Mahler measure of the A-polynomial

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History

In 2000 David Boyd observed (numerically) that the two-variable Mahler measure of A-polynomials were equal to sums of hyperbolic volumes. In many cases it was equal to the volume.

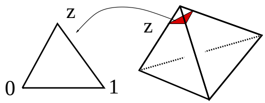
In 2003 Boyd and Rodrigues-Villegas explained this observation and gave a technique to compute the Mahler measures of (tempered) two-variable polynomials.

In this talk I will explain:

- How this technique works for A-polynomials.
- Why A-polynomials are natural examples which work.

Ideal Triangulations

An *ideal tetrahedron* is a geodesic tetrahedron in hyperbolic 3-space $\mathbb{H}^3$ with all its four vertices on the sphere at infinity.



Every edge gets a complex number called the *edge parameter*. Isometry classes $\leftrightarrow \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. An ideal tetrahedron with edge parameter z is denoted by $\triangle(z)$.

An *ideal triangulation* of a cusped hyperbolic 3-manifold N is a decomposition into hyperbolic ideal tetrahedra.

Parameter Space

Let N be one-cusped hyperbolic 3-manifold triangulated with n tetrahedra.

• At every edge the tetrahedra close up and their parameters multiply to 1. This gives **gluing equations:**

$$\prod_{i=1}^{n} z_i^{r'_{ij}}(1-z_i)^{r''_{ij}} = \pm 1,\ j = 1, \ldots n.$$

• The cusp torus gives **completeness equations:**

$$\ell(\mathbf{z}) = \prod_{i=1}^{n} z_i^{l'_i}(1-z_i)^{l''_i} = 1$$

$$m(\mathbf{z}) = \prod_{i=1}^{n} z_i^{m'_i}(1-z_i)^{m''_i} = 1$$

• $P(N) = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n | \text{ satisfy gluing equations}\}$ is called the *parameter space* of N. $P_0(N)$ is the component containing the complete parameter $\mathbf{z}^0$.

$\mathrm{PSL}(2, \mathbb{C})$ A-polynomial

Define $Hol : P_0(N) \to \mathbb{C}^2$ as $Hol(\mathbf{z}) = (\ell(\mathbf{z}), m(\mathbf{z}))$. The image is a curve in $\mathbb{C}^2$ and let $\overline{A}_0(\ell, m)$ be its defining equation.

Thm (C) $\overline{A}_0(\ell, m)$ is the component of the $\mathrm{PSL}(2, \mathbb{C})$ A-polynomial corresponding to the component containing the complete structure.

For knot complements

$$\overline{A}_0(\ell^2, m^2) = A_0(\ell, m) A_0(-\ell, m)$$

In general all four factors of the $\mathrm{SL}(2, \mathbb{C})$ A-polynomial can appear with signs on ℓ and m.

Mahler measure

Let $p(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$. The logarithmic **Mahler measure** of p is defined as

$$m(p) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |p(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$$

- $m(p_1 \cdot p_2) = m(p_1) + m(p_2)$.
- Jensen's formula: $\displaystyle\frac{1}{2\pi i} \int_{S^1} \log |x - \alpha| \frac{dx}{x} = \log^+ |\alpha|$
- Let $p(x) = a_0 \displaystyle\prod_{i=1}^{n} (x - \alpha_i)$. Then $m(p) = \log |a_0| + \displaystyle\sum_{i=1}^{n} \log^+ |\alpha_i|$,

where $\log^+ |\alpha| = max\{0, \log |\alpha|\}$.

Volume Form or Regulator

Let $p(x, y) \in \mathbb{Z}[x, y]$ be irreducible polynomial.
$X = \{(x, y) \in \mathbb{C}^2 | \ p(x, y) = 0\}$
$\tilde{X}$= smooth projective completion of X
$\mathbb{C}(\tilde{X})$= field of meromorphic functions on $\tilde{X}$

For $f, g \in \mathbb{C}(\tilde{X})$, the **Volume form** is defined as

$$\eta(f, g) = \log|f| \ d \arg g - \log|g| \ d \arg f$$

$\eta \in H^1(\tilde{X} - S; \mathbb{R})$ where S= zeros and poles of $f \& g$.

Mahler measure of $p(x, y)$

Write $p(x,y) = a_0(y)\prod_{j=1}^{m}(x - x_j(y))$ where x_j's are algebraic functions of y on $\tilde{X}$. By Jensen's formula

$$\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|x - x_j(y)|\,\frac{dx}{x}\frac{dy}{y} = \frac{1}{2\pi i}\int_{S^1}\log^+|x_j(y)|\,\frac{dy}{y}$$

Let $\gamma_j = \{(x,y)\in\tilde{X} \mid |y| = 1, |x_j| \geq 1\}$ be an oriented path in $\tilde{X}$.

On γ_j, $\dfrac{dy}{y} = d\log|y| + i d\arg y = i d\arg y$.

$$
\begin{aligned}
i\eta(x_j, y) &= i(\log|x_j|\; d\arg y - \log|y|\; d\arg x_j) \\
&= i\log|x_j|\; d\arg y \\
&= \log|x|\frac{dy}{y}
\end{aligned}
$$

Prop $m(p(x,y)) = m(a_0(y)) + \sum_{i=1}^{n} \frac{1}{2\pi}\int_{\gamma_j} \eta(x_j, y)$

Bloch-Wigner dilogarithm

Lobachevsky function: $L(\theta) = -\int_0^\theta \log|2\sin u|\, du$

$\mathrm{vol}(\triangle(z)) = L(\alpha) + L(\beta) + L(\gamma)$ where $\alpha,\ \beta,\ \gamma$ are the dihedral angles of $\triangle(z)$.

Classical dilogarithm: $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},\ |z| < 1$

It can be analytically extended to $\mathbb{C} - (1, \infty)$ as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\, du$$

The **Bloch-Wigner dilogarithm** is defined as

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z)$$

Properties of $D(z)$

- $D(z)$ is real analytic on $\mathbb{C} - \{0, 1\}$.

- $D(e^{i\theta}) = L(\theta)$

- **Thm** $\mathrm{vol}(\triangle(z)) = D(z)$.
This follows from the 5-term relation and other functional equations of $D(z)$.

- **Thm** $\eta(z, 1 - z) = dD(z)$.

If we can express $\eta(x, y)$ in terms of $\eta(z, 1 - z)$'s then we can use Stokes Theorem to evaluate $m(p(x, y))$ in terms of $D(z)$ and get hyperbolic volumes.

Exactness of Volume Form

Let $F = \mathbb{C}(\tilde{X})$, there are maps

$$\wedge^2_{\mathbb{Z}}(F^*) \xrightarrow{sym} K_2(F) \xrightarrow{\eta} H^1(\tilde{X};\mathbb{R})$$

where $sym(f \wedge g) = \{f, g\}$ and $\eta(\{f, g\}) = \eta(f, g)$.

For $x, y, z_i \in F^*$, suppose in $\wedge^2_{\mathbb{Z}}(F^*)$ we can show

$$x \wedge y = \sum_{i=1}^{n} z_i \wedge (1 - z_i)$$

Then $\eta(x, y) = \sum_{i=1}^{n} \eta(z_i, 1 - z_i) = \sum_{i=1}^{n} dD(z)$

Let $X = P_0(N)$ and let $\ell, m, z_i \in F = \mathbb{C}(\widetilde{P_0(N)})$.

Thm(C) In $\wedge^2_{\mathbb{Z}}(F^*)$, $\ell \wedge m = \sum_{i=1}^{n} z_i \wedge (1 - z_i)$.

$$\implies \eta(\ell, m) = d(\sum_{i=1}^{n} D(z_i))$$

$$\sum_{i=1}^{n} D(z_i) = \mathrm{vol}(N(\mathbf{z}))$$

Hence $\eta(\ell, m)$ gives variation of volume under deformation and hence is called the volume form.

Exactness of $\eta(\ell, m)$ was directly shown by Hodgson and Neumann-Zagier.

Mahler measure of $\overline{A}_0(\ell, m)$

Let $\gamma_j = \{|m| = 1, |\ell_j| \geq 1\}$.
Let each γ_j have c_j components.
Let ω^1_{ijk} and ω^2_{ijk} be lifts of the end points of γ_j to $P_0(N)$.

$$\begin{aligned} m(\overline{A}_0(\ell, m)\ ) &= \frac{1}{2\pi} \sum_{j=1}^{m} \int_{\gamma_j} \eta(\ell_j, m) \\ &= \frac{1}{2\pi} \sum_{j=1}^{m} \sum_{k=1}^{c_j} \sum_{i=1}^{n} \left(D(\omega^2_{ijk}) - D(\omega^1_{ijk}) \right) \end{aligned}$$

Remarks

• Since $\overline{A}_0(1,1) = 0$ and $(1,1)$ corresponds to the complete structure, $\mathrm{vol}(N)$ always appears as a summand in above.

• Conjugate lifts of $(1,1)$ to $P_0(N)$ correspond to different complex embeddings of the invariant trace field of N.

These give conjugate volumes in the summand.

• $\displaystyle\sum_{i=1}^{n}[\omega^s_{ijk}]$ are elements of the Bloch group $\mathcal{B}(\mathbb{C})$.

Examples

- $K = 4_1$, $\pi m(\overline{A}_0(\ell, m)\ ) = vol(S^3 - K)$.

- $K = 6_2$, $\pi m(\overline{A}_0(\ell, m)\ ) = vol(S^3 - K) + V_2$, where V_2 is the conjugate volume given by the Borel regulator.

- $K = k5_{15} \cong m240$, $\pi m(\overline{A}_0(\ell, m)\ ) = vol(S^3 - K) + V_2 + V_3$, where $V_2 = \mathrm{vol}(m240(0,1))$ and $V_3 = \mathrm{vol}(m240(0,2))$.

Marc Culler has a program which computes A-polynomials. In addition it also computes the necessary information to compute its Mahler measure (numerically).

Neumann-Zagier matrices

Let $J_{2k} = \begin{pmatrix} 0 & \mathrm{Id}_k \\ -\mathrm{Id}_k & 0 \end{pmatrix}$ be the symplectic matrix.

A $(n+2) \times 2n$ matrix U is called a Neumann-Zagier matrix if it satisfies

$$UJ_{2n}U^t = 2 \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$$

Thm (Neumann-Zagier 85) The exponents of the gluing and completeness euqation satisfy the above condition.

Starting with any NZ matrix U, we can form "gluing" and "completeness" equations to obtain an A-polynomial. We can compute its Mahler measure using this method.

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Thank You Very Much