An Introduction to the Volume Conjecture and its generalizations, I

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Workshop on Volume Conjecture and Related Topics in Knot Theory
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Kauffman bracket

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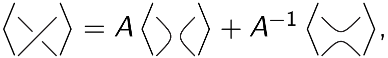
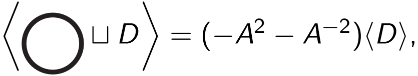
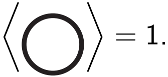
- 

- 

- 

Here $\sqcup$ denotes the disjoint union.

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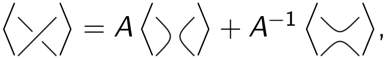
Kauffman bracket $\langle D \rangle$ for a knot diagram D is defined as

- 
- $\left\langle \bigcirc \sqcup D \right\rangle = (-A^2 - A^{-2})\langle D \rangle$,
- $\left\langle \bigcirc \right\rangle = 1$.

Here $\sqcup$ denotes the disjoint union.

Note: $\Bigl\langle \underbrace{\bigcirc \sqcup \bigcirc \sqcup \cdots \sqcup \bigcirc}_{c} \Bigr\rangle = (-A^2 - A^{-2})^{c-1}$.

Kauffman bracket of the trefoil

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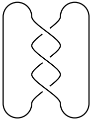
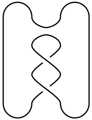
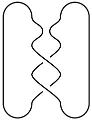
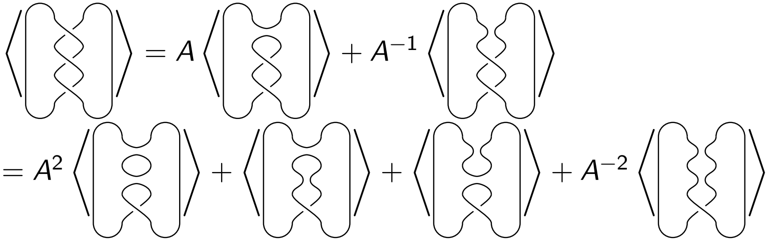
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

Kauffman bracket of the trefoil



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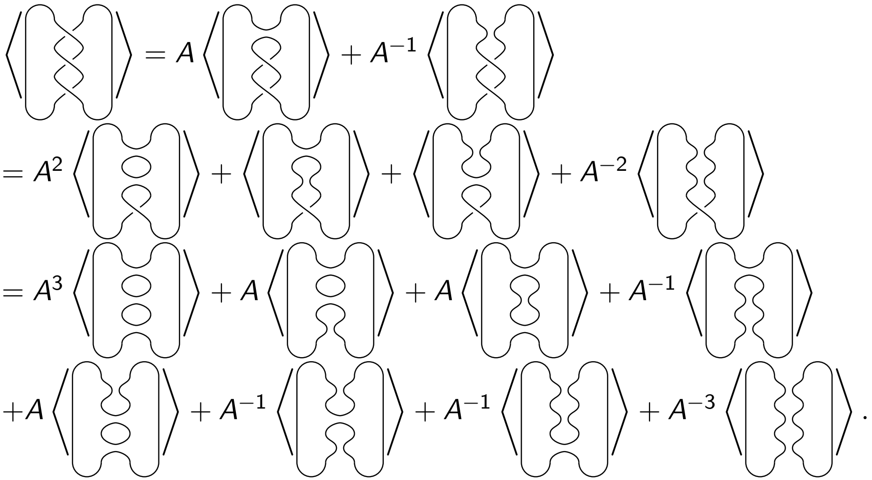

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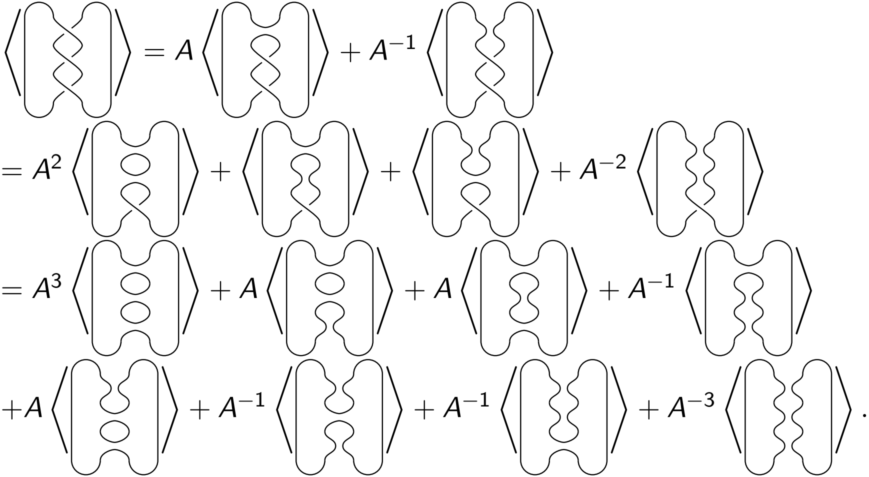
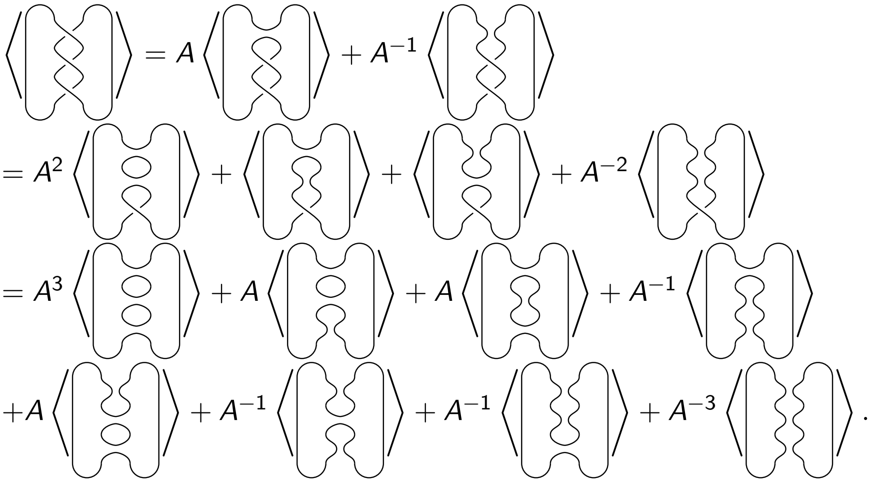


$$= A^3(-A^2 - A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2}) + A^{-1}$$
$$+A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2})$$

Kauffman bracket of the trefoil



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$$+A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2})$$
$$= A^7 - A^3 - A^{-5}.$$

Jones polynomial

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Definition (writhe)

$\vec{D}$: oriented knot diagram.

- $w(\vec{D}) := \sharp$  $- \sharp$ 

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Definition

- K: oriented knot presented by $\vec{D}$.
- D: $\vec{D}$ without orientation.

$$V(K;q) := (-A^3)^{-w(\vec{D})} \langle D \rangle \Big|_{q:=A^{-4}}.$$

$V(K;q)$ is a knot invariant, called the Jones polynomial.

Example of the Jones polynomials

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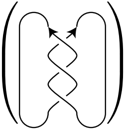
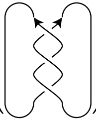
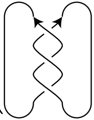
w  $= -3$

Example of the Jones polynomials

$$w\left(\right.$$  $$\left.\right) = -3$$

$$\Rightarrow J\left(\right.$$  $$\left.; q\right) = -A^{9}(A^{7} - A^{3} - A^{-5})\Big|_{q:=A^{-4}} = -q^{-4} + q^{-3} + q^{-1}.$$

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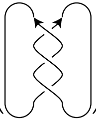
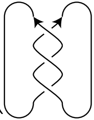
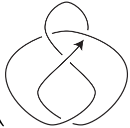
$w\Bigg($  $\Bigg) = -3$

$\Rightarrow J\Bigg($  $; q\Bigg) = -A^9(A^7 - A^3 - A^{-5})\Big|_{q:=A^{-4}} = -q^{-4} + q^{-3} + q^{-1}.$

$J\Bigg($  $; q\Bigg) = q^2 - q + 1 - q^{-1} - q^{-2}.$

Jones–Wenzl idempotent

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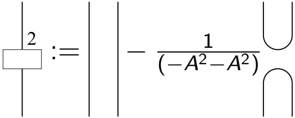
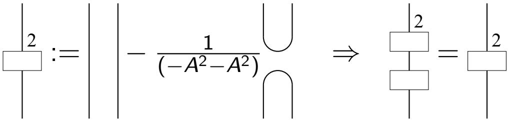
- 

Jones–Wenzl idempotent

- 

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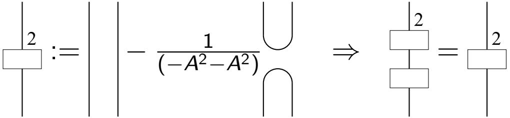
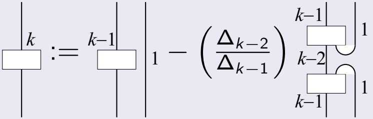
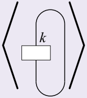
- 

Definition (Jones–Wenzl idempotent)



with $\Delta_k := (-1)^k \frac{A^{2(k+1)}-A^{-2(k+1)}}{A^2-A^{-2}} =$ .

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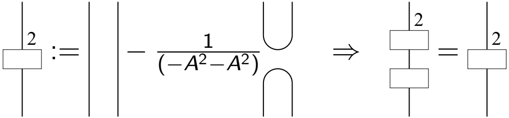
- 

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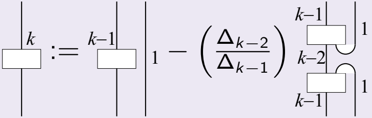
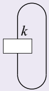


with $\Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}} = \Big\langle$ 

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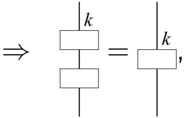

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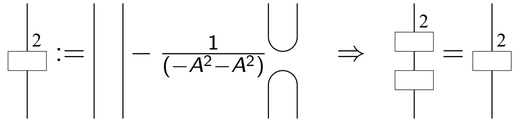
- 

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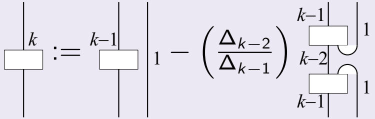
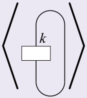


with $\Delta_k := (-1)^k \frac{A^{2(k+1)}-A^{-2(k+1)}}{A^2-A^{-2}} =$ 

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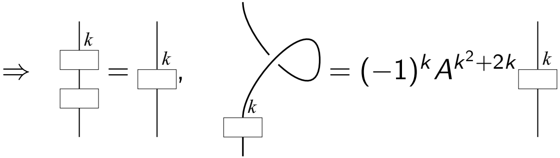

colored Jones polynomial

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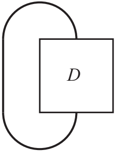
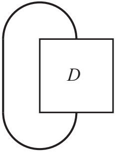


: knot diagram.

colored Jones polynomial



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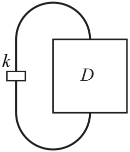
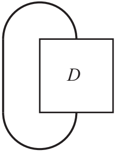
⇒ 

: diagram obtained from D by replacing the string with the Jones–Wenzl idempotent

colored Jones polynomial



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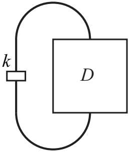
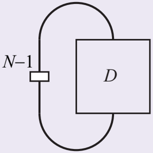
$\Rightarrow$ 

: diagram obtained from D by replacing the string with the Jones–Wenzl idempotent

Definition (N-colored Jones polynomial)

$$J_N(K;q) := \left((-1)^{N-1}A^{N^2-1}\right)^{-w(\vec{D})} \left\langle \quad \right\rangle \Big|_{q:=A^4} .$$



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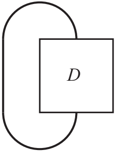

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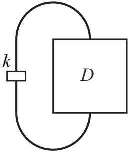
$\Rightarrow$ 

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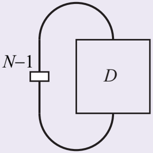


$J_2(K;q) = V(K;q^{-1})$.

colored Jones polynomials of 

colored Jones polynomials of 

$$J_2(\quad; q) = q^1 + q^3 - q^4,$$
$$J_3(\quad; q) = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11},$$
$$J_4(\quad; q) = q^3 + q^7 - q^{10} + q^{11} - q^{13} - q^{14} + q^{15} - q^{17} + q^{19} + q^{20} - q^{21}$$
$$\vdots$$
$$J_N(\quad; q) = \frac{(-1)^{N-1} q^{3(N^2-1)/2}}{q^{N/2} - q^{-N/2}} \times \sum_{k=0}^{N-1} (-1)^k q^{-3(k^2+k)/2} \left(q^{(2k+1)/2} - q^{-(2k+1)/2}\right).$$

(M. Rosso, V. Jones, H. Morton)

colored Jones polynomials of 

colored Jones polynomials of 

$J_2($  $) = q^2 - q + 1 - q^{-1} + q^{-2},$

$J_3($  $) = q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 3 - q^{-1} - q^{-2} + 2q^{-3} - q^{-4} - q^{-5} + q^{-6},$

$J_4($  $) = q^{12} - q^{11} - q^{10} + 2q^8 - 2q^6 + 3q^4 - 3q^2 + 3 - 3q^{-2} + 3q^{-4} - 2q^{-6} + 2q^{-8} - q^{-10} - q^{-11} + q^{-12}$

$\vdots$

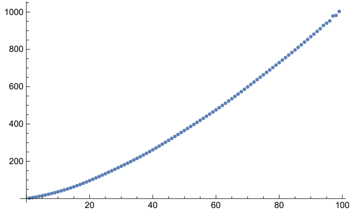
$J_N($  $; q) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \left(q^{(N-k)/2} - q^{-(N-k)/2}\right) \left(q^{(N+k)/2} - q^{-(N+k)/2}\right).$

(K. Habiro, T. Lê)

Colored Jones polynomial at Nth root of unity, 

Colored Jones polynomial at Nth root of unity, [figure-eight knot]

$$\left|J_N\left([\text{figure-eight knot}]; \exp(2\pi\sqrt{-1}/N)\right)\right|$$



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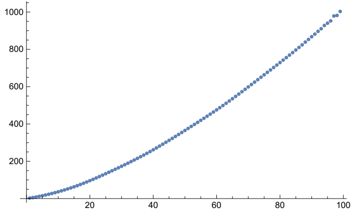
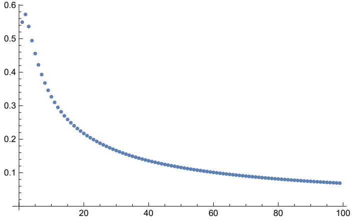


$\frac{1}{N}\log\left|J_N\left(\;;\exp(2\pi\sqrt{-1}/N)\right)\right|$



$$J_N(\;;q) = \frac{(-1)^{N-1}q^{3(N^2-1)/2}}{q^{N/2}-q^{-N/2}}\sum_{k=0}^{N-1}(-1)^k q^{-3(k^2+k)/2}\left(q^{(2k+1)/2}-q^{-(2k+1)/2}\right)$$

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Graph of

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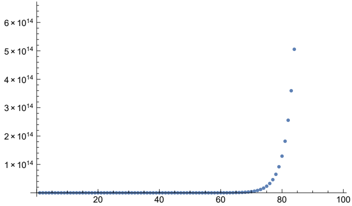

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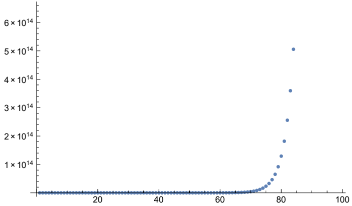

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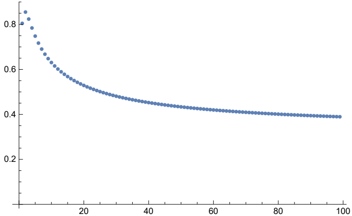
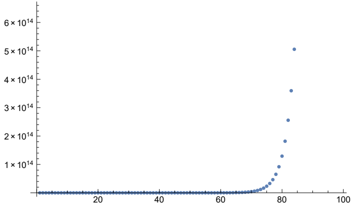
$\frac{1}{N}\log\left|J_N(\;;\exp(2\pi\sqrt{-1}/N))\right|$.



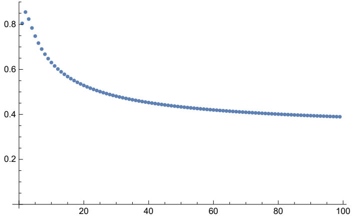
$$J_N(\;;q) = \sum_{j=0}^{N-1}\prod_{k=1}^{j}\left(q^{(N-k)/2}-q^{-(N-k)/2}\right)\left(q^{(N+k)/2}-q^{-(N+k)/2}\right).$$

Colored Jones polynomial at Nth root of unity, 

Graph of $\left|J_N(\right.$  $\left.;\exp(2\pi\sqrt{-1}/N))\right|$.



Graph of $\frac{1}{N}\log\left|J_N(\right.$  $\left.;\exp(2\pi\sqrt{-1}/N))\right|$.



$J_N($  $;q)$

$$= \sum_{j=0}^{N-1}\prod_{k=1}^{j}\left(q^{(N-k)/2}-q^{-(N-k)/2}\right)\left(q^{(N+k)/2}-q^{-(N+k)/2}\right).$$

What is the difference between  and  ?

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev (1997), J. Murakami+H.M. (2001))

K: *knot*

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3 \setminus K).$$

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Definition (Simplicial volume (Gromov norm))

$$\mathrm{Vol}(S^3 \setminus K) := \sum_{H_i:\text{hyperbolic piece}} \text{Hyperbolic Volume of } H_i.$$

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Definition (Simplicial volume (Gromov norm))

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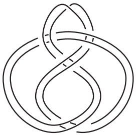
Definition (Jaco–Shalen–Johannson decomposition)

$S^3 \setminus K$ can be uniquely decomposed as

$$S^3 \setminus K = \left(\bigsqcup H_i\right) \sqcup \left(\bigsqcup E_j\right)$$

with H_i hyperbolic and E_j Seifert-fibered.

Example of JSJ decomposition



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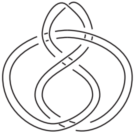

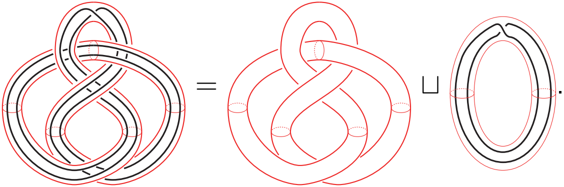

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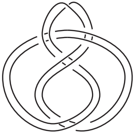

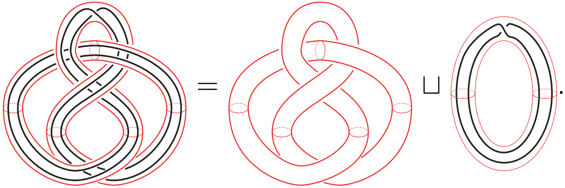

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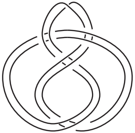

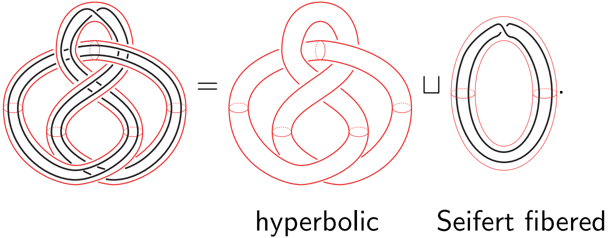

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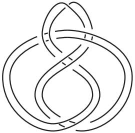
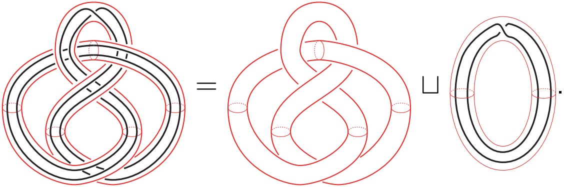
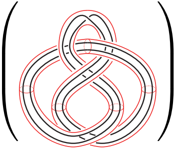
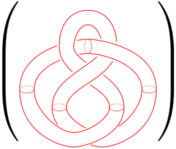




hyperbolic Seifert fibered

Vol  = Vol 

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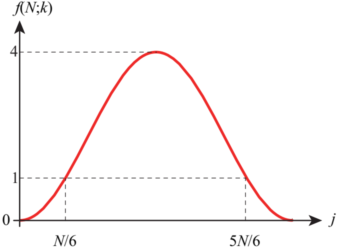

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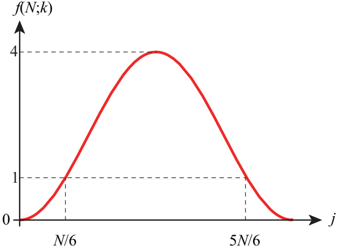

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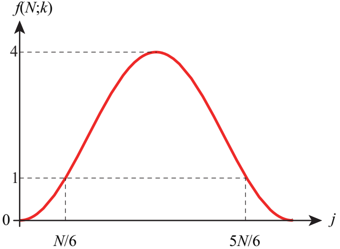

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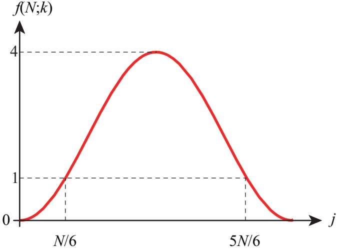

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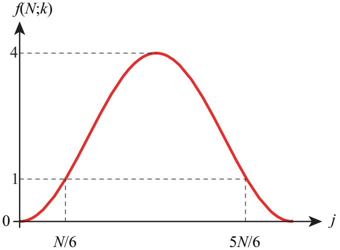

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Decomposition of $S^3 \setminus$  into two tetrahedra

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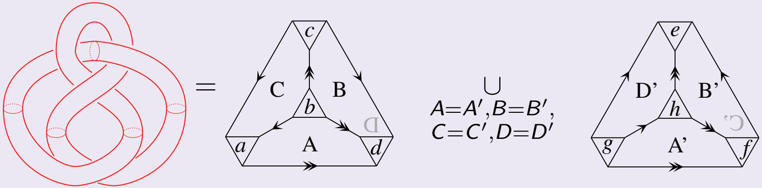

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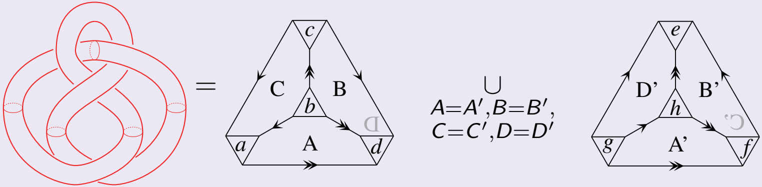

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$\Rightarrow S^3 \setminus$  possesses a complete hyperbolic structure.

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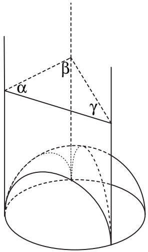
Ideal hyperbolic tetrahedron $\Delta(\alpha, \beta, \gamma)$



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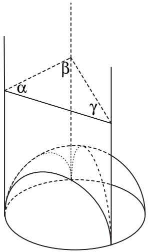
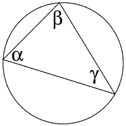


Top view



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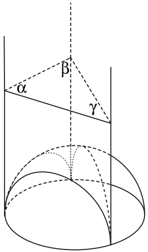

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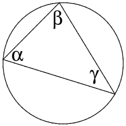

Ideal hyperbolic tetrahedron is defined (up to isometry) by the similarity class of this triangle.

$$\mathrm{Vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

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This amazing fact was first observed by R. Kashaev and proved by T. Ekholm.

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$\Rightarrow \mathrm{Vol}\left(S^3 \setminus\right.$  $\left.\right) = 0$. In fact Kashaev and O. Tirkkonen proved that $2\pi \lim_{N\to\infty} \frac{\log J_N\left(T(p,q);\exp(2\pi\sqrt{-1}/N)\right)}{N} = 0$ for any torus knot $T(p,q)$.

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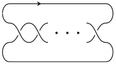
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),

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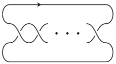
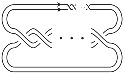
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (Seifert fibered + Seifert fibered) (R. van der Veen (2008)),

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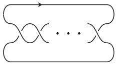
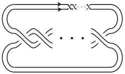
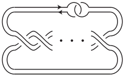
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (Seifert fibered + Seifert fibered) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n+1)$ (Seifert fibered + hyperbolic) (H. Zheng (2007)),

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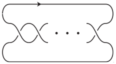
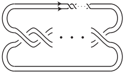
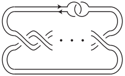
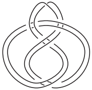
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (Seifert fibered + Seifert fibered) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n+1)$ (Seifert fibered + hyperbolic) (H. Zheng (2007)),
-  $(2, 2m+1)$-cable of the figure-eight knot (hyperbolic + Seifert fibered) (T. Lê and A. Tran (2010)).

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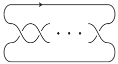
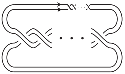
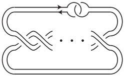
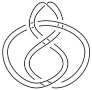
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (Seifert fibered + Seifert fibered) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n+1)$ (Seifert fibered + hyperbolic) (H. Zheng (2007)),
-  $(2, 2m+1)$-cable of the figure-eight knot (hyperbolic + Seifert fibered) (T. Lê and A. Tran (2010)).
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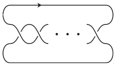
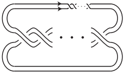
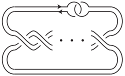
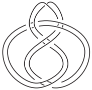
-  torus knots (including ) (Seifert fibered) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (Seifert fibered + Seifert fibered) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n+1)$ (Seifert fibered + hyperbolic) (H. Zheng (2007)),
-  $(2, 2m+1)$-cable of the figure-eight knot (hyperbolic + Seifert fibered) (T. Lê and A. Tran (2010)).
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