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Ribbon graphs

An (oriented) ribbon graph $\mathcal{G}$ is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface, such that its complement is a union of 2-cells.

Example
Algebraic definition

\( \mathbb{G} \) can also be described by a triple of permutations \((\sigma_0, \sigma_1, \sigma_2)\) of the set \(\{1, 2, \ldots, 2n\}\) such that
- \(\sigma_1\) is a fixed point free involution.
- \(\sigma_0 \circ \sigma_1 \circ \sigma_2 = \text{Identity}\)

This triple gives a cell complex structure for the surface of \(\mathbb{G}\) such that
- Orbits of \(\sigma_0\) are vertices.
- Orbits of \(\sigma_1\) are edges.
- Orbits of \(\sigma_2\) are faces.
Example

\[ \sigma_0 = (1234)(56) \]
\[ \sigma_1 = (14)(25)(36) \]
\[ \sigma_2 = (246)(35) \]

\[ \sigma_0 = (1234)(56) \]
\[ \sigma_1 = (13)(26)(45) \]
\[ \sigma_2 = (152364) \]
Spanning trees and regular neighbourhoods

A spanning tree of a graph is a spanning subgraph without any cycles.

For a planar graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example
Quasi-trees

A quasi-tree of a ribbon graph is a spanning ribbon subgraph with one face.

Example
Quasi-trees and spanning trees

Links, Tait graphs and ribbon graphs

Abhijit Champanerkar (USA)  Graphs on surfaces and Khovanov homology
Let $D$ be a connected link diagram, let $G$ be its Tait graph and $\mathbb{G}$ be its all-A ribbon graph.

**Theorem** (C-Kofman-Stoltzfus) Quasi-trees of $\mathbb{G}$ are in one-one correspondence with spanning trees of $G$:

$$Q_j \leftrightarrow T_v \quad \text{where} \quad v + j = \frac{V(G) + E_+(G) - V(\mathbb{G})}{2}$$

$Q_j$ denotes a quasi-tree of genus $j$, and $T_v$ denotes a spanning tree with $v$ positive edges.
Proof

Spanning subgraphs $\mathcal{H}$ $\iff$ Kauffman states of $D$

$\mathcal{H}$ $\iff$ $s(e) = B \iff e \in \mathcal{H}$

$\#$ faces of $\mathcal{H}$ $=$ $\#$ components of $s$

Quasi-trees $\iff$ States with one component

$\iff$ Jordan trails of $D$

$\iff$ Spanning trees of $G$
Spanning trees and Khovanov homology

**Theorem** (C-Kofman’04) For a knot diagram $D$, there exists a spanning tree complex $C(D) = \{ C^u_v(D), \partial \}$ with $\partial : C^u_v \rightarrow C^{u-1}_{v-1}$ that is a deformation retract of the reduced Khovanov complex $\tilde{C}(D)$. In particular,

$$\tilde{H}^{i,j}(D; \mathbb{Z}) \cong H^u_v(C(D); \mathbb{Z})$$

with the indices related as: $u = j - i + C_1$ & $v = j/2 - i + C_2$

- The $u$-grading for spanning trees is obtained from “activities” of edges of the Tait graph.

- Wehrli also proved a similar result.

**Question** Is there an intrinsic way to understand the $u$-grading for quasi-trees?
**Proposition** Every quasi-tree $\mathbb{Q}$ corresponds to the ordered chord diagram $C_\mathbb{Q}$ with consecutive markings in the positive direction given by the permutation:

$$
\sigma(i) = \begin{cases} 
\sigma_0(i) & i \notin \mathbb{Q} \\
\sigma_2^{-1}(i) & i \in \mathbb{Q}
\end{cases}
$$
Quasi-trees and chord diagram

The bigrading on quasi-trees

A chord in $C_Q$ is live if it does not intersect any lower-ordered chords, otherwise it is dead.

An edge of a quasi-tree is live or dead if th corresponding chord chord is live or dead.

**Definition** For any quasi-tree $Q$ of $G$, we define

$$u(Q) = \#\{\text{live edges not in } Q\} - \#\{\text{live edges in } Q\}, \quad v(Q) = -g(Q)$$

Define $C(G) = \bigoplus_{u,v} C^u_v(G)$, where

$$C^u_v(G) = \mathbb{Z}\langle Q \subset G \mid u(Q) = u, \ v(Q) = v \rangle$$
Main Theorem

**Theorem** (C-Kofman-Stoltzfus) For a knot diagram $D$, there exists a quasi-tree complex $C(G) = \{C^u_v(G), \partial\}$ with $\partial : C^u_v \to C^{u-1}_{v-1}$ that is a deformation retract of the reduced Khovanov complex. In particular, the reduced Khovanov homology $\tilde{H}^{i,j}(D; \mathbb{Z})$ is given by

$$\tilde{H}^{i,j}(D; \mathbb{Z}) \cong H^u_v(C(G); \mathbb{Z})$$

with the indices related as follows:

$$u = j - i - w(D) + 1 \quad \text{and} \quad v = j/2 - i + (V(G) - c_+(D))/2$$
Corollaries

For any link $L$, the *Turaev genus* $g_T(L)$ is the minimum value of the genus of the all-A ribbon graphs over all diagrams of $L$.

**Corollary** The thickness of the reduced Khovanov homology of $K$ is less than or equal to $g_T(K) + 1$.

**Corollary** The thickness of the (unreduced) Khovanov homology of $K$ is less than or equal to $g_T(K) + 2$.

**Corollary** The Turaev genus of $(3, q)$-torus links is unbounded.
Thank You Very Much

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Dziekuje