-trees and chord diagrams

The spanning tree expansion of the Tutte polynomial is defined using activity of edges with respect to a spanning tree.

We extend Tutte’s definition of activities to edges of a ribbon graph with respect to a quasi-tree.

**Proposition**

Every quasi-tree $Q$ corresponds to the ordered chord diagram $C_Q$ with consecutive markings in the positive direction given by the permutation:

$$
\sigma(i) = \begin{cases} 
\sigma_0(i) & i \notin Q \\
\sigma_2^{-1}(i) & i \in Q 
\end{cases}
$$
Example
Extending Tutte’s activities to ribbon graphs

**Definition**

Fix a total order on the edges of a connected ribbon graph $G$. An edge $e$ in a quasi-tree $Q$ is **live** if the corresponding chord in $C_Q$ does not intersect any lower-ordered chords and otherwise it is **dead**.

An edge $e$ is **internal** or **external**, according to $e \in Q$ or $e \in G - Q$, respectively.

- For a genus zero ribbon graph, activities using our definition agree with those using Tutte’s definition.
- Any spanning tree of a ribbon graph is also a quasi-tree of genus zero. In this case, the activities using our definition are **different** from those using Tutte’s definition.
Let $\mathcal{H}$ be a spanning ribbon subgraph of $G$ (i.e. contains all vertices and some edges of $G$).

- $v(\mathcal{H}) =$ number of vertices of $\mathcal{H}$;
- $e(\mathcal{H}) =$ number of edges of $\mathcal{H}$;
- $f(\mathcal{H}) =$ number of faces of $\mathcal{H}$;
- $k(\mathcal{H}) =$ number of components of $\mathcal{H}$;
- $n(\mathcal{H}) =$ nullity of $\mathcal{H} = e(\mathcal{H}) - v(\mathcal{H}) + k(\mathcal{H})$;
- $g(\mathcal{H}) =$ genus of $\mathcal{H} = k(\mathcal{H}) + n(\mathcal{H}) - f(\mathcal{H})$.

Bollobás and Riordan extended the Tutte polynomial to an invariant $C(G; X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ of ribbon graphs using the following state sum over all spanning subgraphs:

$$C(G; X, Y, Z) = \sum_{\mathcal{H} \subset G} (X - 1)^{k(\mathcal{H}) - k(G)} Y^{n(\mathcal{H})} Z^{g(\mathcal{H})}$$
Properties

If $G$ is the underlying graph of $\mathbb{G}$ then

$$C(\mathbb{G}; X, Y, 1) = T_G(X, 1 + Y).$$

**Proposition (C-Kofman-Stoltzfus)**

Let $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2})$. Then $q(\mathbb{G}; t, Y)$ is a polynomial in $t$ and $Y$ such that

$$q(\mathbb{G}; t, 0) = \sum_j a_j t^j$$

where $a_j$ is the number of quasi-trees of genus $j$. Consequently, $q(\mathbb{G}; 1, 0)$ equals the number of quasi-trees of $G$. 

Main Theorem

Order the edges of $G$. Let $Q$ be a quasi-tree of $G$.

- Let $ID(Q)$ be dead edges in $Q$;
- Let $IL(Q)$ be the live edges in $Q$;
- Let $EL(Q)$ be the live edges in $G - Q$;
- Let $D_Q$ be the spanning subgraph with edges in $ID(Q)$;
- Let $G_Q$ be the graph whose vertices are components of $D_Q$, and edges are in $IL(Q)$.

**Theorem (C-Kofman-Stoltzfus)**

For any connected ribbon graph $G$,

$$C(G) = \sum_{Q \subset G} Y^{n(D_Q)} Z^{g(D_Q)} (1 + Y)^{|EL(Q)|} T_{G_Q}(X, 1 + YZ)$$
Special cases

**Planar Case**

If $g(G) = 0$, $G$ is a planar graph with underlying graph $G$. Quasi-trees of $G$ are spanning trees of $G$, and each $G_Q$ is a tree with $|IL(Q)|$ edges.

So if $Y = y - 1$ and $Z = 1$, we recover $T_G(x, y) = \sum_T x^i(T) y^j(T)$ from the Theorem.

**One-vertex case**

If $G$ has a single vertex, all edges are loops, so we get

$$C(G) = \sum_{Q \subseteq G} Y^{n(D_Q)} Z^{g(D_Q)} (1 + Y)^{|EL(Q)|} (1 + YZ)^{|IL(Q)|}$$
Idea of proof

\[ \rho : E(G) \rightarrow \{0, 1, *\} \text{ is a partial resolution of } G; \text{ } * \text{-edges are unresolved.} \]
An unresolved edge \( e \) is nugatory if \( \rho(e) = 0 \) or \( \rho(e) = 1 \) has no quasi-trees.

Binary tree \( \mathcal{T} \) of partial resolutions of \( G \) \( \iff \) Skein resolution tree of a link diagram \( D \)

Leaves of \( \mathcal{T} \) \( \iff \) Quasi-trees \( Q \subset G \)

Unresolved edges in \( \rho \) \( \iff \) Live edges in \( G \) wrt \( Q \)

Unresolved edges of a leaf \( \rho \) are nugatory, and can be uniquely resolved to get \( Q \), just like crossings of the twisted unknot.

State sum for \( \mathcal{T} \) is the state sum for poset of all resolutions of \( G \) and this gives the required expansion.
Main Theorem and Examples

Example

\[
\begin{array}{ccc}
Q & \text{Activity} & \text{Weight} \\
001010 & \ell dDdDd & (1 + Y) \\
001100 & \ell dDLdd & X(1 + Y) \\
001111 & \ell dDDDD & Y^2 Z(1 + Y) \\
010010 & \ell LddDd & X(1 + Y) \\
010100 & \ell LdLdd & X^2(1 + Y) \\
010111 & \ell LdDDD & XY^2 Z(1 + Y) \\
011011 & \ell LLdDD & Y(1 + Y)(X + 1 + YZ) \\
011101 & \ell LLLdD & XY(1 + Y)(X + 1 + YZ) \\
011110 & \ell LLDDd & Y(1 + Y)(X + 1 + YZ) \\
111010 & LDLdDd & Y(1 + YZ) \\
111100 & LDDLdd & XY(1 + YZ) \\
111111 & LDDDDD & Y^3 Z(1 + YZ) \\
\end{array}
\]

\[
C(G) = Z^2 Y^4 + 2XZY^3 + 4ZY^3 + X^2 Y^2 + 3XY^2 + 3XZY^2 + 4ZY^2 + 2Y^2 + 2X^2 Y + 6XY + 4Y + X^2 + 2X + 1
\]
### Main Theorem and Examples

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</tr>
<tr>
<td>001111</td>
<td>(\ell dDDDD)</td>
<td>(Y^2Z(1 + Y))</td>
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<td>(LDDDDD)</td>
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</tr>
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\[ Y^n(D_Q) \ Z g(D_Q) (1 + Y)^{|EL(Q)|} \ T_{G_Q}(X, 1 + YZ) = XY(1 + Y)(X + 1 + YZ) \]