Options on Maxima, Drawdown, Trading Gains, and Local Time

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Abstract

Suppose that an investor buys a risky asset at time 0 for $S_0$ dollars. Ignoring compensation for risk and time value of money, the expected profit from a later sale is zero in an efficient market. However, suppose that the investor has perfect foresight of the entire spot price path to a fixed time $T$. Then the expected (and actual) profit from the later sale rises to $M_T - S_0 \geq 0$, where $M_T \equiv \max_{t \in [0,T]} S_t$ is the continuously monitored maximum over $[0,T]$. In the absence of this foresight, the drawdown $D_T \equiv M_T - S_T$ captures the ex post regret from selling the asset for $S_T$ at $T$, rather than selling it when its maximum price $M_T$ was attained. In this paper, we develop new model-free exact hedges for calls paying $(M_T - K_m)^+, (D_T - K_d)^+$, and even $(M_T - K_m)^+ \times (D_T - K_d)^+$. In general, the hedge uses static positions in both standard and barrier options. Since barrier options are not yet liquid in many markets, we also impose some structure on the underlying price dynamics under which hedging involves occasional trading in just standard options. This structure further permits options on local time and options on the gains from binary trading strategies to be semi-statically hedged using standard options.

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I Introduction

Let $S_t$ denote the spot price of some asset which can be monitored continuously over a fixed time interval $[0, T]$. Let $M_T \equiv \max_{t \in [0,T]} S_t$ be the continuously-monitored maximum of this asset price over $[0, T]$. Let $D_T \equiv M_T - S_T$ be the terminal drawdown or just “drawdown” for brevity. The maximum $M_T$ is a measure of the reward for being long an asset over $[0, T]$, while the drawdown $D_T$ is a measure of the risk experienced from this position over $[0, T]$.

A call on the maximum with payoff $(M_T - K_m)^+$ is clearly a bet on the upside of the underlying asset, which limits the maximum loss for the call buyer to the initial premium. A call on the drawdown with payoff $(D_T - K_d)^+$ provides insurance for the call buyer against large drawdown realizations, with the maximum loss again limited to the initial premium. We develop new model-free exact hedges for each of these calls. We can thereby infer the risk-neutral density of each of these underlyings, and hence value any claim on $M_T$ or $D_T$.

In fact, we more generally develop a new model-free exact hedge for a claim which pays the product $(M_T - K_m)^+(D_T - K_d)^+$ at $T$. This bivariate payoff allows us to infer the joint risk-neutral density of $D_T$ and $M_T$ and thereby value any claim on this pair of random variables. The univariate results are then obtained as special cases.

Our model-free exact hedge requires static positions in standard and barrier options. Since we recognize that barrier options are not yet liquid in many markets, we also employ various continuity and symmetry assumptions in order to express barrier options in terms of co-terminal standard options. The net result of applying these assumptions is that (the product of) calls on the maximum and drawdown can be hedged using just standard options. The hedge is semi-robust in that the instantaneous volatility process need not be known, although the symmetry assumptions appear to require that it evolves independently of the Brownian motion(s) driving price.

Once the problem of pricing and hedging exotic options has been reduced to the problem of pricing
and hedging standard options, one can propose further process restrictions under which standard options are replicated via dynamic trading in their underlying asset. To save space, we leave the development of these classical non-robust hedges as an exercise for the reader. As usual, the choice of which kind of hedging strategy to use ultimately depends on the nature of the market.

Recall that we use continuity and symmetry assumptions to reduce barrier options on one share to co-terminal standard options on one share. These assumptions also apply when the underlying of the barrier or standard option is generalized to be the gains from a binary trading strategy, in which shareholdings oscillate randomly between long one share and short one share. As a result, barrier options on such gains processes can be reduced to co-terminal standard options on these gains processes, which can be further reduced to co-terminal standard options written on a static position in one share. The net result of these continuity and symmetry assumptions is that all standard and some exotic options written on the gains from binary trading strategies can be semi-statically hedged using just co-terminal standard options on the underlying asset.

When the binary trading strategy defining the underlying gains process is a particular mean-reverting strategy, then Skorohod’s lemma can be used to hedge and price calls on the local time of the risky asset at its initial level. In fact, this lemma can be used to price the product of this call payoff with the payoff from a call written on the absolute deviation of the terminal price from its initial level. Hence, the joint risk-neutral distribution of local time and this absolute deviation can be inferred from market prices of standard options. The univariate densities of local time and the absolute deviation arise as special cases.

Calls on the maximum and calls on trading gains have occasionally been offered in the industry and are known respectively as lookback and passport options. To our knowledge, no financial intermediary has yet offered options on local time or options on drawdown. However, we argue that both payoff structures represent the next natural step in an ongoing evolution.

Local time is related to the quadratic variation of the underlying process, which finds its financial realization in the form of variance swaps. Variance swaps are over the counter contracts in which one
counterparty pays a fixed amount at expiry and receives in return the quadratic variation (discretely monitored) of the returns of some specified underlying over the contract’s life. Corridor variance swaps have also appeared in which the floating side only receives the increase in the quadratic variation of returns if the underlying stock price is in a corridor. Dividing this floating side payoff by the corridor width and sending the latter to zero produces the local time of the underlying process. The local time of the stock price at the strike price is the hedging error that arises if one attempts to hedge a long straddle with a naive binary delta hedge. Hence, options on local time can be used to keep this hedging error under control.

We also believe that there exists a substantial demand for long positions in drawdown calls, at least partially because this demand can be driven by agents whose utility functions are not directly impacted by drawdown. For example, consider a risk-neutral asset manager who knows in advance that his portfolio risk is being evaluated wholly or in part by the portfolio’s drawdown. In practice, managers who experience large drawdowns often see their funds under management dramatically reduced. Conversely, a portfolio manager whose drawdown realizes to less than expected can probably anticipate fund inflows. However, it is commonly reported that funds flow out by more to a given positive shock to drawdown than they flow in for a negative drawdown shock of the same size. Since performance fees are typically proportional to funds under management, fees have the same asymmetric response to drawdown shocks as fund flows. Given this asymmetric response of performance fees to drawdown shocks, we anticipate that calls on drawdown would again be an attractive investment vehicle for portfolio managers as well.

Since passport and lookback options have traded over the counter, there is much related literature on these exotics. In one of the pioneering applications of group theory to finance, Lipton[15] first points out that passport options and lookback options enjoy the same valuation formulas in the context of the Black Scholes model. The reduction of passport options to lookback options via local time is also pointed out in interesting papers by Henderson and Hobson[13] and by Delbaen and Yor[8]. The current paper differs from those publications by its explicit delineation of robust and semi-robust hedges of these exotics achieved by static and semi-static trading in barrier and standard options. There is much less literature on this topic compared to the valuation of these exotics in particular models. The first paper on hedging

The structure of the remainder of this paper is as follows. The next section presents the robust hedging strategy and pricing formula for the claim paying the product of a call on the maximum and a call on drawdown. The next section introduces weak continuity and symmetry conditions which permit the sale of this product call to be hedged by semi-static trading in just standard options. The following section shows that path-independent options written on the gains from binary trading strategies can be semi-robustly hedged using semi-static trading in standard options. The following section extends this result to barrier options on gains and to calls on the maximum and/or drawdown of gains processes. In the second to last section, we show that the maximum of a particular gains process is proportional to the number of crosses of a given spatial interval. As a result, the semi-static hedge for a call on the maximum of this gains process also synthesizes the payoff to a call on crosses. As we shrink the width of the spatial interval down to zero, we show in the penultimate section how Skorohod’s lemma can be used to find semi-robust hedges for calls on local time. The final section summarizes the paper and presents suggestions for future research.

II Robust Pricing of Calls on Maxima and Drawdown

II-A Product Option Algebra

Recall that \( M_T \equiv \max_{t \in [0, T]} S_t \) is the maximum over \([0, T]\) and \( D_T \equiv M_T - S_T \) is the drawdown over \([0, T]\). Let \( C_{t}^{md}(K_m, K_d, T) \) denote the value at time \( t \in [0, T] \) of a claim paying off:

\[
C_{t}^{md}(K_m, K_d, T) = (M_t - K_m)^+ (D_t - K_d)^+ \tag{1}
\]
at its maturity date $T > 0$. Since $M_T$ and $D_T$ are both nonnegative random variables, we assume that the fixed strikes $K_m$ and $K_d$ in (1) are also nonnegative. Since the two univariate call payoffs $(M_T - K_m)^+$ and $(D_T - K_d)^+$ are multiplied to produce the payoff in (1), we henceforth refer to this payoff as that of a product call.

Each of the univariate call payoffs $(M_T - K_m)^+$ and $(D_T - K_d)^+$ can be obtained from the product payoff in (1). To see why, notice that:

$$- \frac{\partial}{\partial K_m} C_{T}^{md}(K_m, K_d, T) = 1(M_T > K_m) (D_T - K_d)^+$$

is the payoff from an up-and-in drawdown call with barrier $K_m$. Setting $K_m = 0$ in (2) leads to:

$$- \frac{\partial}{\partial K_m} C_{T}^{md}(K_m, K_d, T) \bigg|_{K_m=0} = (D_T - K_d)^+ \equiv C_{T}^{d}(K_d, T),$$

which is the payoff from a drawdown call with strike $K_d$. Likewise:

$$- \frac{\partial}{\partial K_d} C_{T}^{md}(K_m, K_d, T) \bigg|_{K_d=0} = (M_T - K_m)^+ \equiv C_{T}^{d}(K_m, T)$$

is the payoff from a call on the maximum with strike $K_m$.

We can use put call parity to replace either call in the product payoff (1) with a put. For example, suppose that we wish to value the product of a put on the maximum with a call on drawdown. Let $PC_t^{md}(K_m, K_d, T)$ be the arbitrage-free price at $t \in [0, T]$ of this claim which pays off $(K_m - M_T)^+(D_T - K_d)^+$ at its expiry $T$. Evaluating (1) at $K_m = 0$ implies:

$$C_{T}^{md}(0, K_d, T) = M_T(D_T - K_d)^+. \quad (5)$$

Multiplying (3) by $K_m$, subtracting (1) from this product, and adding (1) to this difference generates the desired payoff:

$$PC_t^{md}(K_m, K_d, T) = - \frac{\partial}{\partial K_m} C_t^{md}(K_m, K_d, T) \bigg|_{K_m=0} \times K_m - C_{t}^{md}(0, K_d, T) + C_t^{md}(K_m, K_d, T), \quad (6)$$

since at $t = T$:

$$(K_m - M_T)^+(D_T - K_d)^+ = K_m(D_T - K_d)^+ - M_T(D_T - K_d)^+ + (M_T - K_m)^+(D_T - K_d)^+. \quad (7)$$
As a result, any of the 4 combinations involving puts and calls can be generated from any one. In the
remainder of this paper, we will therefore focus on the product call payoff (1) without loss of generality.

Let \( V_{f_t} \) denote the arbitrage-free value at \( t \in [0, T] \) of a path-independent claim paying \( f(S_T) \) at \( T \). As a consequence of the pioneering work of Breeden and Litzenberger\([3]\), we have that:

\[
V_{f_t} = f(0)B_t(T) + f'(0)C_t(0,T) + \int_0^\infty f''(K)C_t(K,T)dK,
\]

provided that the indicated function values and integral exists. Now let \( V_{g_t} \) denote the arbitrage-free value at \( t \in [0, T] \) of a path-independent claim paying \( g(M_T, D_T) \) at \( T \). The analogous result in terms of calls on the maximum, calls on drawdown, and product calls is:

\[
V_{g_t} = g(0,0)B_t(T) + g_m(0,0)C_{t}^{m}(0,T) + g_d(0,0)C_{t}^{d}(0,T) \\
+ \int_0^\infty D_{K_m}^2g(K_m,0)C_{t}^{m}(K_m,0,T)dK_m + \int_0^\infty D_{K_m}^2D_{K_d}^2g(K_m,K_d)C_{t}^{md}(K_m,K_d,T)dK_m dK_d,
\]

provided once again that the indicated function values and integrals exist. If they do not, eg. for \( g(M, D) = \ln M \ln D \), then it is possible to generalize (9) to use puts and thereby expand the space of functions spanned by static positioning in product options. We remark that all of the results in this section hold for any pair of nonnegative random variables, not just \( M_T \) and \( D_T \). Furthermore, there are extensions of these results to pairs of real-valued random variables, which we do not pursue here.

II-B Robust Price and Hedge for Product Call

We assume frictionless markets and no arbitrage for the remainder of the paper. Consider the problem of finding a trading strategy in some given set of assets which is non-anticipating, self-financing, and which replicates some given target payoff. Such a trading strategy is said to be robust if these three properties all hold irrespective of the dynamics of the traded assets. The only dynamical assumption is that the initial market prices of the hedge instruments are known with certainty and that the final market prices of these
instruments equate to intrinsic value at expiry. In the remainder of this section, we present a robust trading strategy in standard and barrier options which is non-anticipating, self-financing, and which replicates the payoff of a product call.

We anticipate that there is more market interest in either of the univariate calls than in the bivariate payoff in (1). We will nonetheless develop a pricing formula for the product call, since only it can be used to infer the joint risk-neutral density of drawdown and the maximum. The pricing formula and hedging strategy for the product call determines the corresponding quantities for a drawdown call and a call on the maximum. Given the greater interest in the two special cases, we will elucidate pricing formulas and hedging strategies for each of the univariate payoffs.

Let $B_t(T)$ be the price of a default-free zero coupon bond paying one dollar with certainty at $T$. We assume that $B_t > 0$ for all $t \in [0, T]$ and hence no arbitrage implies the existence of a probability measure $Q$ associated with this numeraire. The measure $Q$ is equivalent to the statistical probability measure and hence is usually referred to as an equivalent martingale measure. Under $Q$, the ratios of non-dividend paying asset prices to $B$ are martingales. As a result, we have the following theoretical representation for the time $t$ value of a product call:

$$C_t^{md}(K_m, K_d, T) = B_t(T)E_t^Q[(M_T - K_m)^+(D_T - K_d)^+]$$.

In this section, we will price the product call relative to standard puts, binary puts, up-and-in puts, and up-and-in binary puts. Accordingly, let $P_t(K_p, T) = B_t(T)E_t^Q(K_p - S_T)^+$ denote the value at time $t \in [0, T]$ of a European put with strike $K_p \in \mathbb{R}$ and maturity $T \geq 0$. Let $BP_t(K_b, T) = B_t(T)E_t^Q1(S_T < K_b)$ denote the value at time $t \in [0, T]$ of a European binary put with strike $K_b \in \mathbb{R}$ and maturity $T \geq 0$. Let $\tau_H$ be the first passage time of the process $S$ to a barrier $H > S_0$. Let $UIP_t(K_u, T; H) = B_t(T)E_t^Q1(M_T > H)(K_u - S_T)^+$ denote the value at time $t \in [0, \tau_H]$ of an up-and-in put with strike $K_u \in \mathbb{R}$, maturity $T \geq t$ and barrier $H \geq S_0$. Finally, let $UIBP_t(K_u, T; H) = B_t(T)E_t^Q1(M_T > H, S_T < K_u)$ denote the value at time $t \in [0, \tau_H]$ of an up-and-in binary put with strike $K_u$, maturity $T \geq t$, and barrier $H \geq S_0$.

**Theorem 1: Robust Pricing of Product Call**
Under frictionless markets, no arbitrage implies that for \( t \in [0, T] \), we have \( C_t^{\text{mod}}(K_m, K_d, T) = (M_t - K_m)^+ P_t(M_t - K_d, T) + \int_{M_t \vee K_m}^{\infty} (H - K_m) U I B P_t(H - K_d, T; H) dH + \int_{M_t \vee K_m}^{\infty} U I P_t(H - K_d, T; H) dH. \) (10)

**Proof:** Let \( \gamma_t(K_m, K_d, T) = \frac{C_t^{\text{mod}}(K_m, K_d, T)}{B_t(T)} \) be the forward price at time \( t \in [0, T] \) of the product call. Since \( \gamma \) is a \( Q \) martingale:

\[
\gamma_t(K_m, K_d, T) = E_t^Q \gamma_T(K_m, K_d, T) = E_t^Q (M_T - K_m)^+(D_T - K_d)^+ = E_t^Q (M_T - K_m)^+(M_T - S_T - K_d)^+ = E_t^Q \int_{M_t \vee K_m}^{\infty} \delta(M_T - H)(H - K_m)(H - S_T - K_d)^+ dH, \tag{11}
\]

where \( \delta(\cdot) \) denotes a Dirac delta function. Using integration by parts, let:

\[
u = (H - K_m)(H - S_T - K_d)^+ \quad dv = \delta(M_T - H) dH
\]

\[
 du = [(H - S_T - K_d)^+ + (H - K_m)1(S_T < H - K_d)] dH \quad v = -1(M_T \geq H).
\tag{12}
\]

Hence:

\[
\gamma_t(K_m, K_d, T) = E_t^Q \left\{ -1(M_T \geq H)(H - K_m)(H - S_T - K_d)^+ \bigg|_{H = \infty}^{H = M_t \vee K_m} \right. \\
+ \int_{M_t \vee K_m}^{\infty} 1(M_T \geq H)(H - K_m)1(S_T < H - K_d) + (H - S_T - K_d)^+] dH \right\}
\tag{13}
\]

\[
= (M_t - K_m)^+ E_t^Q (M_t - S_T - K_d)^+ + \int_{M_t \vee K_m}^{\infty} (H - K_m) E_t^Q 1(M_T \geq H)1(S_T < H - K_d) dH \\
+ \int_{M_t \vee K_m}^{\infty} E_t^Q 1(M_T \geq H)(H - K_d - S_T)^+ dH,
\]

assuming that \( \lim_{H \to \infty} E_t^Q 1(M_T \geq H)(H - K_m)(H - S_T - K_d)^+ = 0 \). Multiplying both sides of (13) by \( B_t(T) \):

\[
C_t^{\text{mod}}(K_m, K_d, T) = (M_t - K_m)^+ P_t(M_t - K_d, T) + \int_{M_t \vee K_m}^{\infty} (H - K_m) U I B P_t(H - K_d, T; H) dH \\
+ \int_{M_t \vee K_m}^{\infty} U I P_t(H - K_d, T; H) dH, \quad t \in [0, T]. \tag{14}
\]

QED
In words, Theorem 1 states that the value at \( t \) of the product call decomposes into the value of a position in standard puts and the value of a position in barrier puts (both standard and binary). The first term in (14) is the value at \( t \) of \( (M_t - K_m)^+ \) standard puts struck \( K_d \) dollars below the running maximum \( M_t \). Likewise, the last two terms in (14) are the value at \( t \) of \( (H - K_m)dH \) up-and-in binary puts and \( dH \) up-and-in puts, both struck \( K_d \) dollars below \( H \), for each in barrier \( H > M_t \lor K_m \). The position in standard puts is zero unless \( M_t > K_m \). If \( M_t > K_m \), then the put position at \( t \) provides the desired payoff in the event that the current maximum \( M_t \) is actually the final maximum \( M_T \) at expiry. The position in the barrier puts is there to ensure that the quantity and strike of the standard put position can be rolled up appropriately should the running maximum exceed both its previous value and \( K_m \) before \( T \). At maturity, these up-and-in puts conveniently expire worthless. The position at \( t \in [0, T] \) in standard puts represents the intrinsic value at \( t \) of the product call, while the contemporaneous position in barrier puts represents its time value (also called volatility value). The content of Theorem 1 is that the problem of pricing the product call reduces to the much more standard problem of pricing standard and barrier options.

Differentiating (14) w.r.t. \( K_m \) and negating gives the value of an up-and-in drawdown call:

\[
B_t(T)E_t^Q[1(M_T > K_m)(D_T - K_d)^+] = 1(M_t > K_m)P_t(M_t - K_d, T) + 1(M_t < K_m)UIP_t(K_m - K_d, T; K_m) + \int_{M_t \lor K_m}^{\infty} UIP_t(H - K_d, T; H)dH, \ t \in [0, T], K_m, K_d \geq 0.
\]  

(15)

Differentiating (15) w.r.t. \( K_d \) and negating gives the value of an up-and-in binary drawdown call:

\[
B_t(T)E_t^Q[1(M_T > K_m)1(D_T > K_d)] = 1(M_t > K_m)BP_t(M_t - K_d, T) + 1(M_t < K_m)UIBP_t(K_m - K_d, T; K_m) + \int_{M_t \lor K_m}^{\infty} UIBP_t(H - K_d, T; H)dH, \ t \in [0, T], K_m, K_d \geq 0,
\]  

(16)

where \( UIBS_t(K, T; H) \equiv B_t(T)E_t^Q[1(M_T > H)\delta(S_T - K)] \) denotes the value at \( t \in [0, T] \) of an up-and-in butterfly spread with strike \( K \), maturity \( T \), and barrier \( H \). Dividing by the bond price \( B_t(T) \) yields:

**Corollary 1: Joint Risk-Neutral Probability**

*Under frictionless markets and no arbitrage, the joint risk-neutral probability that the random vector*
\((M_T, D_T) \geq (K_m, K_d)\) is given by:

\[
Q_t \{ M_T > K_m, D_T > K_d \} = \frac{1}{B_t(T)} \left\{ 1(M_t > K_m)BP_t(M_t - K_d, T) + 1(M_t < K_m)UIBP_t(K_m - K_d, T; K_m) + \int_{M_t \vee K_m}^{\infty} UIBS_t(H - K_d, T; H)dH \right\}, \quad t \in [0, T], K_m, K_d \geq 0. \tag{17}
\]

Differentiating once more in \(K_m\) and \(K_d\) yields the joint risk-neutral density of the random vector \((M_T, D_T)\) evaluated at the point \((K_m, K_d)\). This density can be used to value any function \(g(\cdot, \cdot)\) of \((M_T, D_T)\), such as \((D_T M_T - K_d)\) for example. It also follows that one can value any function \(h(\cdot, \cdot)\) of \((M_T, S_T)\).

Setting \(K_m = 0\) in (15) leads to:

**Corollary 2: Robust Pricing of Drawdown Call**

Under frictionless markets, the drawdown call value \(C^d_t(K_d, T) \equiv B_t(T)E^Q_t(D_T - K_d)^+ = P_t(M_t - K_d, T) + \int_{M_t \vee K_m}^{\infty} UIBP_t(H - K_d, T; H)dH, \quad t \in [0, T], K_d \geq 0. \tag{18}\)

In words, a drawdown call with strike \(K_d\) is robustly replicated by keeping a put struck \(K_d\) dollars below the running maximum \(M_t\) and also holding \(dH\) up-and-in binary puts struck \(K_d\) dollars below its in barrier \(H\) for each in-barrier above \(M_t\). If \(M_T = M_t\), then the put provides the desired payoff, while if \(M_T > M_t\), then the up-and-in binary puts which knock in when \(M\) increases are used to roll up the put strike.

Conversely, differentiating (14) w.r.t. \(K_d\) and negating yields the following valuation formula for a call on the maximum which knocks in when running drawdown exceeds a barrier \(K_d\):

\[
B_t(T)E^Q_t 1(D_T > K_d)(M_T - K_m)^+ = (M_t - K_m)^+BP_t(M_t - K_d, T) + \int_{M_t \vee K_m}^{\infty} (H - K_m)UIBS_t(H - K_d, T; H)dH + \int_{M_t \vee K_m}^{\infty} UIBP_t(H - K_d, T; H)dH, \quad t \in [0, T], K_d, K_m \geq 0. \tag{19}\]

Setting \(K_d = 0\) in (19) leads to a representation for the call on the maximum:

\[
B_t(T)E^Q_t (M_T - K_m)^+ = (M_t - K_m)^+BP_t(M_t, T) + \int_{M_t \vee K_m}^{\infty} (H - K_m)UIBS_t(H, T; H)dH + \int_{M_t \vee K_m}^{\infty} UIBP_t(H, T; H)dH, \quad t \in [0, T]. \tag{20}\]
We can actually drop the up-and-in modifier on the butterfly spread since payment requires barrier hit and hence:

\[
\int_{M_t \lor K_m}^{\infty} (H - K_m) UIBS_t(H, T; H) dH = \int_{M_t \lor K_m}^{\infty} (H - K_m) Q_t \{ S_T \in dH \}. \quad (21)
\]

Integrating by parts with \( dv = Q_t \{ S_T \in dH \} \):

\[
\int_{M_t \lor K_m}^{\infty} (H - K_m) UIBS_t(H, T; H) dH = (M_t - K_m)^+ BC_t(M_t, T) + \int_{M_t \lor K_m}^{\infty} BC_t(H, T) dH, \quad (22)
\]

where \( BC_t(K_b, T) \equiv B_t(T) E_t^Q 1(S_T > K_b) \) denotes the value at \( t \in [0, T] \) of a binary call with strike \( K_b \) and expiry \( T \). Substituting (22) in (20) leads to:

**Corollary 3: Robust Pricing of Call on the Maximum**

Under frictionless markets, the value of the call on the maximum \( C_t^m(K_m, T) \equiv B_t(T) E_t^Q (M_T - K_m)^+ = (M_t - K_m)^+ B_t(T) + \int_{M_t \lor K_m}^{\infty} OTPE_t(T; H) dH, \quad t \in [0, T], \ K_m \geq 0, \) \quad (23)

where \( OTPE_t(T; H) = UIBP_t(K, T; H) + BC_t(H, T) \) denotes the value at \( t \in [0, T] \) of a one touch with payment at its expiry \( T \) and with barrier \( H \). In words, a call on the maximum is robustly replicated by keeping its intrinsic value in bonds and keeping its volatility value in \( dH \) one touches for each barrier \( H > (M_t \lor K_m) \). When the maximum increases above \( K_m \), the one touches which knock in finance the required increase in the bond position.

**III Semi-Robust Hedge for Product Call**

In many markets, barrier options do not trade and hence it is of interest to find alternative hedges that just use standard options. There are three known approaches for hedging barrier options in terms of standard options. The first approach is model-free and hence is only able to generate upper and lower bounds. See Brown, Hobson, and Rogers [4] and Neuberger and Hodges[16] for an exposition of this approach. The second approach puts some structure on the underlying price process and as a result is able to generate exact replication. This approach is based on the reflection principle and hence the structure eliminates the possibility of jumps over the barrier and asymmetries after the barrier crossing time (including risk-neutral
(drift). However, this second approach is semi-robust in that it does allow unspecified stochastic volatility and unspecified jumps away from the barrier. See Bowie and Carr[2] for the first paper along these lines and see Carr and Lee[6] for a modern account. The third approach also provides exact replication and is model-based in that it is predicated on the Markov property. The first two approaches value barrier options relative to co-terminal options, while this third approach uses all maturities up to that of the barrier option. See Derman, Ergener, and Kani[9] for the first paper along these lines and see Andersen, Andreasen, and Eliezer[1] for a modern account.

Although no approach dominates any other on all dimensions, we will focus on the second approach in this section. Hence, we will impose some structure on the risk-neutral dynamics of the underlying price, so that the product call hedge reduces to occasional trading in just standard options. In the last section, we had no restrictions on the dynamics of the underlying asset price. For the remainder of this paper, we will require that the price of the underlying asset be a martingale under $Q$. As a result, we think of the underlying as a forward price which we will denote by $F_t$. The implicit maturity date of the forward price can be $T$ or greater.

We also rule out up jumps in the path of the running maximum of $F$. Hence, at each $t \in [0, T]$, we allow the possibility of down jumps in $F_t$ and we allow the possibility of up jumps of limited size in $F_t$, but we give zero probability to up jumps in $F_t$ which are sufficiently large so that $M_t \equiv \max_{s \in [0,t]} F_s$ could increase by a jump.

In the next two subsections, we place alternative assumptions on the symmetry of the risk-neutral process for $F$, so that the sale of a product call can be perfectly hedged by a self-financing non-anticipating replicating trading strategy which just uses standard options.

**III-A Arithmetic Put Call Symmetry**

We say that Arithmetic Put Call Symmetry (APCS) holds at a particular time $t \geq 0$ for a particular maturity $T \geq t$ if a put maturing at $T$ has the same market price as the co-terminal call struck the same
distance away from $F_t$:

$$P_t(K_p, T) = C_t(K_c, T),$$

for all strikes $K_p, K_c$ satisfying: $K_p = F_t + \Delta K, K_c = F_t - \Delta K$, where $\Delta K \in \mathbb{R}$. APCS implies that equally out-of-the-money options have the same value.

In addition to assuming that $M$ never increases by a jump, we now assume that APCS holds at all times $t$ for which the running maximum increases\(^1\).

Recall that we have already required that the underlying price be a $Q$ martingale. It is reasonable to question whether there exists a stochastic process which meets all three requirements. Fortunately, these requirements are all met by the class of Ocone martingales. These martingales have purely continuous sample paths and APCS holds at all times. Standard Brownian motion is the most famous example of an Ocone martingale, but they also include any continuous martingale arising from Brownian motion by performing an independent absolutely continuous time change. Using the language of stochastic differential equations, Ocone martingales solve:

$$dF_t = a_t dW_t, \quad t \in [0, T]$$

where the absolute volatility process $a$ evolves independently of $W$.

With existence of our underlying price process assured, we now define:

**Assumption set A1:** The price process $F$ is a $Q$ martingale whose running maximum is continuous and for which APCS holds at all times $\tau$ when $dM_\tau > 0$.

Note that aside from requiring $Q$ martingality of the underlying, continuity of its running maximum $M$, and APCS to hold at trading times, we are placing no restrictions on asset price processes. In particular, we are allowing unspecified stochastic volatility of the underlying asset price, although it must evolve independently of the standard Brownian motion driving the price in order for APCS to hold.

\(^1\)We recognize that the times for which the running maximum increases are not stopping times. Our motivation for imposing APCS at just these random times is that we wished to have the weakest possible sufficient conditions for our theorems and corollaries. Readers who prefer to work with stopping times can make the stronger assumption that APCS holds at all times when the underlying price is at its running maximum.
We claim that:

**Theorem 2: Semi-Robust Pricing of Product Call Under APCS**

*Under frictionless markets and A1, no arbitrage implies that for \( K_m, K_d \geq 0, t \in [0, T] \):

\[
C^{md}_t(K_m, K_d, T) = (M_t - K_m)^+[P_t(M_t - K_d, T) + C_t(M_t + K_d, T)] + 2 \int_{M_t \vee K_m}^{\infty} C_t(H + K_d, T) dH. \tag{25}
\]

**Proof:** Recall from Theorem 1 that for all \( t \in [0, T] \),

\[
C^{md}_t(K_m, K_d, T) = (M_t - K_m)^+P_t(M_t - K, T) + \int_{M_t \vee K_m}^{\infty} (H - K_m)UIBP_t(H - K_d, T; H) dH + \int_{M_t \vee K_m}^{\infty} UIP_t(H - K_d, T; H) dH. \tag{26}
\]

However, if the paths of \( M \) are continuous and if APCS is holding at times when \( M \) increases, then for all \( t \in [0, T] \), the reflection principle implies that:

\[
UIBP_t(H - K_d, T; H) = BC_t(H + K_d, T), \tag{27}
\]

and:

\[
UIP_t(H - K_d, T; H) = C_t(H + K_d, T). \tag{28}
\]

Substituting (27) and (28) in (26) implies that for all \( t \in [0, T] \),

\[
C^{md}_t(K_m, K_d, T) = (M_t - K_m)^+P_t(M_t - K_d, T) + \int_{M_t}^{\infty} (H - K_m)^+BC_t(H + K_d, T; H) dH + \int_{M_t \vee K_m}^{\infty} C_t(H + K_d, T; H) dH. \tag{29}
\]

Using integration by parts on the first integral:

\[
u = (H - K_m)^+ \quad dv = BC_t(H + K_d, T) dH \quad du = 1(H > K_m) dH \quad v = -C_t(H + K_d, T).
\]

So

\[
C^{md}_t(K_m, K_d, T) = (M_t - K_m)^+[P_t(M_t - K_d, T) + C_t(M_t + K_d, T)] + 2 \int_{M_t \vee K_m}^{\infty} C_t(H + K_d, T) dH. \tag{30}
\]

**QED**

The first term on the RHS of (30) is the value of \((M_t - K_m)^+\) strangles centered at the running maximum \(M_t\) and whose width (distance between call strike and center) is the strike \(K_d \geq 0\) of the drawdown call. Theorem 2 states that the product call always has the same value as \((M_t - K_m)^+\) of these strangles and
$2dK_c$ calls for each strike $K_c > (M_t \lor K_m) - K_d$. The first term in (30) provides the desired payoff if $M_T = M_t$, while the second term is there to finance the rollup of both the number of strangles held and their center whenever we have both $dM > 0$ and $M > K_m$. One can say that the $(M_t - K_m)^+$ strangles provide the product call’s intrinsic value, while the call integral provides the product call’s volatility value.

So long as the price paths of $M$ are continuous and APCS holds when $M$ increases, then when both $dM > 0$ and $M > K_m$, the infinitessimal cost of running up the strikes of the puts in the strangle position is financed by the infinitessimal gain from running up the strikes of the calls in the strangle position. Since $(M_t - K)^+$ rises when $dM > 0$ and $M > K_m$, the number of strangles held also increases infinitessimally. The cost of this increase is financed by selling off the calls in the call integral which are struck inbetween the old maximum and the new one.

If the call integral in (30) is held static to maturity, then it produces the following payoff:

$$f(F_T) \equiv F_T^2 1(F_T > (M_t \lor K_m) - K_d),$$

which is the right arm of a parabola. Interestingly, this call position is also held as the static component of the hedge when replicating the following payoff on the floating side of a corridor price variance swap:

$$\int_t^T 1[F_u > (M_t \lor K_m) - K_d](dF_u)^2.$$

Differentiating (30) w.r.t. $K_m$ and negating yields the following valuation formula for an up-and-in drawdown call:

$$B_t(T)E_t^Q 1(M_T > K_m)(D_T - K_d)^+ = 1(M_t > K_m)[P_t(M_t - K_d, T) + C_t(M_t - K_d, T)] + 1(M_t < K_m)2C_t(K_m + K_d, T)].$$

Differentiating (31) w.r.t. $K_d$ and negating gives the value of an up-and-in binary drawdown call:

$$B_t(T)E_t^Q 1(M_T > K_m)1(D_T > K_d) = 1(M_t > K_m)[BP_t(M_t - K_d, T) + BC_t(M_t + K_d, T)] + 1(M_t < K_m)2BC_t(K_m + K_d, T).$$

\(^2\)This replication is model-free, i.e. it succeeds even under jumps and violations of APCS.
Dividing by the bond price $B_t(T)$ yields the joint risk-neutral probability that $(M_T, D_T) \geq (K_m, K_d)$:

**Corollary 4: Joint Risk Neutral Probability Under APCS**

Under frictionless markets and assumption set A1, no arbitrage implies that for $K_m, K_d \geq 0, t \in [0, T]$, we have

$$Q_t\{M_T > K_m, D_T > K_d\} = \frac{1}{B_t(T)} \{1(M_t > K_m)[BP_t(M_t - K_d, T) + BC_t(M_t + K_d, T)] + 1(M_t < K_m)2BC_t(K_m + K_d, T)\}.$$  \(33\)

Setting $K_m = 0$ in (31) leads to the following corollary:

**Corollary 5: Drawdown Call under APCS**

Under frictionless markets and assumption set A1, no arbitrage implies:

$$C^d(K_d, T) = P_t(M_t - K_d, T) + C_t(M_t + K_d, T), \quad t \in [0, T], K_d \geq 0. \tag{34}$$

In words, a drawdown call is replicated by alway holding a strangle centered at the running maximum $M_t$, and whose width is the strike $K_d$ of the drawdown call. The strategy is self-financing because the cash outflow required to move the put strike up when the running maximum increases infinitessimally is financed by the cash inflow received from moving the call strike up (given that APCS is in fact holding at such times).

Conversely, differentiating (30) w.r.t. $K_d$ and negating yields the following valuation formula for a call on the maximum which knocks in when running drawdown exceeds a barrier $K_d$:

$$B_t(T)E_t^Q(M_T - K_m)^+1(D_T > K_d) = (M_t - K_m)^+[BP_t(M_t - K_d, T) + BC_t(M_t - K_d, T)]$$

$$+2C_t((M_t \vee K_m) - K_d, T). \tag{35}$$

Setting $K_d = 0$ in (35) leads to:

**Corollary 6: Call on the Maximum Under APCS**

Under frictionless markets and assumption set A1, no arbitrage implies:

$$C^m_t(K_m, T) = (M_t - K_m)^+B_t(T) + 2C_t(M_t \vee K_m, T), \quad K_m \geq 0, t \in [0, T]. \tag{36}$$

In words, a call on the maximum is replicated by keeping its intrinsic value in bonds and alway holding two standard calls struck at the larger of the running maximum $M_t$ and the call strike $K_m$. The strategy
is self-financing because each standard call’s strike derivative is one half when the running maximum is above $K_m$ and increases infinitessimally (given that APCS is in fact holding at such times). Hence, the cash generated by moving up the two call strikes is just enough to keep the intrinsic value of the call on the maximum in bonds.

### III-B Geometric Put Call Symmetry

If the price of the underlying asset can more than double over the time horizon of interest, then an unfortunate implication of APCS is that this price can also go negative. To prevent the possibility of negative prices, one can use Geometric Put Call Symmetry (GPCS) instead of APCS. We say that GPCS holds at a particular time $t \geq 0$ for a particular maturity $T \geq t$ if the time $t$ market prices of puts and calls of maturity $T$ are such that:

$$\frac{P_t(K_p, T)}{\sqrt{K_p}} = \frac{C_t(K_c, T)}{\sqrt{K_c}},$$

for all strikes $K_p, K_c$ satisfying $K_c = F_t \times u, K_p = F_t / u$, where $u > 0$.

A sufficient condition on the dynamics of $F$ for engendering GPCS is that:

$$dF_t = F_t \sqrt{V_t} dW_t, \quad t \in [0, T],$$

where $W$ is a $Q$ standard Brownian motion and the stochastic instantaneous variance process $V$ evolves independently of $W$.

With existence assured, we now define:

**Assumption set A2:** The price process $F$ is a $Q$ martingale whose running maximum is continuous and for which GPCS holds at all times $\tau$ when $dM_\tau > 0$.

Appendix 1 proves:

**Theorem 3:** Semi-Robust Pricing of Product Call Under GPCS

\footnote{Note that the process $\ln \left( \frac{F_t}{F_0} \right) + \int_0^t V_s ds$ is an Ocone martingale.}
Under frictionless markets and A2, no arbitrage implies that for \( K_m \geq 0, K_d \in [0, \frac{F_0}{2}], t \in [0, T] \):

\[
C_t^{md}(K_m, K_d, T) = (M_t - K_m)^+ \left[ P_t(M_t - K_d, T) + \frac{M_t - K_d}{M_t - 2K_d} C_t \left( \frac{M_t^2}{M_t - K_d}, T \right) \right] 
+ \int_{\frac{(M_t-K_m)^2}{(M_t+K_m^2)-K_d}}^{\infty} N^c(K_c, K_d) C_t(K_c, T) dK_c, \quad (39)
\]

where:

\[
N^c(K_c, K_d) \equiv \frac{2}{K_c} \frac{H(K_c - K_d)^2}{K_c - 4K_d} + \frac{H(K_c) - K_d (\sqrt{K_c - 4K_d})^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c}}{2(\sqrt{K_c - 4K_d})^3}, \quad (40)
\]

and where:

\[
H(K_c) \equiv \frac{K_c}{2} + \sqrt{\frac{K_c^2}{4} - K_dK_c}. \quad (41)
\]

The reason that the drawdown call strike \( K_d \) cannot exceed \( \frac{F_0}{2} \) is because \( M_t \geq F_0 \) and we want the denominator in the expression \( \frac{M_t-K_d}{M_t-2K_d} \) to be nonnegative. If a product call is sold at \( t = 0 \), then the nature of the hedging strategy is the same under GPCS as it was under APCS. In particular, a growing number of ratioed strangles provides the intrinsic value, where now the geometric mean of the put strike and the call strike is the running maximum. When \( K_d > 0 \), more calls than puts are held in the ratioed strangle because a put on one dollars worth of the underlying stock has greater value than an equally out-of-the-money call on one dollars worth of stock\(^4\). As under APCS, an integral of calls struck above \( M_t \) provides the volatility value, although the number of OTM calls held, \( N^c dK_c \), is now a complicated function of the call strike \( K_c \). As a result, all calls held at expiry finish out-of-the-money. As usual, the \((M_t - K_m)^+ \) puts struck at \( M_T - K_d \) furnishes the product call payoff.

Differentiating (39) w.r.t. \( K_m \), negating, and setting \( K_m = 0 \) yields:

**Corollary 7: Drawdown Call under GPCS**

Under frictionless markets and assumption set A2, no arbitrage implies that for \( K_d \in [0, \frac{F_0}{2}], t \in [0, T] \):

\[
C_t^d(K_d, T) = P_t(M_t - K_d, T) + \frac{M_t - K_d}{M_t - 2K_d} C_t \left( \frac{M_t^2}{M_t - K_d}, T \right) + \int_{\frac{M_t^2}{M_t-K_d}}^{\infty} N^c(K_c, K_d) C_t(K_c, T) dK_c, \quad (42)
\]

\(^4\)In particular, the greater put value arises due to a negative component of the drift in the log price induced by concavity correction.
for $t \in [0, T]$, where:

$$N^c(K_c, K_d) = \frac{(\sqrt{K_c - 4K})^3 + K_c^{3/2} - 6K\sqrt{K_c}}{2K_c(\sqrt{K_c - 4K})^3}. \tag{43}$$

Suppose again that a drawdown call is sold at $t = 0$. In contrast to the hedge of a product call, the the number of ratioed strangles is kept constant, although the center of the strangle can still increase. Since no change in the number of strangles needs to be financed, the expression for the number of OTM calls used to hedge the drawdown call is simpler than the one for the product call. As before, all calls held at expiry finish out-of-the-money, while the put struck at $M_T - K_d$ furnishes the drawdown call payoff.

In principle, differentiating (39) w.r.t. $K_d$, negating, and setting $K_d = 0$ yields the valuation formula for a call on the maximum. As this calculation is tedious, we take a shorter route. Under A2, Bowie and Carr\[2\] show that:

$$OTPE_t(T; H) = 2BC_t(H, T) + \frac{1}{H}C_t(H, T), \quad t \in [0, T \wedge \tau_H), \tag{44}$$

where $\tau_H$ is the first passage time of the forward price $F$ to the upper barrier $H$. Substituting (44) in (23):

$$C_t^m(K_m, T) = (M_t - K_m)^+B_t(T) + \int_{M_t \vee K_m}^\infty \left[2BC_t(H, T) + \frac{1}{H}C_t(H, T)\right]dH, \quad t \in [0, T]. \tag{45}$$

Recalling that $BC_t(H, T) = -\frac{\partial}{\partial H}C_t(H, T)$, the fundamental theorem of calculus implies:

**Corollary 8: Call on the Maximum under GPCS**

*Under frictionless markets and assumption set A2, no arbitrage implies that for $K_m \geq 0, t \in [0, T]$ :*

$$C_t^m(K_m, T) = (M_t - K_m)^+B_t(T) + 2C_t(M_t \vee K_m, T) + \int_{M_t \vee K_m}^\infty \frac{1}{H}C_t(H, T)dH. \tag{46}$$

Comparing this result with the corresponding one (36) under APCS, we see that the hedge has an additional holding in $\frac{dH}{H}$ calls for all strikes $H > M_t \vee K_m$. It can be shown that for any fixed $t \in [0, T]$, holding this call position static to maturity would create the payoff:

$$f(F_T) = \left\{ F_T \ln \left( \frac{F_T}{M_t \vee K_m} \right) - [F_T - (M_t \vee K_m)] \right\}1(F_T > (M_t \vee K_m)). \tag{47}$$
Interestingly, this is also the position in standard calls used to synthesize the payoff for a corridor gamma variance swap, whose floating payoff is:

\[\int_t^T \frac{F_u}{F_t} 1[F_u > (M_t \vee K_m)] \left(\frac{dF_u}{F_u}\right)^2.\]

### IV Path Independent Options on Trading Gains

Consider a dynamic trading strategy which is binary in that the number of shares held can only be ±1. The price of the asset being traded will have zero risk-neutral drift and hence we use \(F\) (for forward price) to denote it. The running P&L \(\pi_t\) of a binary trading strategy in such an asset is defined by:

\[\pi_t \equiv \int_0^t c_s dF_s, \quad t \in [0, T], \quad (48)\]

where \(c_s = \pm 1\). We define the parity at \(t\) of the gains process \(\pi\) as \(c_t\).

Now consider a European standard call option written on the terminal P&L from a binary trading strategy. Letting \(C_\pi^{T}(k,T)\) denote the arbitrage-free value at time \(t \in [0, T]\), this call has a final payoff at its maturity \(T\) of:

\[C_\pi^{T}(k,T) = (\pi_T - k)^+, \quad (49)\]

where \(k \in \mathbb{R}\) is the strike price. We henceforth refer to this call as a passport call or just a passport for brevity.

Many authors have considered the problem of replicating the payoff to a passport option via the classical strategy of dynamically trading the underlying asset. The Black Scholes model is usually assumed and no effort is make to accomodate the reality of jumps or stochastic volatility. In this section, we will address the problem of replicating a passport option payoff under an unspecified process for jumps and stochastic volatility. At any given time, our replicating strategy only requires holding a single European option of maturity \(T\). In contrast to the earlier work on pricing passport options in the Black Scholes model, we will assume zero risk-neutral drift in the underlying asset. In fact, we assume zero interest rates or dividends.

\[\text{This replication assumes only positivity and continuity of the process; it does not require GPCS.}\]
for simplicity. We will furthermore require that at any time $\tau$ that the parity of the strategy changes, the risk-neutral distribution of $F_T$ given $F_\tau$ is symmetric about $F_\tau$. In other words, we require that APCS holds whenever the parity changes.

We formalize these assumptions with:

**Assumption set A3:** Interest rates and dividends are zero. The price process $F$ defining the reference binary strategy is a $\mathbb{Q}$ martingale for which APCS holds at all times $\tau$ when parity changes.

We define the moneyness at $t \in [0, T]$ of the passport as:

$$m_t \equiv \pi_t - k,$$

and note that it can be negative. Analogously, we define the real-valued moneyness at $t \in [0, T]$ of a European call with strike $K_c$ and a European put with strike $K_p$ respectively by:

$$M^c_t \equiv F_t - K^c, \quad M^p_t \equiv K^p - F_t.$$  

(51)

Recall that $C_t(K_c, T)$ and $P_t(K_p, T)$ respectively denote the market call and put prices for strikes $K_c \in \mathbb{R}$ and $K_p \in \mathbb{R}$ at time $t \in [0, T]$. Using this notation, assumption set A3 implies that:

$$C_\tau(F_\tau - M^c_\tau, T) = P_\tau(F_\tau + M^p_\tau, T),$$

(52)

when $M^c_\tau = M^p_\tau$ and for all times $\tau \in [0, T]$ when parity changes.

Appendix 2 proves:

**Theorem 4: Semi-Robust Pricing of Passport Call**

*Under frictionless markets and assumption set A3, no arbitrage implies that for $k \in \mathbb{R}, t \in [0, T] :$

$$C_t^\pi(k, T) = 1(c_t = 1)C_t(F_t^- - m_t^-, T) + 1(c_t = -1)P_t(F_t^- + m_t^-, T),$$

where $F^-$ and $\pi^-$ indicate the forward price and P&L at the time of the last trade at or before $t$.*

To gain intuition on this result, note that at maturity, call and put values converge to their intrinsic values:

$$C_T(K_c, T) = (F_T - K_c)^+, \quad P_T(K_p, T) = (K_p - F_T)^+.$$  

(54)
From (51), these payoffs can also be written in terms of their moneyness at \( t \in [0, T] \) as:

\[
C_T(K, T) = \left( M^c_t + \int_t^T (+1) dF_u \right)^+ \\
P_T(K, T) = \left( M^p_t + \int_t^T (-1) dF_u \right)^+. \tag{55}
\]

Likewise, (49), (48), and (50) imply that the payoff on a passport can be written in terms of its moneyness at \( t \in [0, T] \) as:

\[
C^\pi_T(k, T) = (m_t + \int_t^T c_u dF_u)^+. \tag{56}
\]

At any time \( \tau \) when parity changes, the hedger of a written passport can always find a standard option with the same parity that the underlying gains process changes to. Furthermore, this hedger can always find a standard option whose moneyness matches the moneyness of the passport at \( \tau \). If the parity of the underlying gains process never changes after \( \tau \), then (55) indicates that a static position in the standard option held over \([\tau, T]\) results in the same terminal payoff as the passport. Since the parity of the underlying gains process can change again at any time after \( \tau \), the hedger of the passport just needs to ensure that the transition to the standard option with the right parity and moneyness is costless. As Appendix 2 proves, assumption set \( A3 \) guarantees that this switch is indeed self-financing. Since there will eventually be a time when there are no further parity changes in the underlying gains process, this switching strategy does hedge the liability arising from writing a passport.

Substituting (50) in (53) implies:

\[
C^\pi_t(k, T) = 1(c_t = 1) C_t(F^-_t - \pi^-_t + k, T) + 1(c_t = -1) P_t(F^-_t + \pi^-_t - k, T), \quad t \in [0, T]. \tag{57}
\]

In words, the passport call always has the same value as a standard option with the same moneyness at the last switch time. The parity of the standard option held at \( t \) (i.e. call or put) matches the parity at \( t \) of the underlying gains process.

Now let \( P^\pi_t(k, T) \) denote the arbitrage-free value at time \( t \in [0, T] \) of a put option on the gains from a binary trading strategy. At its expiry \( T \), the payoff is:

\[
P^\pi_t(k, T) = (k - \pi_T)^+, \tag{58}
\]

where \( k \in \mathbb{R} \) is the put strike. One can show that the passport put always has the same value as a standard option with the same moneyness at the last switch time. The parity of the standard option held at \( t \) is
now the opposite of the parity at $t$ of the underlying gains process. As a result, we have:

**Corollary 9: Semi-Robust Pricing of Passport Put**

*Under frictionless markets and assumption set A3, no arbitrage implies that for $k \in \mathbb{R}, t \in [0, T]$:

$$P^n_t(k,T) = 1(c_t = -1)C_t(F^*_t + \pi^*_t - k, T) + 1(c_t = 1)P_t(F^*_t - \pi^*_t + k, T), \quad t \in [0, T].$$

(59)

Since puts and calls on gains can be semi-statically hedged with standard options, it follows that any path-independent payoff on gains can also be semi-statically hedged with standard options.

**V Path Dependent Options on Gains Processes**

The payoff from the passport options discussed in the last section depends only on the terminal P&L realized from a binary trading strategy. Two paths with the same terminal P&L lead to the same payoff.

In this section, we consider the problem of replicating an option payoff which depends on the path taken by the running P&L process. In particular, we consider barrier options written on the gains from a binary trading strategy. We also consider a call on the maximum or a call on the drawdown of the gains from a binary trading strategy. In fact, we also consider the product of the two call payoffs. We show that under a particular assumption, all of these payoffs can be replicated by semi-static trading in standard options. As a result, the joint risk-neutral distribution of the maximum and drawdown of the gains from a binary trading strategy can be inferred from the market prices of these options.

To obtain these results, we make the same assumptions as in the last section and we additionally assume that the underlying price process $F$ is continuous. Although weaker sufficient conditions may be possible, we assume in this section that:

**Assumption set A4: Interest rates and dividends are zero. The price process $F$ defining the reference binary strategy is an Ocone martingale under $Q$.**

Hence, APCS holds at all times $t \in [0, T]$. Assumption set A4 implies that that the gains from a binary trading strategy in $F$ is also an Ocone martingale.
V-A Barrier Options on Trading Gains

In this subsection, we consider barrier options written on the gains from a binary trading strategy. In particular, we illustrate with an up-and-in put. Accordingly, let \( M_T^\pi \equiv \max_{t \in [0,T]} \pi_t \) be the maximum profit that a binary trading strategy earned over \([0,T]\). Let \( \text{UIP}_t^{\pi}(k_u, T; h) \equiv E_t^Q 1(M_T^\pi > h)(k_u - \pi_T)^+ \) be the arbitrage-free value at time \( t \geq 0 \) of an up-and-in put with strike \( k_u \in \mathbb{R} \), maturity \( T \geq t \), and in-barrier \( h \geq 0 \). Let \( \tau_h^\pi \) be the first passage time of the gains process to the barrier \( h \). If \( \tau_h^\pi < T \), then the barrier is hit before maturity. For \( t \geq [\tau_h, T] \), the up-and-in put on gains has knocked into a put on gains with strike \( k \in \mathbb{R} \) and maturity \( T \geq t \).

For \( t \geq [0, \tau_h) \), applying the reflection principle to the underlying gains process implies that the up-and-in put on gains has the same value as a co-terminal call on gains struck the same distance away from the barrier:

\[
\text{UIP}_t^{\pi}(k_u, T; h) = C_t^{\pi}(2h - k_u, T).
\] (60)

By the results of the last section, the call on gains has the same value as the standard option with the same parity and moneyness at the last switch time. Evaluating (57) at \( k = 2h - k_u \) and substituting into (60) implies:

**Theorem 5: Semi-Robust Pricing of Up and In Put on Gains**

*Under frictionless markets and assumption set A4, no arbitrage implies that for \( k, h \in \mathbb{R}, t \in [0,T] \):

\[
\text{UIP}_t^{\pi}(k, T; h) = 1(c_t = 1)C_t(F_t^- - \pi_t^- + 2h - k, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - 2h + k, T).
\] (61)

V-B Product Call on Trading Gains

Theorem 2 showed that when A1 holds for the underlying price, the arbitrage-free value of the product call is given by summing the value of \((M_t - K)^+\) strangles with \(2dH\) standard calls of all strikes greater than \((M_t \lor K_m) - K_d\). Under A4, A1 holds for both the underlying price and for the gains from a binary trading strategy. Hence, the payoff from a product call on gains can also be hedged in terms of standard options, as we now show.
To begin, consider the drawdown on the gains from a binary trading strategy:

\[ D_t^\pi \equiv M_t^\pi - \pi_t. \] (62)

Now let \( C_t^{md\pi}(K_m, K_d, T) \) denote the arbitrage-free value at time \( t \in [0, T] \) of a product call whose underlying is the gains from a binary trading strategy. At its maturity date \( T \), this claim pays off:

\[ C_T^{md\pi}(K_m, K_d, T) = (M_T^\pi - K_m)^+(D_T^\pi - K_d)^+. \] (63)

Applying Theorem 2 when the asset underlying the product call is the gains from a binary trading strategy:

\[ C_t^{md\pi}(K_m, K_d, T) = (M_t^\pi - K_m)^+[P_t^\pi(M_t^\pi - K_d, T) + C_t^\pi(M_t^\pi + K_d, T)] + 2 \int_{M_t^\pi \vee K_m}^\infty C_t^\pi(H + K_d, T) dH. \] (64)

Using (57) and (59) to replace each passport option with its semi-static hedge, we have:

**Theorem 6: Semi-Robust Pricing of Product Call on Gains**

Under frictionless markets and A4, no arbitrage implies that for \( K_m, K_d \geq 0, t \in [0, T], C_t^{md\pi}(K_m, K_d, T) = \)

\[
(M_t^\pi - K_m)^+[1(c_t = -1)C_t(F_t^- + \pi_t^- - M_t^\pi + K_d, T) + 1(c_t = 1)P_t(F_t^- - \pi_t^- + M_t^\pi - K_d, T) + 1(c_t = 1)C_t(F_t^- - \pi_t^- + M_t^\pi + K_d, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - M_t^\pi - K_d, T)] + 2 \int_{M_t^\pi \vee K_m}^\infty [1(c_t = 1)C_t(F_t^- - \pi_t^- + H + K_d, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - H - K_d, T)] dH. \] (65)

Let \( C_t^{d\pi}(K_d, T) \) be the arbitrage-free value of a call on the drawdown of a gains process paying \((D_T^\pi - K_d)^+\) at \( T \). Differentiating (65) w.r.t. \( K_m \), negating, and setting \( K_m = 0 \) recovers the value of this call:

**Corollary 10: Call on Drawdown of Gains**

Under frictionless markets and A4, no arbitrage implies:

\[ C_t^{d\pi}(K_d, T) = 1(c_t = -1)C_t(F_t^- + \pi_t^- - M_t^\pi + K_d, T) + 1(c_t = 1)P_t(F_t^- - \pi_t^- + M_t^\pi - K_d, T) + 1(c_t = 1)C_t(F_t^- - \pi_t^- + M_t^\pi + K_d, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - M_t^\pi - K_d, T). \] (66)

Likewise, let \( C_t^{max\pi}(K_m, T) \) be the arbitrage-free value of a call on the maximum of a gains process paying \((M_T^\pi - K_m)^+\) at \( T \). Differentiating (65) w.r.t. \( K_d \), negating, and setting \( K_d = 0 \) recovers the value of this
Corollary 11: Call on Maximum of Gains

Under frictionless markets and A4, no arbitrage implies:

\[ C^m_t(K_m, T) = (M^\pi_t - K_m)^+ + 1(c_t = 1)BC_t(F_t^- - \pi_t^- - M^\pi_t, T) + 1(c_t = -1)BP_t(F_t^- - \pi_t^- + M^\pi_t, T) \]

\[ + 1(c_t = 1)BC_t(F_t^- - \pi_t^- + M^\pi_t, T) + 1(c_t = -1)BP_t(F_t^- + \pi_t^- - M^\pi_t, T) \]

\[ + 2 \int_{M^\pi_t \vee K_m}^{\infty} [-1(c_t = 1)] \frac{\partial}{\partial K_c} C_t(F_t^- - \pi_t^- + H, T) + 1(c_t = -1) \frac{\partial}{\partial K_p} P_t(F_t^- + \pi_t^- - H, T)] dH. \]

In words, a call on the maximum of the gains from a binary trading strategy is replicated by keeping its intrinsic value in bonds and always holding two standard options of the same parity as the underlying gains process. The strike held is such that at the last time the parity changed, the standard option acquired has the same moneyness as a call on gains struck at the larger of the running maximum \( M_t \) and the call strike \( K_m \). The strategy is self-financing when \( M^\pi_t < K_m \), because the trade just involves changing the parity of the standard option held while preserving moneyness. The strategy is also self-financing when \( M^\pi_t > K_m \) and the maximum increases, because either \( c_t = 1 \), in which case the infinitesimal rollup of the strikes of the 2 ATM calls held finances the bond position, or else \( c_t = -1 \), in which case the infinitesimal rolldown of the strikes of the 2 ATM puts held finances the bond position. In either case, one can keep the intrinsic value of the call on the maximum in bonds.

VI Semi-Robust Hedge of a Call on Crosses

Suppose that we specify a spatial interval \((F_0, F_0 + w)\) with some width \( w > 0 \). The number of crosses of this interval over \([0, T]\) is a nonnegative random variable, which sums upcrosses, downcrosses, and partial crosses realizing at \( T \). So for example, if the spatial interval is \((100,103)\), and starting from \( F_0 = 100 \), the forward price performs 3 upcrosses interlaced with 3 downcrosses, and finishes at 102, then the number of crosses is 6 and 2/3.
Consider a contingent claim that pays the number of crosses of a given spatial interval over a given time interval. We refer to such a claim as a crosser. Crossers are designed to appeal to anyone wishing to speculate on serial correlation contingent on the price being in or around the interval \((F_0, F_0 + w)\). More precisely, someone who predicts that the serial correlation of future returns was very negative when the price was in or around the interval would receive a large payoff from being long a crosser should their forecast turn out to be correct. In contrast, someone who predicts that serial correlation will be positive with the price in or around the interval should sell a crosser. The marketing of crossers can also be phrased in terms of mean-reversion and trending. The ideal buyer of a crosser would be forecasting mean-reversion, while the ideal seller would be forecasting trending.

Now consider a call option written on the product of the number of crosses and the width of the spatial interval. In this section, we will show that this call on crosses arises as a call on the maximum of a gains process for a particular binary trading strategy. As the last section identified a semi-static hedge of the latter, it follows that calls on crosses enjoy the same semi-static hedge.

We define the *contrarian strategy* as a binary trading strategy where shareholdings just depend on \(F\), \(F^-\), and the width \(w\) as follows:

\[
ct = \begin{cases} 
1 & \text{if } F_t \leq F_0, \\
1 & \text{if } F_t \in (F_0, F_0 + w) \text{ and } F^-_t = F_0 \\
-1 & \text{if } F_t \in (F_0, F_0 + w) \text{ and } F^-_t = F_0 + w \\
-1 & \text{if } F_t \geq F_0 + w.
\end{cases}
, \quad t \in [0, T]. \tag{68}
\]

In words, the contrarian investor is long the risky asset if \(F\) is below the spatial interval \((F_0, F_0 + w)\) and short the risky asset if \(F\) is above this interval. When \(F\) is inside the interval, the investor just keeps the position held when this interval was last entered.

We let \(\gamma_t\) denote the gains at \(t \in [0, t]\) from the contrarian trading strategy:

\[
\gamma_t = \begin{cases} 
n^*_t w + F_t - F_0 & \text{if } c_t = 1, \\
n^*_t w + F_0 + w - F_t & \text{if } c_t = -1,
\end{cases} \quad t \in [0, T], \tag{69}
\]

where \(n^*_t\) is the number of completed crosses in \([0, t]\). The running maximum of the gains process is given
by:

\[ M_t^\gamma = \begin{cases} n_t^c w + \max_{t \leq s \leq t} (F_t - F_0) & \text{if } c_t = 1, \\ n_t^c w + \max_{t \leq s \leq t} (F_0 + w - F_t) & \text{if } c_t = -1, \end{cases} \quad t \in [0, T]. \] (70)

Equivalently:

\[ M_t^\gamma = w(n_t^c + f_t), \quad t \in [0, T], \] (71)

where:

\[ f_t \equiv \begin{cases} \max_{t \leq s \leq t} \frac{F_t - F_0}{w} & \text{if } c_t = 1, \\ \max_{t \leq s \leq t} \frac{F_0 + w - F_t}{w} & \text{if } c_t = -1, \end{cases} \quad t \in [0, T], \] (72)

is the fraction of the last cross in \([0, t]\) which has been completed by \(t\).

As with any other risky gains process, no arbitrage forces the gains from a contrarian trading strategy to take possible realizations on both sides of zero. Suppose that we wish to isolate from these gains a nonnegative quantity with the intent of making its terminal value the underlying of a call option. In particular, suppose that we want to extract the number of crosses of the interval \((F_0, F_0 + w)\). At expiry, this quantity is the sum of the total number of completed upcrosses in \([0, T]\), the total number of completed downcrosses in \([0, T]\), and a terminal fraction \(f_T \in [0, 1]\) if there is a partially completed upcross or downcross. This sum is non-negative and from (71), it is proportional to the maximum of the gains from the contrarian trading strategy:

\[ n_t^c + f_t = \frac{M_t^\gamma}{w}, \quad t \in [0, T]. \] (73)

Let \(C_t^c(K_m, T)\) denote the arbitrage-free value of a call option at time \(t \in [0, T]\) written on the product of \(w\) and the sum of the crosses. Let \(K_m \geq 0\) be the call strike and let:

\[ k_m \equiv \frac{K_m}{w}. \] (74)

Thus, the payoff of the call at expiry is:

\[ C_T^c(K_m, T) = w(n_T^c + f_T - k_m)^+ = (M_T^\gamma - K_m)^+. \] (75)

Now:

\[ F_t^- = \begin{cases} F_0 & \text{if and only if } c_t = 1 \\ F_0 + w & \text{if and only if } c_t = -1 \end{cases} \quad t \in [0, T]. \] (76)
Evaluating (69) at $t-$ and substituting in (76):

$$\gamma_t^{-} = w \times n^c_t, \quad t \in [0, T].$$

As each cross is completed, the quantity $\gamma_t^{-}$ jumps up by $w$ dollars.

Substituting (71), (76), and (77) in (67) implies that for $t \in [0, T), k_m \geq 0$:

$$C_t^{\text{mns}}(k_m w, T) = w(n_t^c + f_t - k_m)^{+} B_t(T)$$

$$+ 21(F^-_t = F_0)C_t(F_t + w f_t + w[k_m - (n_t^c + f_t)]^{+}, T)$$

$$+ 21(F^-_t = F_0 + w)P_t(F_t + w - w f_t - w[k_m - (n_t^c + f_t)]^{+}, T).$$

Suppose that at $t = 0$, an investor sells a call on the product of $w$ and the number of crosses of the interval $(F_0, F_0 + w)$. The hedging strategy has either one or two regimes, corresponding to whether or not the call on crosses finishes out-of-the-money. In the first regime, options are traded at the completion of each cross. If there is a second regime, options are traded each time the running maximum rises. In both regimes, the option trading strategy is semi-static.

Since $n^c_0 = 0$, $f_0 = 0$, and $F^-_0 = F_0$, (78) indicates that the initial hedge requires buying 2 calls struck $K_m = w k_m$ dollars above $F_0$. On the completion of each cross, the investor switches the polarity of the option held. While the number of crosses $n_t^c + f_t < k_m$, the completion of each upcross involves selling 2 calls and buying 2 puts. Since $F = F_0 + w$ at the completion of each upcross, the 2 puts purchased are struck $2w$ dollars closer to $F_0$ than the 2 calls sold. While $n_t^c < k_m$, the completion of each downcross involves selling 2 puts and buying 2 calls. Since $F = F_0$ at the completion of each downcross, the 2 puts sold are struck the same distance away from $F_0$ as the 2 calls bought. In this regime, each round trip forces the strikes held to move in towards $F_0$ by $2w$. As 2 crosses occur per round trip, one is free to think of both strikes as moving in towards $F_0$ by $w$ on each cross, so long as one remembers that only one parity is held at a time\(^6\).

If the number of crosses at expiry $n^c_T + f_T$ is less than $k_m$, then the call on crosses expires worthless.

\(^6\)If one adopts this mnemonic, the initial put strike should be set at $F_0 - w k_m$. 29
as does its hedge. On the other hand, if the number of crosses \( n_c^c \) exceeds \( k_m \) at some point prior to \( T \), then the first time that \( n_c^c \geq k_m \), the hedger enters the second regime, where the only strikes held creep through the interval \( (F_0, F_0 + w) \). Changes in parity of the option held continue to be caused by changes in the number of completed crosses. If the reference trading strategy is long \( (F^-_t = F_0) \), the hedger holds 2 calls struck \( wF_t \) dollars above \( F_0 \). If the reference trading strategy is short \( (F^-_t = F_0 + w) \), the hedger holds 2 puts struck \( wF_t \) dollars below \( F_0 + w \). All changes in strike are caused by an increase in the fraction of the cross completed. As the strikes of the 2 calls move up or the strikes of the 2 puts move down, cash is generated and used to purchase bonds. The total cash generated allows the investor to buy \( w(n_c^c + f_T - k_m)^+ \) bonds, which provides the desired payoff at expiry.

Setting \( t = 0 \) in (78):

\[
C_0^{m\gamma}(k_m w, T) = 2C_0(F_0 + wk_m, T). \tag{79}
\]

Hence, the initial fair value of a call paying \( w \) times \( (n_c^c + f_T - k_m)^+ \) at \( T \) is simply twice the value of a standard call struck \( K_m = wk_m \) dollars above \( F_0 \).

VII Semi-Robust Hedge for Call on Local Time

Suppose that a stock price process is continuous, and for simplicity suppose that we have zero interest rates and dividends. Suppose that an investor initially sells an at-the-money (ATM) straddle and pockets a positive premium. The standard approach for ensuring that no payout is made at expiry is to assume that volatility depends on at most the stock price path and then to delta hedge with the underlying stock.

As the initial cost of this hedge is positive, suppose that the investor tries an alternative hedging strategy which appears to be costless. This trading strategy is simply to be short two units of whichever option is presently out-of-the-money (OTM). Hence if the stock price rises initially, then since \( S_{0+} > S_0 \), the investor buys one call and sells one put, both struck at \( S_0 \). Conversely, if the stock price falls initially, then since \( S_{0+} < S_0 \), the investor buys one put and sells one call, both struck at \( S_0 \). If the stock price returns to \( S_0 \), the investor returns to a short ATM straddle position. If the return is from above, then the
investor buys one put and sells one call, both struck at $S_0$. If the return is from below, then the investor buys one call and sells one put, both struck at $S_0$. Under zero interest rates and dividends, put call parity implies that the trading done upon returning to $S_0$ is self-financing. Whenever a stock move does not involve leaving or returning to $S_0$, no trade is required.

At expiry, the investor is holding an ATM or OTM option position and hence no liability arises. As the strategy is static when $S$ is comfortably away from $S_0$, the only issue determining whether or not this trading strategy is an arbitrage opportunity is the determination of whether or not this strategy is self-financing when options are bought and sold around $S_0$. In fact, if the continuous underlying price process also has bounded variation, then the strategy is both replicating and self-financing. Given that a positive premium was collected initially, the suggested strategy is an arbitrage opportunity. No arbitrage therefore requires that the initial premium collected from selling the ATM straddle vanish.

In practice, ATM straddles with positive time to maturity carry positive premia. This easily observed phenomenon suggests that a realistic model for the underlying stock price dynamics must have either unbounded variation sample paths or jumps or both. Following the route that most of the option pricing literature has gone, suppose we assume that the underlying price process is continuous over time and that all sample paths have unbounded variation. Then the above option trading strategy need not be self-financing whenever the stock price leaves $S_0$. In fact, a mathematical result called the Tanaka Meyer formula tells us that losses accumulate according to the so-called local time of the stock price at its initial level. As a result, no arbitrage requires that the initial straddle premium is just the initial risk-neutral expectation of the local time at expiry.

While at-the-money options tell us the mean local time, it is of interest to know whether or not the implied volatility smile of a fixed maturity can tell us the whole risk-neutral distribution of local time at the options’ expiry. This section shows that the smile does have this information content so long as the underlying price is an Ocone martingale under $Q$. We also explicitly show how to value a call option on local time using just one option price. Furthermore, we indicate the hedging strategy for the sale of a call on local time, which just involves changing the strike and parity of an option held whenever it goes
We define the \textit{mean-reverting strategy} as a binary trading strategy where shareholdings just depend on the underlying Ocone martingale $F$ as follows:

$$c_t = -\text{sgn}(F_t - F_0), \quad t \in [0, T],$$  \hfill (80)

where:

$$\text{sgn}(x) \equiv \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x \leq 0.
\end{cases}$$

In words, an investor following the mean-reverting strategy is short the asset if its price is at or above its initial mean $F_0$ and long it otherwise. Notice that our sgn(x) definition is asymmetric for simplicity, but it will not matter how we define sgn(x) at $x = 0$. The mean-reverting strategy in (80) is clearly the limit of the contrarian strategy (68) as the width $w \downarrow 0$. Recall that for each $w > 0$, the maximum of the gains from the latter strategy is the product of $w$ and the number of crosses of the interval $(F_0, F_0 + w)$. As $w \downarrow 0$, the number of crosses approaches infinity and hence it is not clear ex ante what this maximum process converges to. As first shown by Lévy in the Brownian case, this maximum process will converge to the local time of the $F$ process at $F_0$.

We let $G_t$ denote the gains to $t$ from adopting the mean-reverting trading strategy (80):

$$G_t \equiv -\int_0^t \text{sgn}(F_t - F_0) dF_t, \quad t \in [0, T].$$  \hfill (81)

Since $F$ is an Ocone martingale, so is $G$. Let $M_t^G \equiv \max_{s \in [0,t]} G_s$ denote the maximum gain from the mean-reverting trading strategy employed over $[0, t]$. As $G$ is just the gains from a particular binary trading strategy, we can specialize (65) to:

$$C_t^{mad}(K_m, K_d, T) =$$

$$(M_t^G - K_m)^+ [1(F_t > F_0)C_t(F_t^- + G_t^- - M_t^G + K_d, T) + 1(F_t \leq F_0)P_t(F_t^- - G_t^- + M_t^G - K_d, T)]$$

$+$

$$1(F_t \leq F_0)C_t(F_t^- - G_t^- + M_t^G + K_d, T) + 1(F_t > F_0)P_t(F_t^- + G_t^- - M_t^G - K_d, T)]$$

$+$

$$2 \int_{M_t^G \vee K_m} [1(F_t \leq F_0)C_t(F_t^- - G_t^- + H + K_d, T) + 1(F_t > F_0)P_t(F_t^- + G_t^- - H - K_d, T)] dH.$$  \hfill (82)

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We define the running absolute deviation as the process:

\[ A^F_t \equiv |F_t - F_0|, \quad t \in [0,T]. \] (83)

Let \( L^F_t \) denote the local time at \( t \in [0,T] \) of the \( F \) process at its initial level \( F_0 \). The Tanaka Meyer formula relates these two processes by:

\[ A^F_t = L^F_t - G_t, \quad t \in [0,T]. \] (84)

To analyze \( L^F \), we need the following lemma due to Skorohod (see for example [17], page 239).

**Skorohod’s Lemma:** Let \( y(t) \) be a real-valued continuous function of \( t \in [0, \infty) \) such that \( y(0) \geq 0 \). There exists a unique pair \((z, a)\) of functions of \( t \in [0, \infty) \) such that:

1. \( z(t) = a(t) + y(t) \)

2. \( z(t) \geq 0 \)

3. \( a \) is increasing, continuous, \( a(0) = 0 \), and the corresponding measure \( da \) is carried by \( \{ t : z(t) = 0 \} \).

Moreover, the function \( a \) is given by \( a(t) = \sup_{s \in [0,t]} (-y(s) \lor 0) \).

Consider some realization of the common process on the two sides of (84). Suppose that we attempt to apply Skorohod’s lemma to this realization by attempting to identify \( y(t) \) with the realization of \( -G_t \). The process \( -G \) is real-valued, continuous, and has \( G(0) \geq 0 \), so all of the requirements for \( y \) are met. Now suppose that we attempt to identify \( z(t) \) with the realization of \( A^F_t \) and \( a(t) \) with the realization of \( L^F_t \). The first condition of Skorohod’s lemma is met because of (84). The second condition of Skorohod’s lemma is met because \( A^F_t \geq 0 \). The third condition of Skorohod’s lemma is met because \( L^F_t \) is increasing, continuous, \( L^F(0) = 0 \), and the corresponding measure \( dL \) is carried by \( \{ t : A^F_t = 0 \} \). As a result, we may conclude that:

\[ L^F_t = \sup_{s \in [0,t]} [-(G_s) \lor 0] = M^G_t, \quad t \in [0,T]. \] (85)

Hence, the running local time of the \( F \) process at its initial level is identical to the running maximum of the gains from the mean-reverting strategy.
Recall that $F_t$ denotes the forward price at the time of the last change in parity. Since the mean-reverting strategy only changes parity when $F_t = F_0$, we have:

$$F_t^- = F_0, \quad t \in [0, T].$$  \hfill (86)

Since $M^G = L^F$, $M^G$ is a continuous increasing processes which increases only when $D^G = 0$, i.e. $M^G_t = G_t$. From (84):

$$G_t^- = (L_t^F)^- - (A_t^F)^- = (M_t^G)^- - 0 = M_t^G, \quad t \in [0, T].$$  \hfill (87)

Substituting (86) and (87) in (82) implies that:

$$C_{mdg}^t(K_m, K_d, T) = (M_t^G - K_m)^+ [P_t(F_0 - K_d, T) + C_t(F_0 + K_d, T)]$$

$$+ 21(F_t > F_0) \int_{M_t^G \vee K_m} P_t(F_0 + M_t^G - H - K_d, T) dH$$

$$+ 21(F_t \leq F_0) \int_{M_t^G \vee K_m} C_t(F_0 - M_t^G + H + K_d, T) dH. \hfill (88)$$

Moreover, suppose that we define the running drawdown of $G$ by:

$$D_t^G \equiv M_t^G - G_t, \quad t \in [0, T].$$  \hfill (89)

Comparing (84) with (89), we have by the uniqueness of the solution to Skorohod’s lemma that the bivariate process $(M^G, D^G)$ is identical to $(L^F, A^F)$. Let $C_{la}^t(K_m, K_d, T)$ be the arbitrage-free value of a claim paying $(L_T^F - K_m)^+(A_T^F - K_d)^+$ at $T$. Since $L_T^F$ and $A_T^F$ are both nonnegative, so are the strikes $K_m$ and $K_d$. Since the bivariate process $(L^F, A^F)$ is identical to $(M^G, D^G)$, we have from (88) that:

**Theorem 10: Semi-Robust Pricing of Product Call on Local Time and Absolute Deviation**

Under frictionless markets and $A4$, no arbitrage implies that for $K_m, K_d \geq 0, t \in [0, T], C_{la}^t(K_m, K_d, T) = \hfill (90)$

$$(L_t^F - K_m)^+ [P_t(F_0 - K_d, T) + C_t(F_0 + K_d, T)]$$

$$+ 21(F_t > F_0) \int_{L_t^F \vee K_m} P_t(F_0 + L_t^F - H - K_d, T) dH + 21(F_t \leq F_0) \int_{L_t^F \vee K_m} C_t(F_0 - L_t^F + H + K_d, T) dH.$$
In words, the sale of a product call at \( t = 0 \) is initially hedged by establishing an infinitessimal position in an array of OTM options. If \( F_t > F_0 \) and with \( L_t^F \leq K_m \), the hedger just holds \( 2dK_p \) puts for all strikes \( K_p < F_0 - (K_m - L_t^F) - K_d \). If \( F_t \leq F_0 \) and with \( L_t^F \leq K_m \), the hedger just holds \( 2dK_c \) calls for all strikes \( K_c > F_0 + K_m - L_t^F + K_d \). The transitions that occur between option parities that arise around \( F_0 \) are self-financing. If \( L_t^F \) never reaches \( K_m \) by \( T \), then the holdings in the OTM options expire worthless. On the other hand, if \( L_t^F \) reaches \( K_m \) before \( T \), then as soon as \( L_t^F = K_m \), the hedger initiates a discrete and growing position in strangles centered at \( F_0 \) and with width \( K_d \). This position is combined with the existing infinitessimal position in an array of OTM options. At expiry, the strangle position provides the payoff \( (L_T^F - K_m)^+ + (F_T - F_0 - K_d)^+ = (L_T^F - K_m)^+ (\Delta_T^F - K_d)^+ \) as desired. At expiry, the holdings in the OTM options again expire worthless. The strangle position at \( t \) represents the intrinsic value at \( t \) of the product call, while the OTM option position at \( t \) captures its volatility value.

Let \( C_t^l(K_d, T) \) be the arbitrage-free value of a call on local time paying \( (L_T^F - K_m)^+ \) at \( T \). Differentiating (90) w.r.t. \( K_d \), negating, and setting \( K_d = 0 \) recovers the value of this call:

\[
C_t^l(K_m, T) = (L_t^F - K_m)^+ B_t(T) + 21(F_t > F_0) P_t(F_0 - (K_m - L_t^F)^+, T) + 21(F_t \leq F_0) C_t(F_0 + (K_m - L_t^F)^+, T).
\]

Suppose that an investor sells a call on local time at time 0 and that the investor wishes to replicate the payoff via semi-static option trading. Since the local time is initially zero, (82) indicates that the replicating strategy starts by holding 2 calls struck \( K_m \) dollars above \( F_0 \). As the hedger moves through calendar time with \( L_t^F \leq K_m \), the hedger always holds two units of whichever standard option is out-of-the-money. The strike of the option held is \( K_m - L_t^F \geq 0 \) dollars away from \( F_0 \). If \( F_t = F_0 \) with \( L_t^F \leq K_m \), then the hedger holds two puts with strike \( K_m - L_t^F \geq 0 \) dollars below \( F_0 \). The reason that the strikes move in towards \( F_0 \) is that all option trades are made just as \( F \) moves away from \( F_0 \) in such a way that the two options held become ITM. Each such trade involves a disposition of the two ITM options and an acquisition of two OTM options with the same price. As a result, the strike of the two acquired options must necessarily be closer to \( F_0 \). This strategy is self-financing by design. If the running local time \( L \) never reaches \( K_m \) before \( T \), the hedger just ends up holding 2 worthless OTM options. On the other hand, if \( L \) reaches \( K_m \) before \( T \), the hedger just ends up holding 2 worthless OTM options.
Then at the first passage time of $L$ to $K_m$, the strike of the 2 options held first reaches $F_0$. Between this first passage time and expiry, the hedger holds 2 puts with strike $F_0$ if $F_t > F_0$, and 2 calls with strike $F_0$ otherwise. As this strategy is just the reverse of the one described at the beginning of this section, it generates a positive cash flow each time that $F$ crosses $F_0$, whose magnitude at $T$ accumulates to $L_T^F - K_m$.

Evaluating (91) at $t = 0$ implies:

$$C^d_0(K_m, T) = 2C_0(F_0 + K_m, T) .$$  \hfill (92)

Hence at initiation, a call on local time with strike $K_m$ has twice the value of a standard call struck $K_m$ dollars out-of-the-money. Due to APCS, a call on local time with strike $K_m$ also has twice the value initially as a standard put struck $K_m$ dollars out-of-the-money. More symmetrically, at initiation, a call on local time with strike $K_m$ has the same value as a centered strangle of width $K_m$.

Now consider a call on the absolute deviation $A_T^F ≡ |F_T - F_0|$ with strike $K_d ≥ 0$. This call has the same payoff as a strangle centered at $F_0$ with width $K_d$. As expected, differentiating (82) w.r.t. $K_m$, negating, and setting $K_m = 0$ recovers the value of this strangle:

$$-\frac{\partial}{\partial K_m} C^{da}_t(0, K_d, T) = P_t(F_0 - K_d, T) + C_t(F_0 + K_d, T) .$$  \hfill (93)

**VIII Summary and Future Research**

We showed that the problem of pricing and hedging a claim paying the product of call payoffs on the maximum and the drawdown reduces to the problem of pricing and hedging simple standard and barrier options. We then gave alternative sufficient conditions under which the hedge just involves holdings in standard options. We also showed that these continuity and sufficient conditions permit all path-independent and some path-dependent options on trading gains to be semi-statically hedged using standard options. We showed that a call option on the maximum of a particular contrarian trading strategy is equivalent to a call written on the total number of crosses of a given spatial interval. By shrinking the width of this interval down to zero, we showed that Skorohod’s Lemma can be used to find the semi-static
hedge of options on local time.

For future research, one can experiment with the allowed dynamics. One can treat the two symmetry conditions explored here as special cases arising from a more general family of symmetries. One can also explore adding the possibility of default to the regimes which assume a continuous underlying asset price process. We note that the payoff \((M_T - S_0 - K_m)^+ (M_T - S_T - K_d)^+\) looks the same when the flow of time is reversed.

For future research, one can also experiment with the allowed payoffs. For example, consider the following generalization of the product call payoff:

\[
h(M_T, S_T) = (M_T - K_m)^+ (\lambda M_T - S_T - K_d)^+,
\]

where \(\lambda \in [0, 1]\). If we define generalized drawdown \(D_T(\lambda) \equiv \lambda M_T - S_T\), then this payoff is just the product of a call on the maximum and a call on generalized drawdown. When \(\lambda \in [0, 1]\), this generalized drawdown can be negative, in contrast to the standard drawdown obtained when \(\lambda = 1\). In fact, Grossman and Zhou[12] and Cvitanic and Karatzas[7] consider portfolio optimization under the constraint that \(\lambda \in (0, 1)\) and that this generalized drawdown must be negative.

The generalization of Theorem 1 to generalized drawdown is:

\[
C_{md}^m(K_m, K_d, \lambda, T) = B_t(T) E_t^Q h(M_T, S_T)
\]

\[
= (M_t - K_m)^+ P_t(\lambda M_t - K_d, T) + \lambda \int_{M_t \lor K_m}^{\infty} (H - K_m) UIBP_t(\lambda H - K_d, T; H) dH
\]

\[
+ \int_{M_t \lor K_m}^{\infty} UIP_t(\lambda H - K_d, T; H) dH, \quad t \in [0, T]. \tag{95}
\]

The univariate call on generalized drawdown is obtained by applying the operator \(-\frac{\partial}{\partial K_m} \bigg|_{K_m=0}\) to (95):

\[
C_d^d(K_d, \lambda, T) \equiv B_t(T) E_t^Q (\lambda M_T - S_T - K_d)^+
\]

\[
= P_t(\lambda M_t - K_d, T) + \lambda \int_{M_t}^{\infty} UIBP_t(\lambda H - K_d, T; H) dH, \quad t \in [0, T], K_d \geq 0, \lambda \in [0, 1]. \tag{96}
\]
Other payoffs such as \( \left( \frac{M_T - S_T}{M_T} - K_d \right)^+ \) and \( (M_T - \lambda S_T - K_d)^+ \) and combinations of the two can also be priced and hedged using the methods in this paper.

A risk measure that is related but distinct from terminal drawdown is called maximum drawdown. Recalling that \( D_t \equiv M_t - S_t \) denotes the running drawdown, maximum drawdown is defined as \( \max_{t \in [0,T]} D_t \). Future research should be directed towards extending these results to forward contracts and to options on maximum drawdown. In the interests of brevity, this captivating possibility is not pursued here.
Appendix 1: Derivation of Theorem 3

Suppose that we write the result of Theorem 1 as:

\[ C^{md}_t(K_m, K_d, T) = (M_t - K_m)^+ P_t(M_t - K_d, T) + I, \]  

where:

\[ I \equiv \int_{M_t}^\infty (H - K_m)^+ UIBP_t(H - K_d, T; H) dH + \int_{M_t}^\infty 1(H > K_m) UIP_t(H - K_d, T; H) dH. \]  

This integral has the following simpler representation:

\[ I = - \left( \frac{\partial}{\partial K_m} + \frac{\partial}{\partial K_d} \right) \int_{M_t}^\infty (H - K_m)^+ UIP_t(H - K_d, T; H) dH. \]  

Under A2, the results of Bowie and Carr[2] imply that the value of an up-and-in put written on a forward price can be related to the value of a call:

\[ UIP_t(H - K_d, T; H) = \frac{H - K_d}{H} C_t \left( \frac{H^2}{H - K_d}, T \right), \quad t \in [0, T \wedge \tau_H], \]  

where \( \tau_H \) is the first passage time to the upper barrier \( H \). Notice that we now require \( K_d \in [0, H) \) in order for (100) to be well-defined. Substituting (100) in (99) implies:

\[ I = - \left( \frac{\partial}{\partial K_m} + \frac{\partial}{\partial K_d} \right) \int_{M_t}^\infty (H - K_m)^+ \frac{H - K_d}{H} C_t \left( \frac{H^2}{H - K_d}, T \right) dH. \]  

Let:

\[ K_c \equiv \frac{H^2}{H - K_d} \]  

be a change of variable in the integral. Hence:

\[ \frac{H^2}{2} - \frac{K_c}{2} H + \frac{K_c K_d}{2} = 0. \]  

Using the quadratic root formula:

\[ H(K_c) \equiv \frac{K_c}{2} + \sqrt{\frac{K_c^2}{4} - K_d K_c}, \]
where the + in ± has been chosen to ensure that $H = K_c$ when $K_d = 0$. Hence:

$$
H'(K_c) = \frac{1}{2} \frac{K_c/2 - K_d}{\sqrt{K_c^2/4 - K_d K_c}}.
$$

$$
= \frac{K_c/2 - K_d + \sqrt{K_c^2/4 - K_d K_c}}{2 \sqrt{K_c^2/4 - K_d K_c}}.
$$

$$
= \frac{H(K_c) - K_d}{2H(K_c) - K_c}, \quad (105)
$$

from (104). Making the indicated change of variable, (101) becomes:

$$
I = \left( \frac{\partial}{\partial K_m} + \frac{\partial}{\partial K_d} \right) \int_{\frac{M^2}{M_t - K_d}}^{\infty} \left[ H(K_c) - K_m \right] \frac{H(K_c) - K_d}{H(K_c)} H'(K_c) C_t (K_c, T) dk_c
$$

$$
= L + J, \quad (106)
$$

where from the fundamental theorem of calculus and the chain rule:

$$
L \equiv \left[ H(K_c) - K_m \right] + \frac{H(K_c) - K_d}{H(K_c)} H'(K_c) C_t (K_c, T) \bigg|_{K_c = \frac{M^2}{M_t - K_d}} \frac{M^2}{(M_t - K_d)^2} \quad (107)
$$

and:

$$
J \equiv \int_{\frac{M^2}{M_t - K_d}}^{\infty} \left( \frac{\partial}{\partial K_m} + \frac{\partial}{\partial K_d} \right) \left[ H(K_c) - K_m \right] + \frac{H(K_c) - K_d}{H(K_c)} H'(K_c) C_t (K_c, T) dk_c. \quad (108)
$$

Substituting (105) in (107):

$$
L = \left[ H(K_c) - K_m \right] + \frac{[H(K_c) - K_d]^2}{H(K_c)[2H(K_c) - K_c]} C_t (K_c, T) \bigg|_{K_c = \frac{M^2}{M_t - K_d}} \frac{M^2}{(M_t - K_d)^2}
$$

$$
= [M_t - K_m] + \frac{M_t - K_d}{M_t \left[ 2M_t - \frac{M^2}{M_t - K_d} \right]} C_t \left( \frac{M^2}{M_t - K_d}, T \right) \frac{M^2}{(M_t - K_d)^2}
$$

$$
= (M_t - K_m) + \frac{M_t - K_d}{M_t - 2K_d} C_t \left( \frac{M^2}{M_t - K_d}, T \right). \quad (109)
$$

To simplify the expression for $J$, first note that (102) implies that:

$$
\frac{H(K_c) - K_d}{H(K_c)} = \frac{H(K_c)}{K_c}. \quad (110)
$$
Substituting (110) in (108):

\[
J = \int_{\frac{M^2}{M_t-K_d}}^{\infty} - \left( \frac{\partial}{\partial K_m} + \frac{\partial}{\partial K_d} \right) [H(K_c) - K_m] + \frac{H(K_c)}{K_c} H'(K_c) C_t(K_c, T) dK_c. \tag{111}
\]

Distributing the differential operators in (111):

\[
J = \int_{\frac{M^2}{M_t-K_d}}^{\infty} \left[ 1[H(K_c) > K_m] \frac{H(K_c)}{K_c} H'(K_c) - 1[H(K_c) > K_m] \frac{dH(K_c)}{dK_d} \frac{H(K_c)}{K_c} H'(K_c) \right.
+ \left. [H(K_c) - K_m]^+ \left( -\frac{\partial}{\partial K_d} \right) \frac{H(K_c)}{K_c} H'(K_c) \right] C_t(K_c, T) dK_c. \tag{112}
\]

Distributing the integral:

\[
J = J_1 + J_2, \tag{113}
\]

where:

\[
J_1 \equiv \int_{\frac{M^2}{M_t-K_d}}^{\infty} 1[H(K_c) > K_m] \left[ 1 - \frac{dH(K_c)}{dK_d} \right] \frac{H(K_c)}{K_c} H'(K_c) C_t(K_c, T) dK_c, \tag{114}
\]

and:

\[
J_2 \equiv \int_{\frac{M^2}{M_t-K_d}}^{\infty} [H(K_c) - K_m]^+ \left( -\frac{\partial}{\partial K_d} \right) \frac{H(K_c)}{K_c} H'(K_c) C_t(K_c, T) dK_c. \tag{115}
\]

To do \( J_1 \), differentiate (103) w.r.t. \( K_d \) holding \( K_c \) constant:

\[
H(K_c) \frac{dH(K_c)}{dK_d} - \frac{K_c}{2} \frac{dH(K_c)}{dK_d} + \frac{K_c}{2} = 0. \tag{116}
\]

So:

\[
\frac{dH(K_c)}{dK_d} = \frac{K_c/2}{K_c/2 - H(K_c)}. \tag{117}
\]

Hence:

\[
1 - \frac{dH(K_c)}{dK_d} = 1 + \frac{K_c}{2H(K_c) - K_c} = \frac{2H(K_c)}{2H(K_c) - K_c}. \tag{118}
\]

However, (104) implies:

\[
2H(K_c) - K_c = \sqrt{K_c} \sqrt{K_c - 4K_d}. \tag{119}
\]
Substituting (119) in (118) implies:

\[ 1 - \frac{dH(K_c)}{dK_d} = \frac{2H(K_c)}{\sqrt{K_c\sqrt{K_c - 4K_d}}}. \tag{120} \]

Furthermore, substituting (119) in (105) implies:

\[ H'(K_c) = \frac{H(K_c) - K_d}{\sqrt{K_c\sqrt{K_c - 4K_d}}}. \tag{121} \]

Substituting (120) and (121) in (114):

\[
J_1 = \int \limits_{\frac{M^2}{m_t-K_d}}^{\infty} 1[H(K_c) > K_m] \frac{2H(K_c)H(K_c) - H(K_c) - K_d}{K_c - 4K_d} C_t(K_c, T) \frac{dK_c}{K_c}
\]

\[
= \int \limits_{\frac{M^2}{m_t-K_d}}^{\infty} 1[H(K_c) > K_m] \frac{2H(K_c)H(K_c) - K_d}{K_c - 4K_d} C_t(K_c, T) \frac{dK_c}{K_c}
\]

\[
= \int \limits_{\frac{M^2}{m_t-K_d}}^{\infty} 1[H(K_c) > K_m] 2[H(K_c) - K_d]^2 \frac{dK_c}{K_c - 4K_d}, \tag{122}
\]

from re-arranging (110).

Turning to \( J_2 \), first note from (104) and (105) that:

\[
\frac{H(K_c)}{K_c}H'(K_c) = \frac{1}{K_c} \left[ \frac{K_c}{2} + \sqrt{K_c^2/4 - K_cK_d} \right] \left[ \frac{1}{2} + \frac{K_c/2 - K_d}{2\sqrt{K_c^2/4 - K_cK_d}} \right]
\]

\[
= \frac{1}{K_c} \left[ \frac{K_c}{2} + \frac{\sqrt{K_c}}{2\sqrt{K_c - 4K_d}} \right] \left[ \frac{1}{2} + \frac{K_c/2 - K_d}{\sqrt{K_c\sqrt{K_c - 4K_d}}} \right]. \tag{123}
\]

Hence:

\[
- \frac{\partial}{\partial K_d} \left[ \frac{H(K_c)}{K_c}H'(K_c) \right] = \frac{1}{K_c} \left[ \frac{4\sqrt{K_c}}{2\sqrt{K_c - 4K_d}} \left[ \frac{1}{2} + \frac{K_c/2 - K_d}{\sqrt{K_c\sqrt{K_c - 4K_d}}} \right] \right. + \left. \frac{1}{K_c} \left[ \frac{K_c}{2} + \frac{\sqrt{K_c}}{2\sqrt{K_c - 4K_d}} \right] \left[ \frac{1}{\sqrt{K_c\sqrt{K_c - 4K_d}}} - \frac{(K_c/2 - K_d)4}{2\sqrt{K_c(\sqrt{K_c - 4K_d})^3}} \right]. \]

Distributing in the obvious way produces six terms, of which two terms combine and two terms cancel
leaving:

\[- \frac{\partial}{\partial K_d} \left[ \frac{H(K_c)}{K_c} H'(K_c) \right] = \frac{1}{K_c} \left[ \frac{\sqrt{K_c}}{\sqrt{K_c - 4K_d}} + \frac{1}{2} - \frac{\sqrt{K_c}(K_c - 2K_d)}{2(\sqrt{K_c - 4K_d})^3} \right] \]

\[= \frac{1}{K_c} \left[ 2\sqrt{K_c}(K_c - 4K_d) + (\sqrt{K_c - 4K_d})^3 - \sqrt{K_c}(K_c - 2K_d) \right] \]

\[= \frac{1}{K_c} \left[ (\sqrt{K_c - 4K_d})^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c} \right] \]

(124)

Substituting (124) in (115):

\[J_2 = \int_{\frac{M_t-K_m}{K_t}}^{\infty} \left\{ [H(K_c) - K_m]^+ \left[ \frac{\sqrt{K_c - 4K_d}^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c}}{2(\sqrt{K_c - 4K_d})^3} \right] \right\} C_t(K_c, T) \frac{dK_c}{K_c}. \]

(125)

Substituting (122) and (125) in (113):

\[J = \int_{\frac{(M_t+K_m)^2}{(M_t+K_m)-K_d}}^{\infty} \left\{ \frac{2[H(K_c) - K_d]^2}{K_c - 4K_d} + [H(K_c) - K_m] \left[ \frac{\sqrt{K_c - 4K_d}^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c}}{2(\sqrt{K_c - 4K_d})^3} \right] \right\} C_t(K_c, T) \frac{dK_c}{K_c}, \]

(126)

where recall from (104):

\[H(K_c) = \frac{K_c}{2} + \sqrt{\frac{K_c^2}{4} - K_dK_c}. \]

(127)

Substituting the respective expressions (109) and (126) for L and J in (106) implies:

\[I = (M_t - K_m)^+ \frac{M_t - K_d}{M_t - 2K_d} C_t \left( \frac{M_t^2}{M_t - K_d} \right) T \]

\[+ \int_{\frac{(M_t+K_m)^2}{(M_t+K_m)-K_d}}^{\infty} \left\{ \frac{2[H(K_c) - K_d]^2}{K_c - 4K_d} + [H(K_c) - K_m] \left[ \frac{\sqrt{K_c - 4K_d}^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c}}{2(\sqrt{K_c - 4K_d})^3} \right] \right\} C_t(K_c, T) \frac{dK_c}{K_c}. \]

(128)

Finally, substituting (128) in (97) implies that \( C_t^{md}(K_m, K_d, T) = \)

\[(M_t - K_m)^+ P_t(M_t - K_d, T) + (M_t - K_m)^+ \frac{M_t - K_d}{M_t - 2K_d} C_t \left( \frac{M_t^2}{M_t - K_d} \right) T + \]

\[\int_{\frac{(M_t+K_m)^2}{(M_t+K_m)-K_d}}^{\infty} \left\{ \frac{2[H(K_c) - K_d]^2}{K_c - 4K_d} + [H(K_c) - K_m] \left[ \frac{\sqrt{K_c - 4K_d}^3 + (\sqrt{K_c})^3 - 6K_d\sqrt{K_c}}{2(\sqrt{K_c - 4K_d})^3} \right] \right\} C_t(K_c, T) \frac{dK_c}{K_c}, \]

(129)

where \( H(K_c) \) is given in (127). This proves Theorem 3.
Appendix 2: Proof of Theorem 4

Let $F_t^-$, and $m_t^-$ respectively denote the stock price and the passport moneyness at the last switch time at or before time $t$. Thus, given our assumption that switches do not occur continuously over time, $F_t^-$, and $m_t^-$ will almost always be pure jump processes, which are RCLL. Let $N_t^c(K)$ and $N_t^p(K)$ respectively denote the number of calls and puts of strike $K \in \mathbb{R}$ and maturity $T$ which are held at time $t \in [0,T]$. Consider the following dynamic trading strategy in calls and puts:

\[ N_t^c(K) = 1(c_t = 1)\delta(K - (F_t^- - m_t^-)) \quad N_t^p(K) = 1(c_t = -1)\delta(K - (F_t^- + m_t^-)), \quad t \in [0,T], \quad (130) \]

where $\delta(\cdot)$ denotes the Dirac delta function. Since the trading strategy depends only on information available at time $t$, it is not anticipating. Since $c$ is either $\pm 1$ and the delta function is nonzero at only one strike, only one option is held at each time. Finally, since $F_t^-$, and $m_t^-$ only change values at switch times, the trading strategy is semi-static, in that it is static between switch times.

Given this trading strategy, the value of the hedge portfolio is defined as:

\[
V_t \equiv \int_{-\infty}^{\infty} N_t^c(K)C_t(K,T)dK + \int_{-\infty}^{\infty} N_t^p(K)P_t(K,T)dK \\
= \int_{-\infty}^{\infty} 1(c_t = 1)\delta(K - (F_t^- - m_t^-))C_t(K,T)dK + \int_{-\infty}^{\infty} 1(c_t = -1)\delta(K - (F_t^- + m_t^-))P_t(K,T)dK \\
= 1(c_t = 1)C_t(F_t^- - m_t^-, T) + 1(c_t = -1)P_t(F_t^- + m_t^-, T), \quad (131)
\]

from the sifting property of delta functions. We can paraphrase this strategy by saying that the hedger always holds the option whose parity matches that of the gains process and whose moneyness at the last switch time matched the moneyness of the passport then. At a switch time $\tau_i$, the standard option held at $\tau_i$ has the same parity as the gains process at $\tau_i$. The standard option held at $\tau_i$ also has the same moneyness as the passport at $\tau_i$. 
Whether or not the APCS condition holds, this portfolio is replicating:

\[ V_T = 1(c_T = 1)C_T(F_T^{-} - m_T^{-}, T) + 1(c_T = -1)P_T(F_T^{+} + m_T^{+}, T) \]

\[ = 1(c_T = 1)(F_T^{-} - (F_T^{-} - m_T^{-}))^{+} + 1(c_T = -1)(F_T^{+} + m_T^{+} - F_T^{+})^{+} \]

\[ = 1(c_T = 1)(\pi_T^{-} + F_T^{-} - F_T^{-} - k)^{+} + 1(c_T = -1)(\pi_T^{+} - (F_T^{+} - F_T^{-}) - k)^{+} \]

\[ = 1(c_T = 1)(\pi_T^{-} - k)^{+} + 1(c_T = -1)(\pi_T^{-} - k)^{+} \]

\[ = (\pi_T^{-} - k)^{+}, \tag{132} \]

since \( 1(c_t = 1) + 1(c_t = -1) = 1 \) for all \( t \in [0, T] \). The initial cost of setting up this replicating portfolio is obtained by setting \( t = 0 \) in (131):

\[ V_0 = 1(c_0 = 1)C_0(F_0 + k, T) + 1(c_0 = -1)P_0(F_0 - k, T) = C_0(F_0 + k, T), \tag{133} \]

by APCS (52) and since \( 1(c_0 = 1) + 1(c_0 = -1) = 1 \). Hence, the replicating portfolio has the same initial cost as the call or put with the same initial moneyness as the passport.

For simplicity, we assume that the total number of parity changes over \([0, T]\) is countable. Let \( \{N_t : t \in [0, T]\} \) be the counting process keeping track of the total number of parity changes over \([0, t]\). Let \( \tau_i, i = 1, 2, \ldots, N_T, \) be the random switch times at which parity changes. For notational ease, we also define \( \tau_0 = 0 \) and \( \tau_{N_T + 1} \equiv T \). We will henceforth adopt the notational convention of replacing subscripts \( \tau_i \) by just \( i \). Using this notation, we have the following updating equations:

\[ \pi_{i+1} = \pi_i + c_i(F_{i+1} - F_i), \quad \pi_{i+1} = -c_i, \tag{134} \]

for \( i = 0, 1, 2, \ldots, N_T, \) where \( \pi_0 = 0 \).

Trivially we also have:

\[ V_T = V_0 + \sum_{i=1}^{N_T}[V_{i+1} - V_i]. \tag{135} \]

Substituting (132), (133), and (130) in (135) implies:

\[ (\pi_T^{-} - k)^{+} = C_0(F_0 + k, T) + \sum_{i=1}^{N_T} \{ 1(c_{i+1} = 1)C_{i+1}(F_{i+1} - m_{i+1}, T) + 1(c_{i+1} = -1)P_{i+1}(F_{i+1} + m_{i+1}, T) \]

\[ - [1(c_i = 1)C_i(F_i - m_i) + 1(c_i = -1)P_i(F_i + m_i)] \}. \tag{136} \]
Add and subtract $1(c_i = 1)C_{i+1}(F_{i+1} - m_{i+1}, T) + 1(c_i = -1)P_{i+1}(F_{i+1} + m_{i+1}, T)$:

$$\pi_T - k^+ = C_0(F_0 + k, T) +$$

$$\sum_{i=1}^{N_T} \{1(c_i = 1)[C_{i+1}(F_{i+1} - m_{i+1}, T) - C_i(F_i - m_i, T)] + 1(c_i = -1)[P_{i+1}(F_{i+1} + m_{i+1}, T) - P_i(F_i + m_i, T)]$$

$$+ \sum_{i=1}^{N_T} [C_{i+1}(F_{i+1} - m_{i+1}, T) - P_i(F_{i+1} + m_{i+1}, T)][1(c_{i+1} = 1) - 1(c_i = 1)] \) = 1(c_{i+1} = 1) - 1(c_i = 1)\right].$$

Substituting (138) in (137) implies:

$$\pi_T - k^+ = C_0(F_0 + k, T) +$$

$$\sum_{i=1}^{N_T} \{1(c_i = 1)[C_{i+1}(F_{i+1} - m_{i+1}, T) - C_i(F_i - m_i, T)] + 1(c_i = -1)[P_{i+1}(F_{i+1} + m_{i+1}, T) - P_i(F_i + m_i, T)]$$

$$+ \sum_{i=1}^{N_T} [C_{i+1}(F_{i+1} - m_{i+1}, T) - P_i(F_{i+1} + m_{i+1}, T)][1(c_{i+1} = 1) - 1(c_i = 1)] \} = 1(c_{i+1} = 1) - 1(c_i = 1)\right].$$

Substituting (140) and (141) in (139) implies that the trading strategy (130) is self-financing. As we have identified a non-anticipating replicating self-financing trading strategy, the arbitrage-free value of the passport call is just the contemporaneous value of the hedge portfolio:

$$C^S_t(k, T) = V_t = 1(c_t = 1)C_t(F_t^m = m_T^m, T) + 1(c_t = -1)P_t(F_t^m - m_T^m, T).$$

Q.E.D. 

(142)
References


