

An Introduction to the Volume Conjecture and its generalizations, I

Hitoshi Murakami

Tohoku University 

Workshop on Volume Conjecture and Related Topics in Knot Theory
Indian Institute of Science Education and Research, Pune
19th December, 2018

- 1 colored Jones polynomial
- 2 Examples of the colored Jones polynomials
- 3 Volume conjecture
- 4 Volume conjecture for the figure-eight knot
- 5 VC is proved for ...

Kauffman bracket

Kauffman bracket

Kauffman bracket $\langle D \rangle$ for a knot diagram D is defined as

Kauffman bracket

Kauffman bracket $\langle D \rangle$ for a knot diagram D is defined as

- $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle,$
- $\langle \bigcirc \sqcup D \rangle = (-A^2 - A^{-2}) \langle D \rangle,$
- $\langle \bigcirc \rangle = 1.$

Here \sqcup denotes the disjoint union.

Kauffman bracket

Kauffman bracket $\langle D \rangle$ for a knot diagram D is defined as

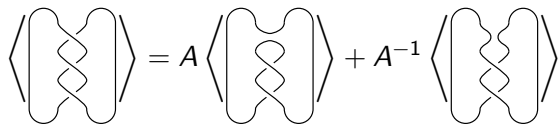
- $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle,$
- $\langle \bigcirc \sqcup D \rangle = (-A^2 - A^{-2}) \langle D \rangle,$
- $\langle \bigcirc \rangle = 1.$

Here \sqcup denotes the disjoint union.

Note: $\langle \underbrace{\bigcirc \sqcup \bigcirc \sqcup \dots \sqcup \bigcirc}_c \rangle = (-A^2 - A^{-2})^{c-1}.$

Kauffman bracket of the trefoil

Kauffman bracket of the trefoil



The diagram illustrates the Kauffman bracket expansion of a trefoil knot. On the left is the trefoil knot, a closed loop with three crossings. This is equal to the sum of two terms: the first term is the trefoil knot with a crossing smoothed in the positive direction (indicated by the coefficient A), and the second term is the trefoil knot with a crossing smoothed in the negative direction (indicated by the coefficient A^{-1}).

$$\langle \text{trefoil} \rangle = A \langle \text{trefoil with positive smoothing} \rangle + A^{-1} \langle \text{trefoil with negative smoothing} \rangle$$

Kauffman bracket of the trefoil

$$\begin{aligned}
 \langle \text{trefoil} \rangle &= A \langle \text{trefoil}_A \rangle + A^{-1} \langle \text{trefoil}_{A^{-1}} \rangle \\
 &= A^2 \langle \text{trefoil}_{A^2} \rangle + \langle \text{trefoil}_A \rangle + \langle \text{trefoil}_{A^{-1}} \rangle + A^{-2} \langle \text{trefoil}_{A^{-2}} \rangle
 \end{aligned}$$

Kauffman bracket of the trefoil

$$\begin{aligned}
 & \langle \text{trefoil} \rangle = A \langle \text{trefoil}_1 \rangle + A^{-1} \langle \text{trefoil}_2 \rangle \\
 & = A^2 \langle \text{trefoil}_3 \rangle + \langle \text{trefoil}_4 \rangle + \langle \text{trefoil}_5 \rangle + A^{-2} \langle \text{trefoil}_6 \rangle \\
 & = A^3 \langle \text{trefoil}_7 \rangle + A \langle \text{trefoil}_8 \rangle + A \langle \text{trefoil}_9 \rangle + A^{-1} \langle \text{trefoil}_{10} \rangle \\
 & + A \langle \text{trefoil}_{11} \rangle + A^{-1} \langle \text{trefoil}_{12} \rangle + A^{-1} \langle \text{trefoil}_{13} \rangle + A^{-3} \langle \text{trefoil}_{14} \rangle .
 \end{aligned}$$

Kauffman bracket of the trefoil

$$\begin{aligned}
& \langle \text{trefoil} \rangle = A \langle \text{trefoil}_1 \rangle + A^{-1} \langle \text{trefoil}_2 \rangle \\
& = A^2 \langle \text{trefoil}_3 \rangle + \langle \text{trefoil}_4 \rangle + \langle \text{trefoil}_5 \rangle + A^{-2} \langle \text{trefoil}_6 \rangle \\
& = A^3 \langle \text{trefoil}_7 \rangle + A \langle \text{trefoil}_8 \rangle + A \langle \text{trefoil}_9 \rangle + A^{-1} \langle \text{trefoil}_{10} \rangle \\
& + A \langle \text{trefoil}_{11} \rangle + A^{-1} \langle \text{trefoil}_{12} \rangle + A^{-1} \langle \text{trefoil}_{13} \rangle + A^{-3} \langle \text{trefoil}_{14} \rangle . \\
& = A^3(-A^2 - A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2}) + A^{-1} \\
& + A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2})
\end{aligned}$$

Kauffman bracket of the trefoil

$$\begin{aligned}
& \langle \text{trefoil} \rangle = A \langle \text{trefoil}_1 \rangle + A^{-1} \langle \text{trefoil}_2 \rangle \\
& = A^2 \langle \text{trefoil}_3 \rangle + \langle \text{trefoil}_4 \rangle + \langle \text{trefoil}_5 \rangle + A^{-2} \langle \text{trefoil}_6 \rangle \\
& = A^3 \langle \text{trefoil}_7 \rangle + A \langle \text{trefoil}_8 \rangle + A \langle \text{trefoil}_9 \rangle + A^{-1} \langle \text{trefoil}_{10} \rangle \\
& + A \langle \text{trefoil}_{11} \rangle + A^{-1} \langle \text{trefoil}_{12} \rangle + A^{-1} \langle \text{trefoil}_{13} \rangle + A^{-3} \langle \text{trefoil}_{14} \rangle . \\
& = A^3(-A^2 - A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2}) + A^{-1} \\
& + A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2}) \\
& = A^7 - A^3 - A^{-5} .
\end{aligned}$$

Jones polynomial

Jones polynomial

Definition (writhe)

\vec{D} : oriented knot diagram.

- $w(\vec{D}) := \# \begin{array}{c} \nearrow \\ \nwarrow \end{array} - \# \begin{array}{c} \nwarrow \\ \nearrow \end{array}$

Jones polynomial

Definition (writhe)

\vec{D} : oriented knot diagram.

- $w(\vec{D}) := \# \begin{array}{c} \nearrow \\ \searrow \end{array} - \# \begin{array}{c} \nwarrow \\ \swarrow \end{array}$

Definition

Jones polynomial

Definition (writhe)

\vec{D} : **oriented** knot diagram.

- $w(\vec{D}) := \# \begin{array}{c} \nearrow \\ \searrow \end{array} - \# \begin{array}{c} \nwarrow \\ \nearrow \end{array}$

Definition

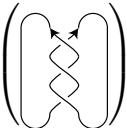
- K : oriented knot presented by \vec{D} .
- D : \vec{D} without orientation.

$$V(K; q) := (-A^3)^{-w(\vec{D})} \langle D \rangle \Big|_{q:=A^{-4}}.$$

$V(K; q)$ is a knot invariant, called the Jones polynomial.

Example of the Jones polynomials

Example of the Jones polynomials

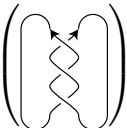
$$w \left(\left(\begin{array}{c} \text{link diagram} \end{array} \right) \right) = -3$$


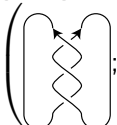
Example of the Jones polynomials

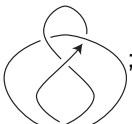
$$w \left(\left(\text{link diagram} \right) \right) = -3$$

$$\Rightarrow J \left(\left(\text{link diagram} \right); q \right) = -A^9(A^7 - A^3 - A^{-5}) \Big|_{q:=A^{-4}} = -q^{-4} + q^{-3} + q^{-1}.$$

Example of the Jones polynomials

$$w \left(\text{link diagram} \right) = -3$$


$$\Rightarrow J \left(\text{link diagram}; q \right) = -A^9(A^7 - A^3 - A^{-5}) \Big|_{q:=A^{-4}} = -q^{-4} + q^{-3} + q^{-1}.$$


$$J \left(\text{link diagram}; q \right) = q^2 - q + 1 - q^{-1} - q^{-2}.$$


Jones–Wenzl idempotent

Jones–Wenzl idempotent

- $$\begin{array}{c} | \\ \boxed{2} \\ | \end{array} \quad | \quad := \quad | \quad - \frac{1}{(-A^2 - A^{-2})} \begin{array}{c} \cup \\ | \\ \cap \end{array}$$

Jones–Wenzl idempotent

•
$$\begin{array}{c} | \\ \boxed{2} \\ | \end{array} := \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \frac{1}{(-A^2 - A^{-2})} \begin{array}{c} \cup \\ \cup \\ \cup \end{array} \Rightarrow \begin{array}{c} | \\ \boxed{2} \\ | \\ \boxed{2} \\ | \end{array} = \begin{array}{c} | \\ \boxed{2} \\ | \end{array}$$

Jones–Wenzl idempotent

$$\bullet \quad \begin{array}{c} | \\ \boxed{2} \\ | \end{array} := \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \frac{1}{(-A^2 - A^{-2})} \begin{array}{c} \cup \\ \cap \end{array} \Rightarrow \begin{array}{c} | \\ \boxed{2} \\ | \\ \boxed{2} \\ | \end{array} = \begin{array}{c} | \\ \boxed{2} \\ | \end{array}$$

Definition (Jones–Wenzl idempotent)

$$\begin{array}{c} | \\ \boxed{k} \\ | \end{array} := \begin{array}{c} | \\ \boxed{k-1} \\ | \end{array} \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \left(\frac{\Delta_{k-2}}{\Delta_{k-1}} \right) \begin{array}{c} | \\ \boxed{k-1} \\ | \\ \cup \\ | \\ \boxed{k-2} \\ | \\ \cap \\ | \\ \boxed{k-1} \\ | \\ | \end{array}$$

$$\text{with } \Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}} = \left\langle \begin{array}{c} \cup \\ \cap \end{array} \right\rangle_k.$$

$$\Rightarrow \begin{array}{c} | \\ \boxed{k} \\ | \\ \boxed{k} \\ | \end{array} = \begin{array}{c} | \\ \boxed{k} \\ | \end{array},$$

Jones–Wenzl idempotent

$$\bullet \quad \begin{array}{c} | \\ \boxed{2} \\ | \end{array} := \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \frac{1}{(-A^2 - A^{-2})} \begin{array}{c} \cup \\ \cap \end{array} \Rightarrow \begin{array}{c} | \\ \boxed{2} \\ \boxed{2} \\ | \end{array} = \begin{array}{c} | \\ \boxed{2} \\ | \end{array}$$

Definition (Jones–Wenzl idempotent)

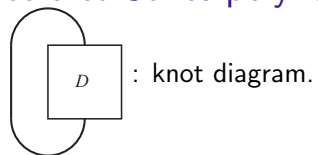
$$\begin{array}{c} | \\ \boxed{k} \\ | \end{array} := \begin{array}{c} | \\ \boxed{k-1} \\ | \end{array} \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \left(\frac{\Delta_{k-2}}{\Delta_{k-1}} \right) \begin{array}{c} | \\ \boxed{k-1} \\ | \end{array} \begin{array}{c} | \\ \cup \\ | \end{array} \begin{array}{c} | \\ \cap \\ | \end{array} \begin{array}{c} | \\ \boxed{k-1} \\ | \end{array}$$

$$\text{with } \Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}} = \left\langle \begin{array}{c} | \\ \boxed{k} \\ | \end{array} \right\rangle.$$

$$\Rightarrow \begin{array}{c} | \\ \boxed{k} \\ \boxed{k} \\ | \end{array} = \begin{array}{c} | \\ \boxed{k} \\ | \end{array}, \quad \begin{array}{c} | \\ \cup \\ | \end{array} \begin{array}{c} | \\ \cap \\ | \end{array} = (-1)^k A^{k^2+2k} \begin{array}{c} | \\ \boxed{k} \\ | \end{array}$$

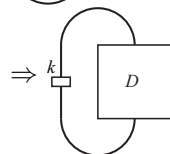
colored Jones polynomial

colored Jones polynomial

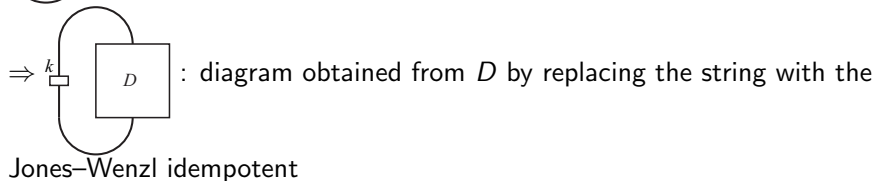
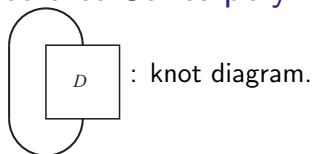


colored Jones polynomial


 D : knot diagram.

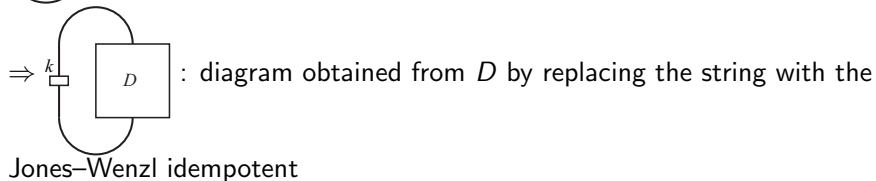

 \Rightarrow k : diagram obtained from D by replacing the string with the Jones–Wenzl idempotent

colored Jones polynomial

Definition (N -colored Jones polynomial)

$$J_N(K; q) := \left((-1)^{N-1} A^{N^2-1} \right)^{-w(\vec{D})} \left\langle \begin{array}{c} N-1 \\ \text{Jones-Wenzl idempotent} \\ D \end{array} \right\rangle \Big|_{q:=A^4} .$$

colored Jones polynomial

Definition (N -colored Jones polynomial)

$$J_N(K; q) := \left((-1)^{N-1} A^{N^2-1} \right)^{-w(\vec{D})} \left\langle \begin{array}{c} N-1 \\ \text{[Diagram: box } D \text{ with loop and idempotent} \end{array} \right\rangle \Big|_{q:=A^4} .$$

$$J_2(K; q) = V(K; q^{-1}).$$

colored Jones polynomials of 

colored Jones polynomials of 

$$J_2(\text{link}; q) = q^1 + q^3 - q^4,$$

$$J_3(\text{link}; q) = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11},$$

$$J_4(\text{link}; q) = q^3 + q^7 - q^{10} + q^{11} - q^{13} - q^{14} + q^{15} - q^{17} + q^{19} \\ + q^{20} - q^{21}$$

$$\vdots$$

$$J_N(\text{link}; q) = \frac{(-1)^{N-1} q^{3(N^2-1)/2}}{q^{N/2} - q^{-N/2}} \\ \times \sum_{k=0}^{N-1} (-1)^k q^{-3(k^2+k)/2} (q^{(2k+1)/2} - q^{-(2k+1)/2}).$$

(M. Rosso, V. Jones, H. Morton)

colored Jones polynomials of 

colored Jones polynomials of 

$$J_2(\text{trefoil}) = q^2 - q + 1 - q^{-1} + q^{-2},$$

$$J_3(\text{trefoil}) = q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 3 - q^{-1} - q^{-2} + 2q^{-3} - q^{-4} \\ - q^{-5} + q^{-6},$$

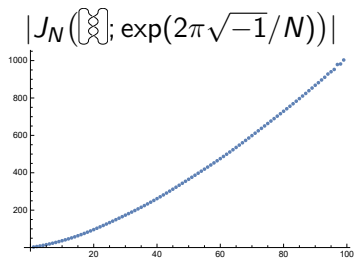
$$J_4(\text{trefoil}) = q^{12} - q^{11} - q^{10} + 2q^8 - 2q^6 + 3q^4 - 3q^2 + 3 - 3q^{-2} + 3q^{-4} \\ - 2q^{-6} + 2q^{-8} - q^{-10} - q^{-11} + q^{-12}$$

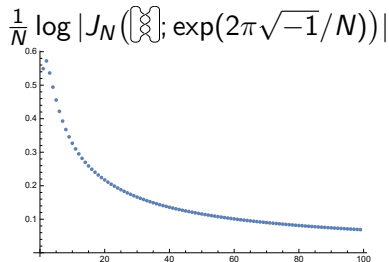
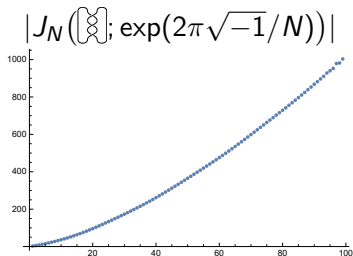
$$\vdots$$

$$J_N(\text{trefoil}; q) = \sum_{j=0}^{N-1} \prod_{k=1}^j \left(q^{(N-k)/2} - q^{-(N-k)/2} \right) \left(q^{(N+k)/2} - q^{-(N+k)/2} \right).$$

(K. Habiro, T. Lê)

Colored Jones polynomial at N th root of unity,

Colored Jones polynomial at N th root of unity, 

Colored Jones polynomial at N th root of unity, 

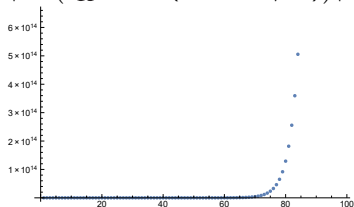
$$\begin{aligned}
 & J_N(\text{trefoil}; q) \\
 &= \frac{(-1)^{N-1} q^{3(N^2-1)/2}}{q^{N/2} - q^{-N/2}} \sum_{k=0}^{N-1} (-1)^k q^{-3(k^2+k)/2} (q^{(2k+1)/2} - q^{-(2k+1)/2})
 \end{aligned}$$

Colored Jones polynomial at N th root of unity,

Colored Jones polynomial at N th root of unity, 

Graph of

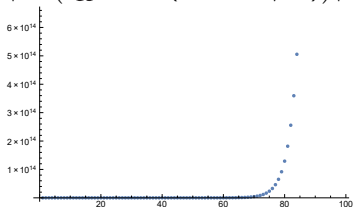
$$|J_N(\text{trefoil}; \exp(2\pi\sqrt{-1}/N))|.$$



Colored Jones polynomial at N th root of unity, 

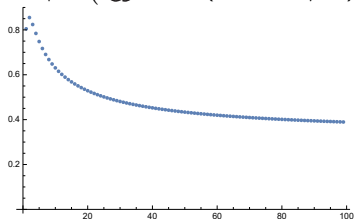
Graph of

$$|J_N(\text{trefoil}; \exp(2\pi\sqrt{-1}/N))|.$$



Graph of

$$\frac{1}{N} \log |J_N(\text{trefoil}; \exp(2\pi\sqrt{-1}/N))|.$$

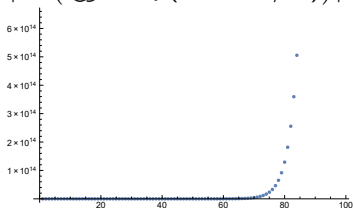


$$J_N(\text{trefoil}; q) = \sum_{j=0}^{N-1} \prod_{k=1}^j (q^{(N-k)/2} - q^{-(N-k)/2}) (q^{(N+k)/2} - q^{-(N+k)/2}).$$

Colored Jones polynomial at N th root of unity, 

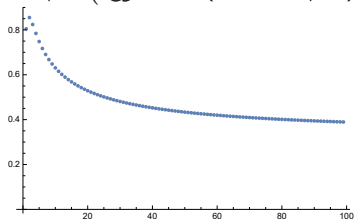
Graph of

$$|J_N(\text{trefoil}; \exp(2\pi\sqrt{-1}/N))|.$$





Graph of

$$\frac{1}{N} \log |J_N(\text{trefoil}; \exp(2\pi\sqrt{-1}/N))|.$$



$$J_N(\text{trefoil}; q) = \sum_{j=0}^{N-1} \prod_{k=1}^j (q^{(N-k)/2} - q^{-(N-k)/2}) (q^{(N+k)/2} - q^{-(N+k)/2}).$$

What is the difference between  and  ?

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev (1997),
J. Murakami+H.M. (2001))

K : knot

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \text{Vol}(S^3 \setminus K).$$

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev (1997),
J. Murakami+H.M. (2001))

K : knot

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \text{Vol}(S^3 \setminus K).$$

Definition (Simplicial volume (Gromov norm))

$$\text{Vol}(S^3 \setminus K) := \sum_{H_i: \text{hyperbolic piece}} \text{Hyperbolic Volume of } H_i.$$

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev (1997),
J. Murakami+H.M. (2001))

K : knot

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \text{Vol}(S^3 \setminus K).$$

Definition (Simplicial volume (Gromov norm))

$$\text{Vol}(S^3 \setminus K) := \sum_{H_i: \text{hyperbolic piece}} \text{Hyperbolic Volume of } H_i.$$

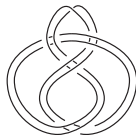
Definition (Jaco–Shalen–Johannson decomposition)

$S^3 \setminus K$ can be uniquely decomposed as

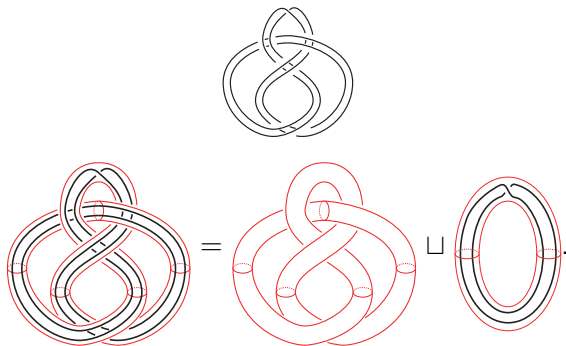
$$S^3 \setminus K = \left(\bigsqcup H_i \right) \sqcup \left(\bigsqcup E_j \right)$$

with H_i hyperbolic and E_j Seifert-fibered.

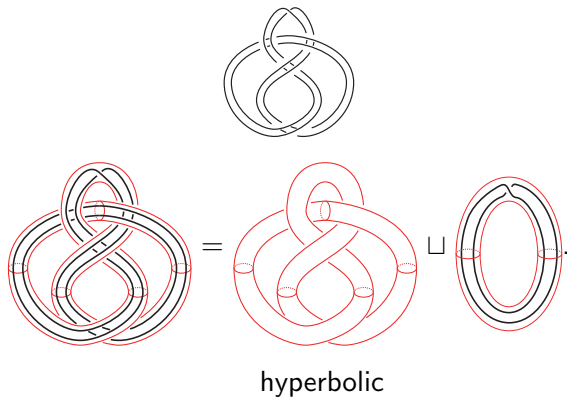
Example of JSJ decomposition



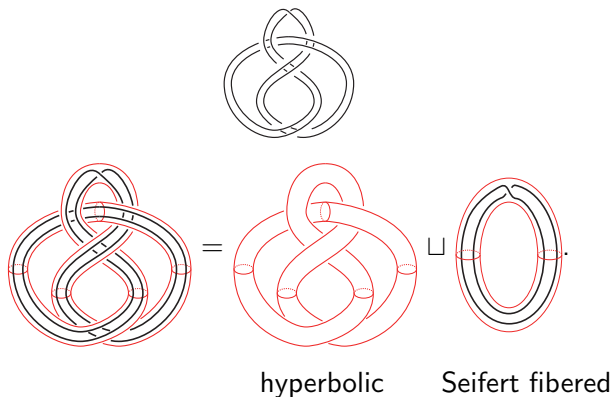
Example of JSJ decomposition



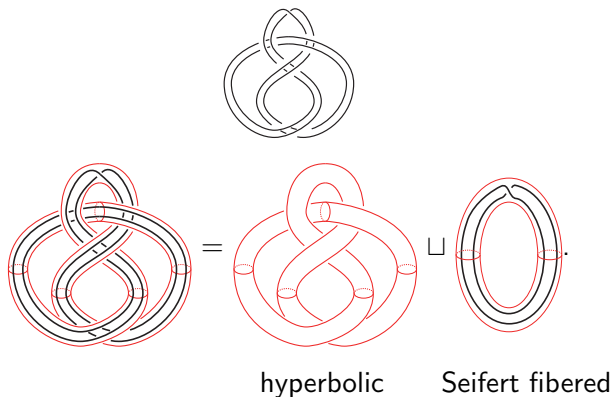
Example of JSJ decomposition



Example of JSJ decomposition




Example of JSJ decomposition




$$\text{Vol} \left(\text{Knot} \right) = \text{Vol} \left(\text{Hyperbolic part} \right)$$

Colored Jones polynomial of

Proof of the VC for  is given by T. Ekholm in 1999.


Colored Jones polynomial of 

Proof of the VC for  is given by T. Ekholm in 1999.

Theorem (K. Habiro, T. Lê)

$$J_N \left(\text{figure-eight knot}; q \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j \left(q^{(N-k)/2} - q^{-(N-k)/2} \right) \left(q^{(N+k)/2} - q^{-(N+k)/2} \right).$$

Colored Jones polynomial of 

Proof of the VC for  is given by T. Ekholm in 1999.

Theorem (K. Habiro, T. Lê)

$$J_N \left(\text{figure-eight knot}; q \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j \left(q^{(N-k)/2} - q^{-(N-k)/2} \right) \left(q^{(N+k)/2} - q^{-(N+k)/2} \right).$$

$$q \mapsto \exp(2\pi\sqrt{-1}/N)$$

$$J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k)$$

with $f(N; k) := 4 \sin^2(k\pi/N)$.

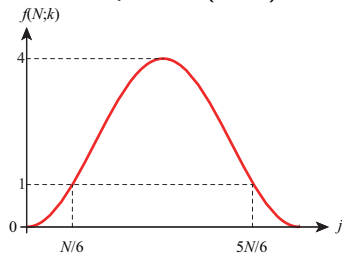
Find the maximum of the summands

Find the maximum of the summands

$$J_N \left(\text{figure-eight knot} ; e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

Find the maximum of the summands

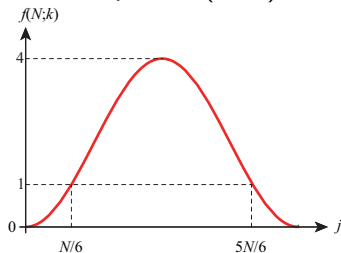
$$J_N \left(\left(\text{figure-eight knot} \right); e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

Graph of $f(N; k)$ 

Find the maximum of the summands

$$J_N \left(\text{figure-eight knot}; e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

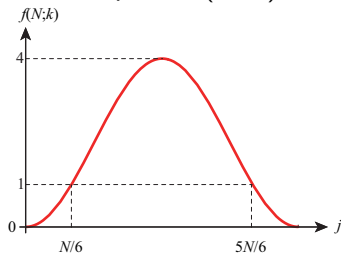
Graph of $f(N; k)$



Put $g(N; j) := \prod_{k=1}^j f(N; k)$.

Find the maximum of the summands

$$J_N \left(\text{figure-eight knot}; e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

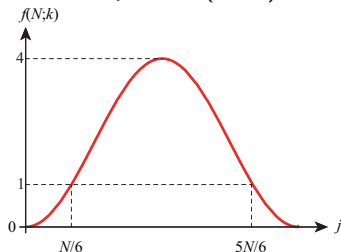
Graph of $f(N; k)$ 

Put $g(N; j) := \prod_{k=1}^j f(N; k)$.

j	0	...	$N/6$...	$5N/6$...	1
-----	---	-----	-------	-----	--------	-----	---

Find the maximum of the summands

$$J_N \left(\text{figure-eight knot}; e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

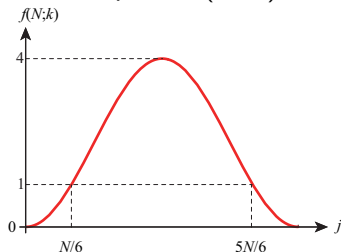
Graph of $f(N; k)$ 

Put $g(N; j) := \prod_{k=1}^j f(N; k)$.

j	0	...	$N/6$...	$5N/6$...	1
$f(N; k)$		< 1	1	> 1	1	< 1	

Find the maximum of the summands

$$J_N \left(\text{figure-eight knot}; e^{2\pi\sqrt{-1}/N} \right) = \sum_{j=0}^{N-1} \prod_{k=1}^j f(N; k) \text{ with } f(N; k) := 4 \sin^2(k\pi/N).$$

Graph of $f(N; k)$ 

Put $g(N; j) := \prod_{k=1}^j f(N; k)$.

j	0	...	$N/6$...	$5N/6$...	1
$f(N; k)$		< 1	1	> 1	1	< 1	
$g(N; j)$	1	\searrow		\nearrow	maximum	\searrow	

Limit of the sum is the limit of the maximum

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

↓

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

$$\Downarrow$$

$$g(N; 5N/6) \leq J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) \leq N \times g(N; 5N/6)$$

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

$$\Downarrow$$

$$g(N; 5N/6) \leq J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) \leq N \times g(N; 5N/6)$$

$$\Downarrow$$

$$\frac{\log g(N; 5N/6)}{N} \leq \frac{\log J_N}{N} \leq \frac{\log N}{N} + \frac{\log g(N; 5N/6)}{N}$$

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

$$\Downarrow$$

$$g(N; 5N/6) \leq J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) \leq N \times g(N; 5N/6)$$

$$\Downarrow$$

$$\frac{\log g(N; 5N/6)}{N} \leq \frac{\log J_N}{N} \leq \frac{\log N}{N} + \frac{\log g(N; 5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \leq \lim_{N \rightarrow \infty} \frac{\log J_N}{N} \leq \lim_{N \rightarrow \infty} \frac{\log N}{N} + \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N}$$

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N; j)\}_{0 \leq j \leq N-1}$ is $g(N; 5N/6)$.
- $J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) = \sum_{j=0}^{N-1} g(N; j)$ and $g(N; j) > 0$.

$$\Downarrow$$

$$g(N; 5N/6) \leq J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) \leq N \times g(N; 5N/6)$$

$$\Downarrow$$

$$\frac{\log g(N; 5N/6)}{N} \leq \frac{\log J_N}{N} \leq \frac{\log N}{N} + \frac{\log g(N; 5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \leq \lim_{N \rightarrow \infty} \frac{\log J_N}{N} \leq \lim_{N \rightarrow \infty} \frac{\log N}{N} + \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N \rightarrow \infty} \frac{\log J_N}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N}$$

Calculation of the limit of the maximum

Calculation of the limit of the maximum

$$\lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot} ; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N}$$

Calculation of the limit of the maximum

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} &= \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) \end{aligned}$$

Calculation of the limit of the maximum

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
 & = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N))
 \end{aligned}$$

Calculation of the limit of the maximum

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N)) \\
&= \frac{2}{\pi} \int_0^{5\pi/6} \log(2 \sin x) dx
\end{aligned}$$

Calculation of the limit of the maximum

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N)) \\
&= \frac{2}{\pi} \int_0^{5\pi/6} \log(2 \sin x) dx = -\frac{2}{\pi} \Lambda(5\pi/6)
\end{aligned}$$

Calculation of the limit of the maximum

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N)) \\
&= \frac{2}{\pi} \int_0^{5\pi/6} \log(2 \sin x) dx = -\frac{2}{\pi} \Lambda(5\pi/6) = 0.323066 \dots,
\end{aligned}$$

Calculation of the limit of the maximum

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N)) \\
&= \frac{2}{\pi} \int_0^{5\pi/6} \log(2 \sin x) dx = -\frac{2}{\pi} \Lambda(5\pi/6) = 0.323066 \dots,
\end{aligned}$$

where $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$ is the Lobachevsky function.

Calculation of the limit of the maximum

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} = \lim_{N \rightarrow \infty} \frac{\log g(N; 5N/6)}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N; k) = 2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2 \sin(k\pi/N)) \\
&= \frac{2}{\pi} \int_0^{5\pi/6} \log(2 \sin x) dx = -\frac{2}{\pi} \Lambda(5\pi/6) = 0.323066 \dots,
\end{aligned}$$

where $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$ is the Lobachevsky function.
 What is $\Lambda(5\pi/6)$?

Lobachevsky function $\Lambda(\theta)$

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2 \sin x \cos x.$$

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2 \sin x \cos x.$$

$$\Rightarrow \log |2 \sin(2x)| = \log |2 \sin x| + \log |2 \sin(x + \pi/2)|. \quad \square$$

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2 \sin x \cos x.$$

$$\Rightarrow \log |2 \sin(2x)| = \log |2 \sin x| + \log |2 \sin(x + \pi/2)|. \quad \square$$

So we have

$$\Lambda(5\pi/6) = -\Lambda(\pi/6)$$

$$\Lambda(\pi/3) = 2\Lambda(\pi/6) + 2\Lambda(2\pi/3) = 2\Lambda(\pi/6) - 2\Lambda(\pi/3)$$

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2 \sin x \cos x.$$

$$\Rightarrow \log |2 \sin(2x)| = \log |2 \sin x| + \log |2 \sin(x + \pi/2)|. \quad \square$$

So we have

$$\Lambda(5\pi/6) = -\Lambda(\pi/6)$$

$$\Lambda(\pi/3) = 2\Lambda(\pi/6) + 2\Lambda(2\pi/3) = 2\Lambda(\pi/6) - 2\Lambda(\pi/3)$$

$$\Rightarrow \Lambda(5\pi/6) = -\frac{3}{2}\Lambda(\pi/3).$$

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda(\theta) := -\int_0^\theta \log |2 \sin x| dx$.

- Λ is an odd function and has period π .
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$.
 ($\Lambda(n\theta) = n \sum_{k=1}^{n-1} \Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2 \sin x \cos x.$$

$$\Rightarrow \log |2 \sin(2x)| = \log |2 \sin x| + \log |2 \sin(x + \pi/2)|. \quad \square$$

So we have

$$\Lambda(5\pi/6) = -\Lambda(\pi/6)$$

$$\Lambda(\pi/3) = 2\Lambda(\pi/6) + 2\Lambda(2\pi/3) = 2\Lambda(\pi/6) - 2\Lambda(\pi/3)$$

$$\Rightarrow \Lambda(5\pi/6) = -\frac{3}{2}\Lambda(\pi/3).$$

$$\Rightarrow 2\pi \lim_{N \rightarrow \infty} \log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right) / N = 6\Lambda(\pi/3)$$

Decomposition of $S^3 \setminus \text{\textcircled{8}}$ into two tetrahedra

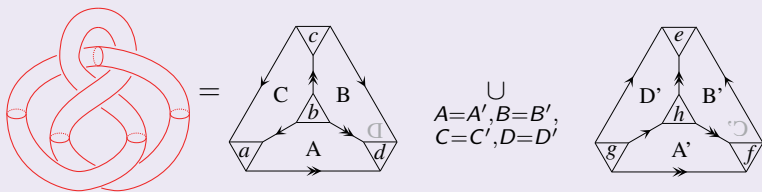
Decomposition of $S^3 \setminus \text{figure-eight knot}$ into two tetrahedra

What is $6\Lambda(\pi/3)$?

Decomposition of $S^3 \setminus \text{figure-eight knot}$ into two tetrahedra

What is $6\Lambda(\pi/3)$?

Theorem (W. Thurston)

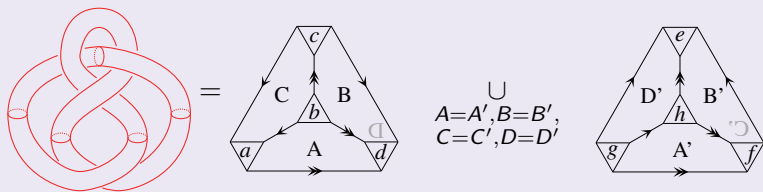


We can regard both pieces in the right hand side as regular ideal hyperbolic tetrahedra.

Decomposition of $S^3 \setminus \text{figure-eight knot}$ into two tetrahedra

What is $6\Lambda(\pi/3)$?

Theorem (W. Thurston)



We can regard both pieces in the right hand side as regular ideal hyperbolic tetrahedra.

$\Rightarrow S^3 \setminus \text{figure-eight knot}$ possesses a complete hyperbolic structure.

Ideal hyperbolic tetrahedron

Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.

Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.
- Ideal hyperbolic tetrahedron: tetrahedron with geodesic faces with four vertices in the boundary at infinity.

Ideal hyperbolic tetrahedron

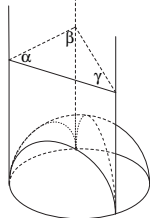
- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.
 - Ideal hyperbolic tetrahedron: tetrahedron with geodesic faces with four vertices in the boundary at infinity.
 - We may assume
 - ▶ One vertex is at ∞ .
 - ▶ The other three are on xy -plane.
-

Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.
- Ideal hyperbolic tetrahedron: tetrahedron with geodesic faces with four vertices in the boundary at infinity.
- We may assume
 - ▶ One vertex is at ∞ .
 - ▶ The other three are on xy -plane.

Ideal hyperbolic
tetrahedron

$\Delta(\alpha, \beta, \gamma)$

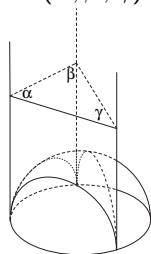


Ideal hyperbolic tetrahedron

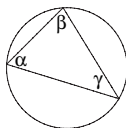
- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.
- Ideal hyperbolic tetrahedron: tetrahedron with geodesic faces with four vertices in the boundary at infinity.
- We may assume
 - ▶ One vertex is at ∞ .
 - ▶ The other three are on xy -plane.

Ideal hyperbolic
tetrahedron

$\Delta(\alpha, \beta, \gamma)$



Top view



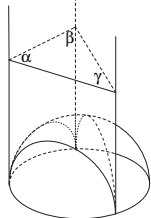
Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, t) \mid t > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$.
- Ideal hyperbolic tetrahedron: tetrahedron with geodesic faces with four vertices in the boundary at infinity.
- We may assume
 - ▶ One vertex is at ∞ .
 - ▶ The other three are on xy -plane.

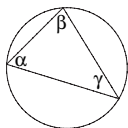
Ideal hyperbolic

tetrahedron

$\Delta(\alpha, \beta, \gamma)$



Top view



Ideal hyperbolic tetrahedron is defined (up to isometry) by the similarity class of this triangle.

$$\text{Vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Proof of VC - conclusion

Proof of VC - conclusion

$$2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot} ; \exp(2\pi\sqrt{-1}/N) \right)}{N}$$

Proof of VC - conclusion

$$2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot} ; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\ = 6\Lambda(\pi/3)$$

Proof of VC - conclusion

$$\begin{aligned}
& 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot} ; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\
&= 6\Lambda(\pi/3) \\
&= 2 \text{Vol}(\text{regular ideal hyperbolic tetrahedron})
\end{aligned}$$

Proof of VC - conclusion

$$\begin{aligned}
& 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\
&= 6\Lambda(\pi/3) \\
&= 2 \text{Vol}(\text{regular ideal hyperbolic tetrahedron}) \\
&= \text{Vol} \left(S^3 \setminus \text{figure-eight knot} \right).
\end{aligned}$$

Proof of VC - conclusion


$$\begin{aligned}
& 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\
&= 6\Lambda(\pi/3) \\
&= 2 \text{Vol}(\text{regular ideal hyperbolic tetrahedron}) \\
&= \text{Vol} \left(S^3 \setminus \text{figure-eight knot} \right).
\end{aligned}$$

This amazing fact was first observed by R. Kashaev and proved by T. Ekholm.

Proof of VC - conclusion

$$\begin{aligned}
& 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\
&= 6\Lambda(\pi/3) \\
&= 2 \text{Vol}(\text{regular ideal hyperbolic tetrahedron}) \\
&= \text{Vol} \left(S^3 \setminus \text{figure-eight knot} \right).
\end{aligned}$$

This amazing fact was first observed by R. Kashaev and proved by T. Ekholm.

On the other hand the complement of  is a Seifert fibered space, that is, it has a geometry of surface \times circle.

Proof of VC - conclusion

$$\begin{aligned}
& 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(\text{figure-eight knot}; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\
&= 6\Lambda(\pi/3) \\
&= 2 \text{Vol}(\text{regular ideal hyperbolic tetrahedron}) \\
&= \text{Vol} \left(S^3 \setminus \text{figure-eight knot} \right).
\end{aligned}$$

This amazing fact was first observed by R. Kashaev and proved by T. Ekholm.

On the other hand the complement of figure-eight knot is a Seifert fibered space, that is, it has a geometry of surface \times circle.

$\Rightarrow \text{Vol} \left(S^3 \setminus \text{figure-eight knot} \right) = 0$. In fact Kashaev and O. Tirkkonen proved that


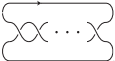

$$2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(T(p, q); \exp(2\pi\sqrt{-1}/N) \right)}{N} = 0 \text{ for any torus knot } T(p, q).$$

So far the Volume Conjecture is proved for


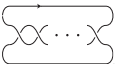

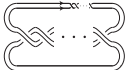
So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))


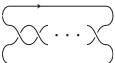

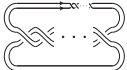

So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),


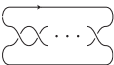


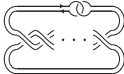

So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (**Seifert fibered + Seifert fibered**) (R. van der Veen (2008)),


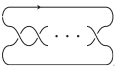

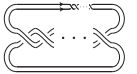
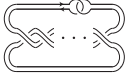


So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (**Seifert fibered + Seifert fibered**) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n + 1)$ (**Seifert fibered + hyperbolic**) (H. Zheng (2007)),


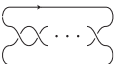


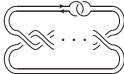



So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (**Seifert fibered + Seifert fibered**) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n + 1)$ (**Seifert fibered + hyperbolic**) (H. Zheng (2007)),
-  $(2, 2m + 1)$ -cable of the figure-eight knot (**hyperbolic + Seifert fibered**) (T. Lê and A. Tran (2010)).

So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (**Seifert fibered + Seifert fibered**) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n + 1)$ (**Seifert fibered + hyperbolic**) (H. Zheng (2007)),
-  $(2, 2m + 1)$ -cable of the figure-eight knot (**hyperbolic + Seifert fibered**) (T. Lê and A. Tran (2010)).
-  5_2 knot (**hyperbolic**) (R. Kashaev and Y. Yokota, T. Oh-tsuki)

So far the Volume Conjecture is proved for

-  figure-eight knot (**hyperbolic**) (T. Ekholm (1999))
-  torus knots (including ) (**Seifert fibered**) (R. Kashaev + O. Tirkkonen (2000)),
-  iterated torus knots (**Seifert fibered + Seifert fibered**) (R. van der Veen (2008)),
-  Whitehead doubles of torus knots of $(2, 2n + 1)$ (**Seifert fibered + hyperbolic**) (H. Zheng (2007)),
-  $(2, 2m + 1)$ -cable of the figure-eight knot (**hyperbolic + Seifert fibered**) (T. Lê and A. Tran (2010)).
-  5_2 knot (**hyperbolic**) (R. Kashaev and Y. Yokota, T. Ohtsuki)
-  **hyperbolic knots** with six crossings (T. Ohtsuki and Y. Yokota)