An Introduction to the Volume Conjecture and its generalizations, II

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Tohoku University

Workshop on Volume Conjecture and Related Topics in Knot Theory
Indian Institute of Science Education and Research, Pune
20th December, 2018

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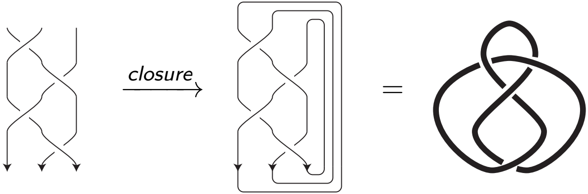

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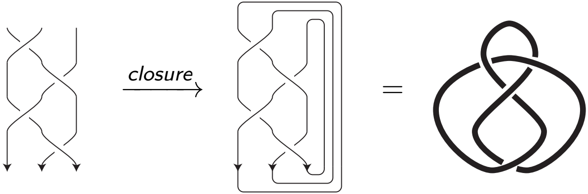

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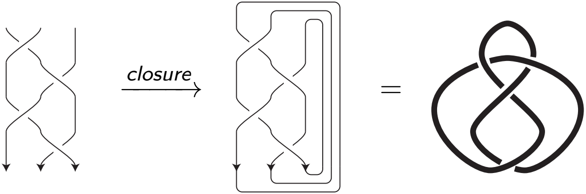

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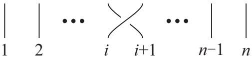
- generators: σ_i $(i = 1, 2, \ldots, n-1)$: 

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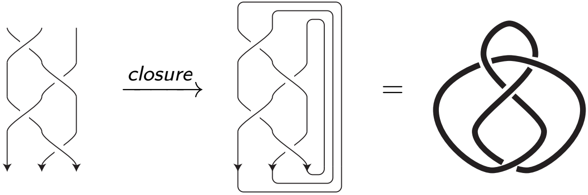

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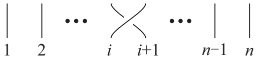
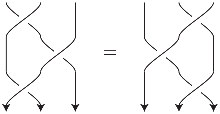
- generators: σ_i $(i = 1, 2, \ldots, n-1)$: 

- relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ $(|i-j| > 1)$, 

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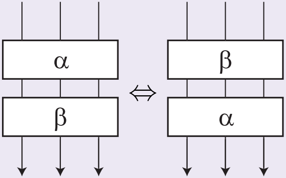

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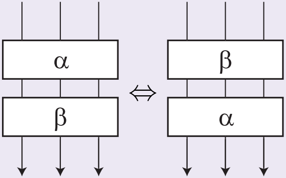
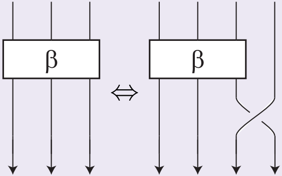


- *stabilization ($\beta \Leftrightarrow \beta\sigma_n^{\pm 1}$):*



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$\mathrm{Tr}_2\colon V \otimes V \to V$ is the operator trace. (For $M \in \mathrm{End}(V \otimes V)$ given by a matrix M^{ij}_{kl}, $\mathrm{Tr}_2(M)$ is given by $\sum_m M^{im}_{km}$.)

Braid $\Rightarrow$ endomorphism

Replace

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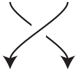
Replace  with

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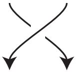
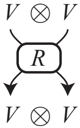
Replace  with 

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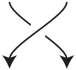
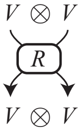
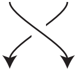
Replace  with 

, and  with

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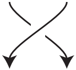
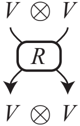
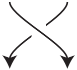
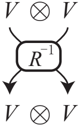
Replace  with 

, and  with 

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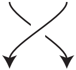
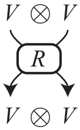
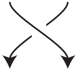
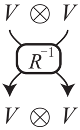
Replace  with 

, and  with 

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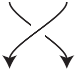
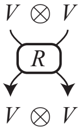
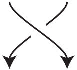
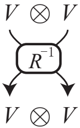
Replace  with 

, and  with 

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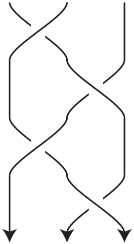

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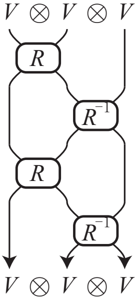

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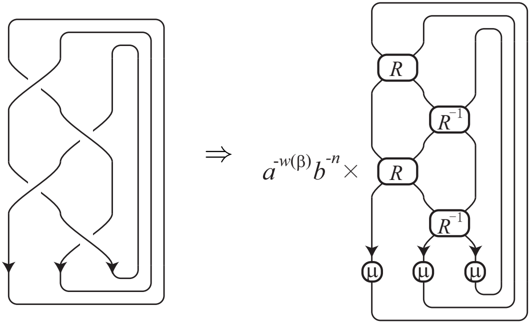

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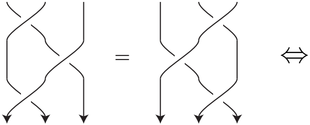

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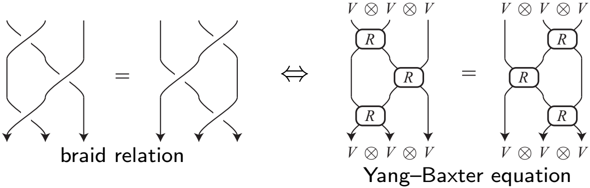

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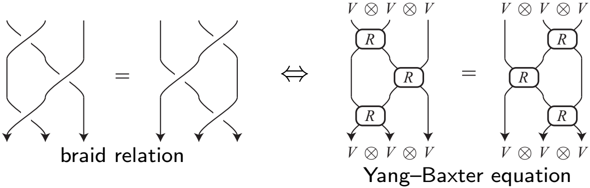

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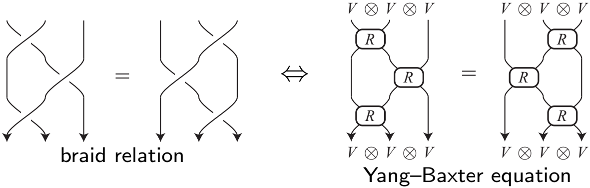

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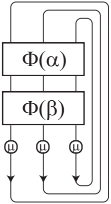

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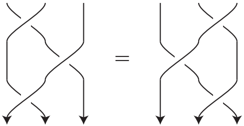

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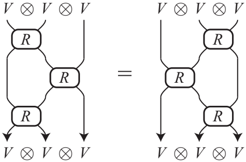

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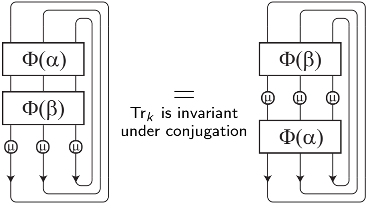

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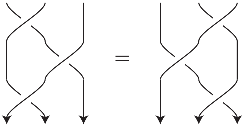

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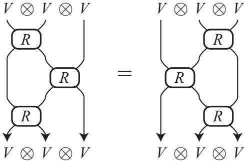

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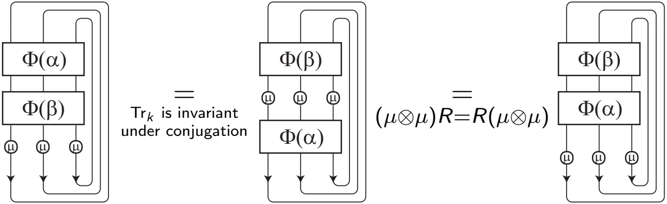

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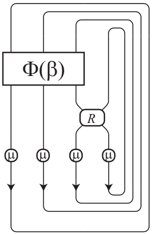

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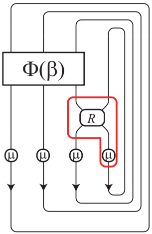

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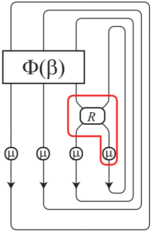

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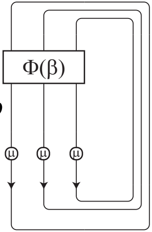
$\mathrm{Tr}_2\left(R(\mathrm{Id}_V \otimes \mu)\right) = ab\,\mathrm{Id}_V$

ab



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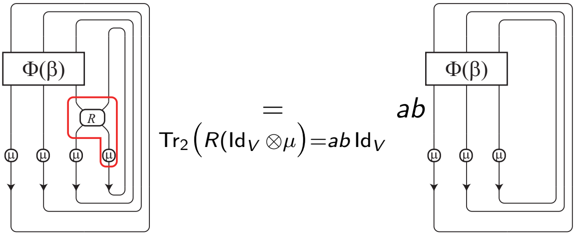

Theorem (Turaev)

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Definition

The quantum $(sl(2, \mathbb{C}), V_N)$ invariant is called the N-dimensional colored Jones polynomial $J_N(L; q)$.

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Note: $T_{(R,\mu,q^{(N^2-1)/4},1)}(\bigcirc) = \mathrm{Tr}_1(\mu) = q^{1-N} + q^{3-N} + \cdots + q^{N-1} = \frac{\{N\}}{\{1\}}$.

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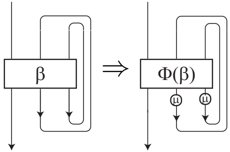

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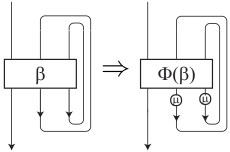

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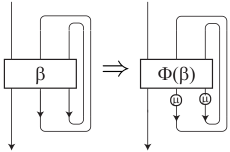

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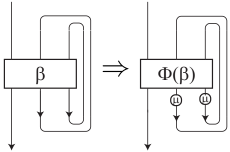

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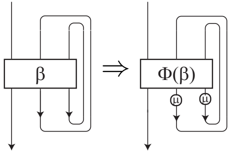

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we have $J_N(L;q) = q^{-w(\beta)(N^2-1)/4}S$.

How to label arcs (due to T. Le)

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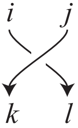

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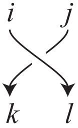

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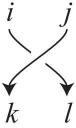

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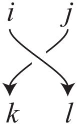

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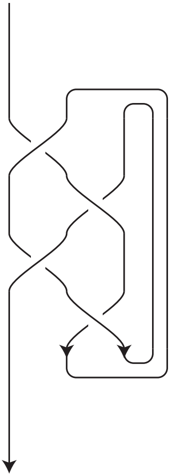

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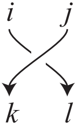

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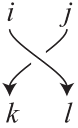

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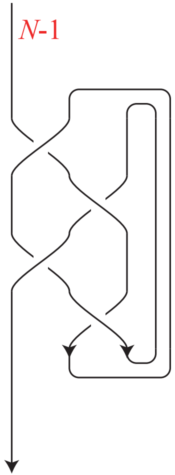

Label the incoming arc with $N-1$.

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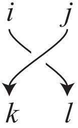

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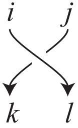

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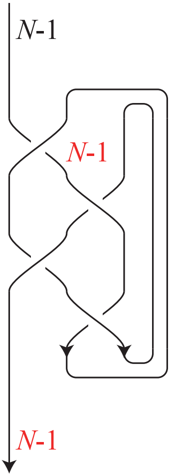

The last one should be $N - 1$ by Schur's lemma. The next one should also be $N - 1$, since it is $\geq N - 1$.

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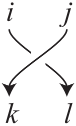

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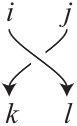

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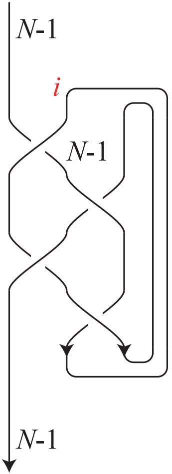

Choose i.

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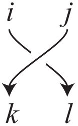

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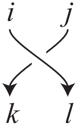

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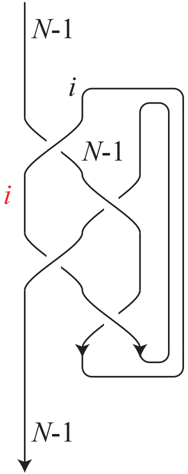

This is also i, since the sum of the labels of the incoming arcs equals the sum of the labels of the outgoing arcs.

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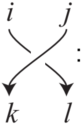

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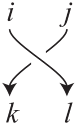

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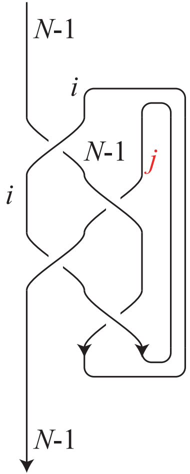
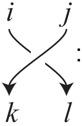


Choose j.

How to label arcs (due to T. Le)



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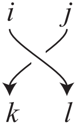
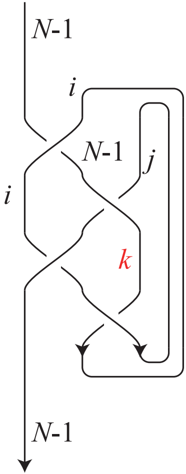


: $\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$



Choose k.

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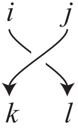

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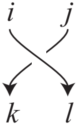

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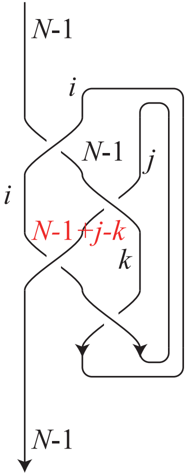

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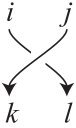

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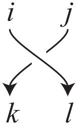

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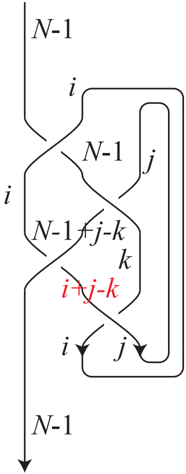

It should be $i+j-k$ by the same reason.
$i+j-k\geq i$ and $N-1\leq N-1+j-k$
$\Rightarrow j=k$.

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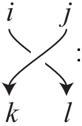

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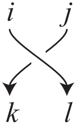

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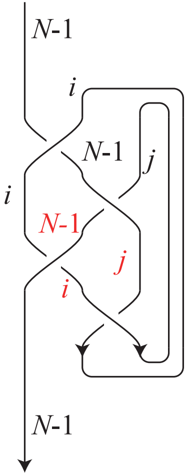

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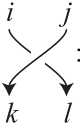

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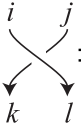

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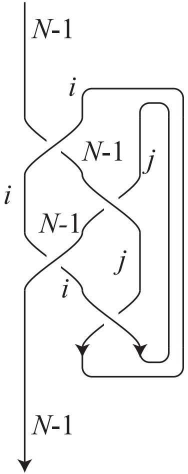

$J_N($  $;q)$

$$
= \sum_{i \ge j} R^{N-1,i}_{i,N-1} \, (R^{-1})^{N-1,j}_{N-1,j} \, R^{i,N-1}_{N-1,i} \, (R^{-1})^{i,j}_{i,j} \, \mu^{j}_{j} \, \mu^{i}_{i}
$$

$$
= \sum_{i \ge j} (-1)^{N-1+i} \frac{\{N-1\}!\{i\}!\{N-1-j\}!}{(\{j\}!)^2\{i-j\}!\{N-1-i\}!}
$$

$$
\times \, q^{(-i-i^2-2ij-2j^2+3N+6Ni+2Nj-3N^2)/4}
$$

A better way of labeling

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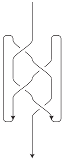

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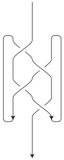

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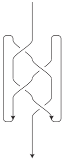

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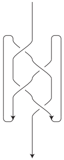

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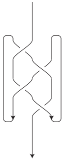

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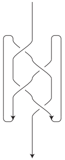

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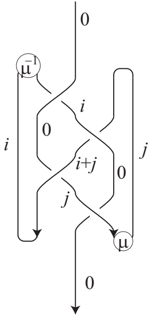

The colored Jones polynomial of 

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$J_N($$;q)$

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$J_N($  $;q)$

$$= \sum_{\substack{0\le i\le N-1,\,0\le j\le N-1\\ 0\le i+j\le N-1}} R^{i,0}_{0,i}\,(R^{-1})^{i,j}_{i+j,0}\,R^{0,i+j}_{i,j}\,(R^{-1})^{j,0}_{0,j}\,(\mu^{-1})^{i}_{i}\,\mu^{j}_{j}$$

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$$= \sum_{\substack{0\le i\le N-1, 0\le j\le N-1 \\ 0\le i+j\le N-1}} (-1)^i \frac{\{i+j\}!\{N-1\}!}{\{i\}!\{j\}!\{N-1-i-j\}!} q^{-(N-1)i/2+(N-1)j/2-i^2/4+j^2/4-3i/4+3j/4}$$

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$$\underset{k:=i+j}{=} \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} q^{k^2/4+Nk/2+k/4} \left(\sum_{i=0}^{k} (-1)^i \frac{\{k\}!}{\{i\}!\{k-i\}!} q^{-Ni-ik/2-i/2} \right).$$

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Using the formula $\sum_{i=0}^{k}(-1)^i q^{li/2} \frac{\{k\}!}{\{i\}!\{k-i\}!} = \prod_{g=1}^{k}(1-q^{(l+k+1)/2-g})$,

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$J_N($  $;q) = \dfrac{1}{\{N\}} \displaystyle\sum_{k=0}^{N-1} \dfrac{\{N+k\}!}{\{N-1-k\}!}.$

Quantum factorial at the N-th root of unity

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- $\{k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^+}$,
  $\{N-1-k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^-}$.

R-matrix as a product of quantum factorial

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$$R^{ij}_{kl} = \sum_{m} \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}$$

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$\Rightarrow$

$$J_N(K;\zeta_N) = \sum_{\substack{\text{labellings} \\ i,j,k,l \\ \text{on arcs}}} \left(\prod_{\pm\text{-crossings}} \frac{\pm(\text{a power of } \zeta_N) \times N^{\pm 2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^\pm}(\zeta_N)_{k^\pm}(\zeta_N)_{j^\mp}(\zeta_N)_{l^\mp}} \right)$$

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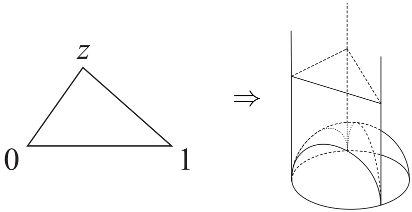

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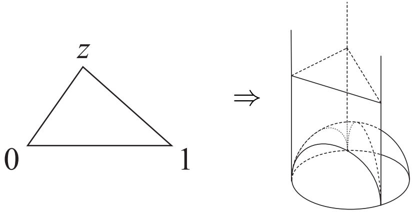

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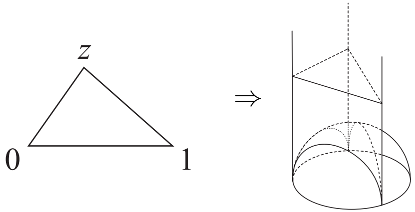

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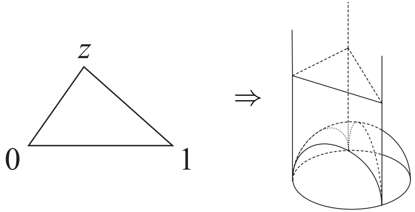
Approximation of the quantum factorial by dilogarithm

- Dilogarithm $\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

  (Recall: $\mathrm{Li}_1(z) := -\log(1-z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.)

  Vol(tetrahedron parametrized by z) $= \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z)$.



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$$\zeta_N^{k^2} = \exp\left(\frac{N}{2\pi\sqrt{-1}}\left(\frac{2\pi\sqrt{-1}k}{N}\right)^2\right) = \exp\left[\frac{N}{2\pi\sqrt{-1}}(\log\zeta_N^k)^2\right].$$

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where $J_1, \dots, J_c$ are some contours.

Saddle Point Method

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Assume that

1. $d\,h(z_0)/dz = 0$ *and* $d^2\,h(z_0)/dz^2 \neq 0$.
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To apply the saddle point method we usually change the contour so that it passes though the saddle point z_0 where $h'(z_0) = 0$.

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- How to apply the saddle point method. In particular, which saddle point to choose. In general, we have many solutions to the system of equations.

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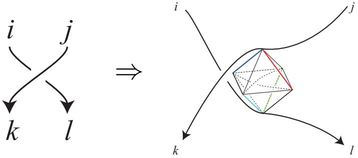

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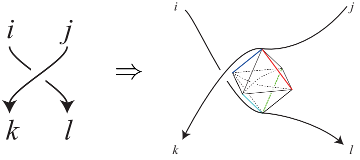
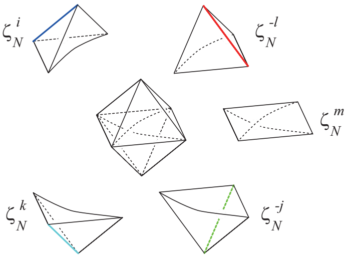


- Decompose the octahedron into five tetrahedra:



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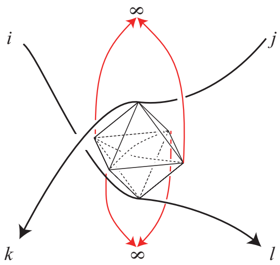

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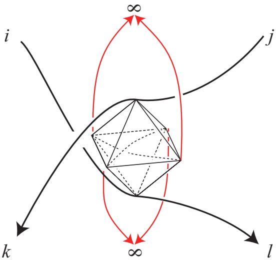

- $S^3 \setminus K$ is now decomposed into topological, truncated tetrahedra, decorated with complex numbers $\zeta_N^{i_k}$.

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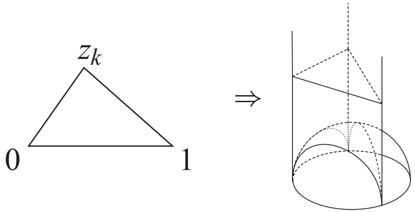
- Regard the tetrahedron decorated with z_k as an hyperbolic, ideal tetrahedron parametrized by z_k.



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- Then, what does $V(x_1, \dots, x_c)(= 2\pi\sqrt{-1} \lim_{N\to\infty} \frac{\log J_N(K, \zeta_N)}{N})$ mean?

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$\Rightarrow$

$$2\pi \lim_{N\to\infty} \frac{\log|J_N(K, \zeta_N)|}{N} = \mathrm{Vol}(S^3 \setminus K),$$

which is the Volume Conjecture.