Spherical Triangles and Girard's Theorem

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Spherical geometry

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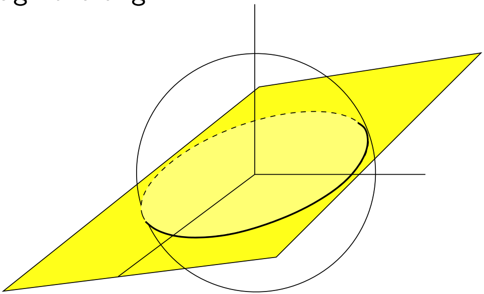

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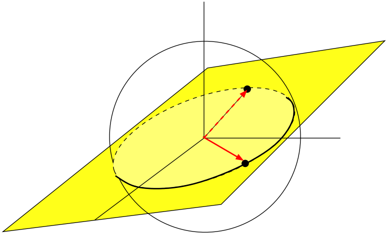

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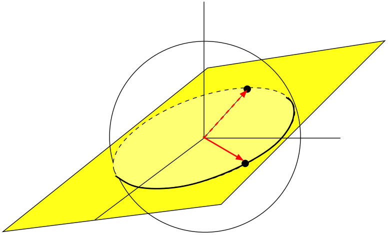

You can similarly verify the other three Euclid's posulates for geometry.

Diangles

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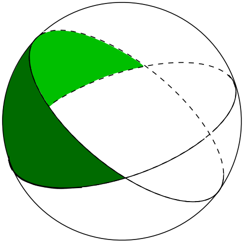

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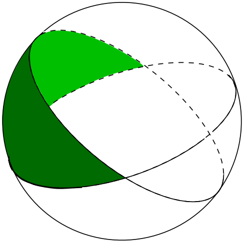

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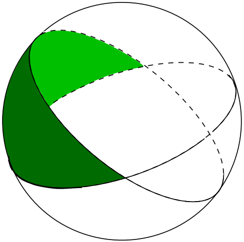

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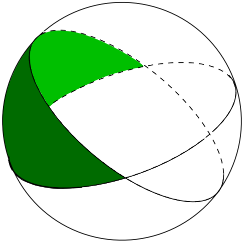

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The angle at both the vertices are equal. Both diangles bounded by two great circles are congruent to each other.

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If the radius of the sphere is r then the area of the diangle is $2\theta r^2$.

This is very similar to the formula for the length of an arc of the unit circle which subtends an angle θ is θ.

Spherical polygons

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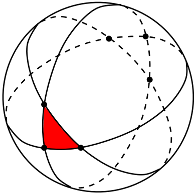
More Examples. Take ballon, ball and draw on it.



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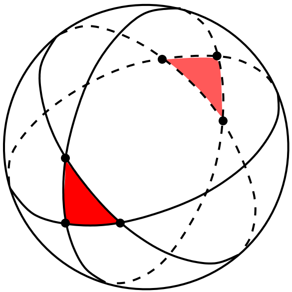

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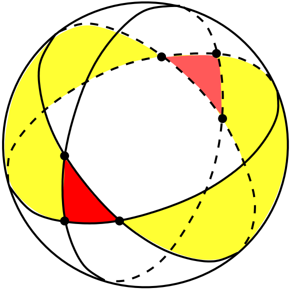

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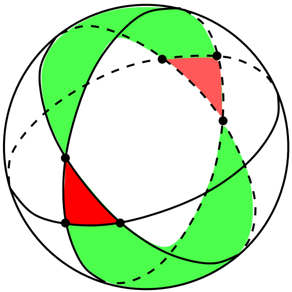

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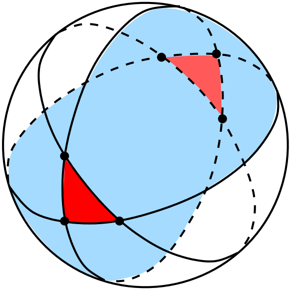

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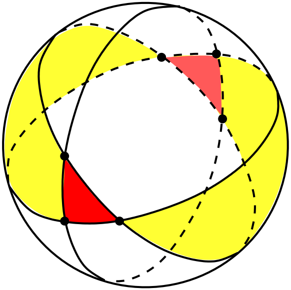

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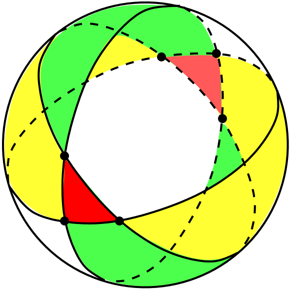

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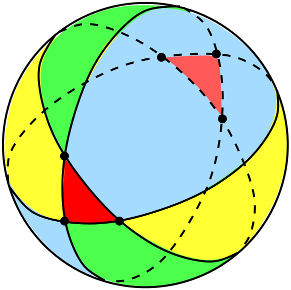

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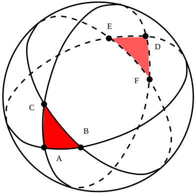

$\triangle ABC$ as shown above is formed by the intersection of three great circles.

Vertices A and D are antipodal to each other and hence have the same angle. Similarly for vertices B, E and C, F. Hence the triangles $\triangle ABC$ and $\triangle DEF$ are antipodal (opposite) triangles and have the same area.

Assume angles at vertices A, B and C to be α, β and γ respectively.

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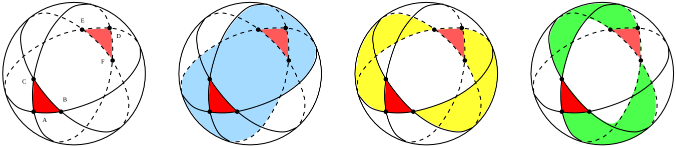

$\triangle ABC$ $\qquad$ R_{AD} $\qquad$ R_{BE} $\qquad$ R_{CF}

Let R_{AD}, R_{BE} and R_{CF} denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.

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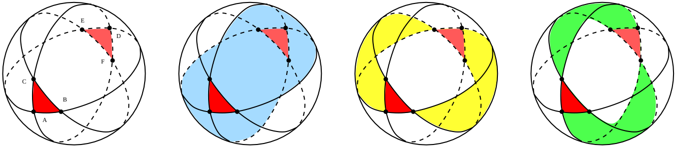

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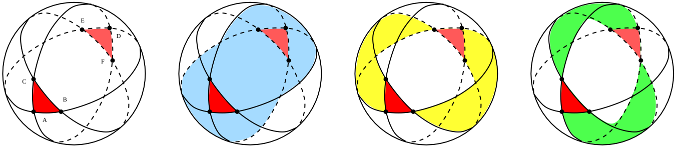

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$$
\begin{aligned}
\mathrm{Area}(S^2) &= \mathrm{Area}(R_{AD}) + \mathrm{Area}(R_{BE}) + \mathrm{Area}(R_{CF}) - 4X \\
4\pi &= 4\alpha + 4\beta + 4\gamma - 4X \\
X &= \alpha + \beta + \gamma - \pi
\end{aligned}
$$

Area of a spherical polygon

Corollary

Let R be a spherical polygon with n vertices and n sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\mathrm{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

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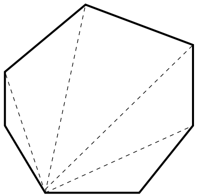
Proof: Any polygon with n sides for $n \geq 4$ can be divided into $n-2$ triangles.



The result follows as the angles of these triangles add up to the interior angles of the polygon. ∎