On Schrödinger's equation, 3-dimensional Bessel bridges, and passage time problems

Abstract: The main aim of this work is finding an explicit representation of the density φ_f of the first time T that a one-dimensional Brownian process B reaches the moving boundary f, where

$$f(t) := a + \int_0^t f'(s)ds$$

and

$$T := \inf\{t \geq 0 | B_t = f(t)\}$$

given that $f''(t) > 0$, $\forall t \geq 0$. We do so, by first finding the expected value of the following 3-dimesional Bessel bridge $\tilde{X}$ functional

$$\mathbb{E}\left[\exp\left\{-\int_0^s f''(u)\tilde{X}_u du\right\}\right],$$

[the reader may consult for instance Chapter 11 in Revuz and Yor (2005) for a general overview of this process] and exploiting its relationship with first-passage time problems as pointed out by Kardaras (2007). It turns out that this problem is related to Schrödinger's equation with time-dependent linear potential, see Feng (2001).

As a by-product we solve for a family of Volterra integral equations, which were previously only treated numerically, see Peskir (2001).

References

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