Options on Maxima, Drawdown, Trading Gains, and Local Time

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Overview

- There are three parts to this talk:
  1. Introduction
  2. Calls on everything but the kitchen sink
  3. Robust and semi-robust hedges.

Part I - Introduction: Static Position in a European Put

- A static position in a put with strike K pays off $(K - S_T)^+$ at its maturity T:

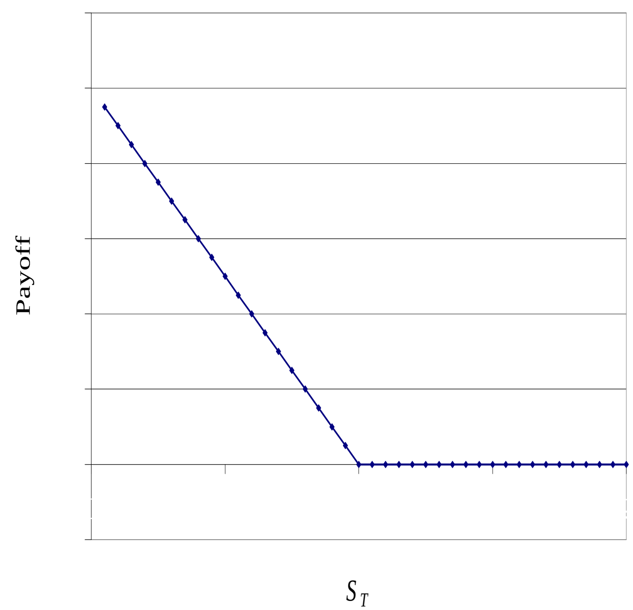

Figure 1: Put Payoff

Static Position in a European Call

- A static position in a call with strike K pays off $(S_T - K)^+$ at its maturity T:

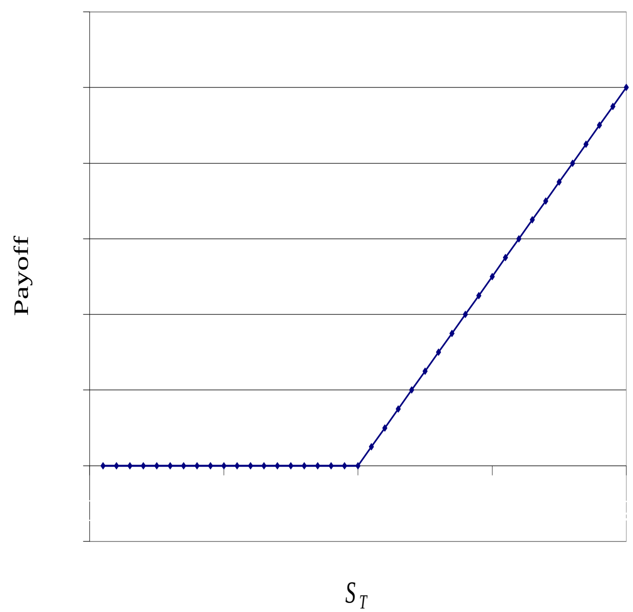

Figure 2: Call Payoff

Classical Hedging

- Beginning with the seminal work of Black Scholes (1973) and Merton (1973), many papers have focussed on the problem of replicating the payoffs to a static position in calls or puts by dynamically trading the underlying asset.
- Merton (1973) also showed that the payoff to a static position in a barrier option can also be replicated in this way.
- The payoff to static positions in many other exotic options such as lookbacks (eg. Goldman Sosin, and Gatto (1979)) or Asian options (see tomorrow's program) can also be replicated in this way.
- In all of these cases, the hedging strategy is model-dependent.

Options as Hedge Instruments

- In 1978, Breeden and Litzenberger pointed out that the payoff to univariate path-independent claims can be statically hedged using a portfolio of co-terminal standard options written on the same underlying asset.
- The hedge is *robust*, i.e. model independent.
- In 1979, Goldman Sosin, and Gatto developed a pricing formula for a floating strike lookback put in the context of the Black Scholes model.
- They noticed that their formula for the initial value reduced to the Black Scholes value of a co-terminal at-the-money straddle when log price has zero risk-neutral drift.
- They then presented a novel hedging strategy that involves rolling up the strike of a straddle whenever the maximum increases.
- As presented, this strategy is both semi-static and model-dependent.

Semi-robust Hedging Strategies

- Suppose that one wishes to add additional state variables to the Black Scholes model to accomodate the observation that interest rates and/or volatilities appear to evolve stochastically over time.
- The time-honored approach is to impose additional processes and then dynamically trade additional hedge instruments.
- An alternative is to do a stochastic time change, which may or may not be continuous, and which may or may not be independent of the Brownian motion driving the price of the underlying asset.
- If one does a continuous independent stochastic time change to Black Scholes, it turns out that the original GSG semi-static hedge still replicates the payoff to a floating strike lookback put.
- The same simple semi-static hedge succeeds even though the stochastic process governing the time change is not specified.
- As a consequence, such a strategy is said to be semi-robust.

Semi-robust and Semi-static Hedging Strategies

- Recall that Merton 1973 showed that the payoff to a down-and-out call can be replicated under the Black Scholes model by dynamically trading in the underlying stock.
- In 1994, Bowie and Carr considered the special case of a down-and-out call written on a forward price with strike equal to barrier. They showed that the payoff to this contract can be replicated by a semi-static position in a forward contract, provided that jumps over the barrier are not possible. Aside from this skipfree condition, no additional assumptions are required, i.e the hedge is semi-robust.
- They also showed that there exists a simple semi-static hedge for all barrier options on forward prices, provided that one additionally assumes that Bates' put call symmetry condition holds at the barrier crossing time.
- These hedges are also semi-robust, in that the volatility of the underlying can follow an unspecified stochastic process so long as it evolves independently of the Brownian motion driving the forward price.

Exotic Option Generation

- While originally considered as exotic, barrier options such as one touches and up-and-in puts are now commonplace in currency options markets.
- As a result, FX barrier options are often referred to as first generation exotics.
- Newer exotics such as passport options are referred to as second generation exotics.
- In other markets such as commodities, barrier options are not that common and one instead finds other exotics such as Asians trading liquidly.

Part II: Calls on Everything but the Kitchen Sink

- We develop robust and semi-robust hedging strategies for various kinds of exotics, both traded and non-traded.
- Some of our robust hedging strategies use static positions in barrier options as part of the hedge.
- Since we recognize that barrier options are not yet liquid in some markets, we impose drift, continuity, and symmetry conditions under which the hedges just involve semi-static trading in standard options.
- We never explicitly consider dynamic replication using the underlying asset, but pricing formulas and hedges in this context will often easily arise out of our analysis.

Calls on Maximum and Drawdown

- Suppose that an investor buys a risky asset at time 0 for S_0 dollars.
- Let $M_T \equiv \max_{t \in [0,T]} S_t$ be the continuously monitored maximum over $[0,T]$.
- Armed with perfect foresight of the path to T, the profit from an optimal selling strategy is $M_T - S_0$.
- In the absence of this foresight, the *drawdown* $D_T \equiv M_T - S_T$ captures the ex post regret from selling the asset for S_T at T, rather than selling it when its maximum price M_T was attained.
- In this paper, we develop new model-free exact hedges for calls paying $(M_T - K_m)^+$, $(D_T - K_d)^+$, and even $(M_T - K_m)^+ \times (D_T - K_d)^+$.

Calls on Trading Gains

- Consider a dynamic trading strategy in a single risky asset for which shareholdings oscillate randomly between ± 1.
- We consider call options written on the gains from such a binary trading strategy.
- Under martingality and symmetry assumptions, we show how the payoff to calls on gains from binary trading strategies can be replicated via semi-static trading in standard options.
- Passport options are over the counter options written on the gains from a dynamic strategy for which shareholdings can vary in the interval $[-1, 1]$.
- So long as the payoff is convex, the optimal trading strategy is binary and hence passport options are also covered.
- We make no assumption concerning jumps.

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EURUSD Passport Option

Receive the net cumulative profit from spot positions of up to EUR10mio in EURUSD, with losses underwritten

Counterparty A:	Merrill Lynch Capital Services
Counterparty B:	
Notional:	EUR10mio
Trade Date:	30th April, 2004
Settlement Date:	4th May, 2004
Expiry Date:	2nd August, 2004
Maturity Date:	4th August, 2004
Counterparty B pays:	2.95% * Notional on Settlement Date
Counterparty A pays	max (0, Net USD value of permitted spot trades) on Maturity Date
Permitted Spot Trades	Counterparty B may have long or short position of no greater than Notional in EURUSD, from the Trade date to the Valuation Date, subject to the trading conditions.
Trading Conditions	Counterparty B may alter this position up to 3 times a day. Counterparty B must execute all position amendments through Counterparty A All amendments must be carried out between 8am and 5pm London time, or on an order basis
Calculation Agent:	Merrill Lynch Capital Services

Calls on the Maximum and Drawdown of Trading Gains

- We also consider calls written on the drawdown or maximum of the trading gains from a binary trading strategy.
- If we additionally assume that this maximum never increases by a jump, then both kinds of calls can be semi-statically hedged using standard options.
- In fact, we can also hedge the product of the call payoffs and thereby determine the joint risk-neutral density of the drawdown and maximum of gains.

Calls on (Up)Crosses

- When the sign of the position in a binary trading strategy changes when an upcross or a downcross of an interval completes, then the running maximum of the gains becomes proportional to the number of crosses of the interval.
- As a result, we can also hedge calls written on the number of crosses of a given spatial interval.
- Since about half of the crosses from this binary trading strategy are upcrosses, these can also be the underlying of the call.

Calls on Local Time

- Consider the number of crosses of a given spatial interval as the width of the interval shrinks down to zero.
- If the process is a continuous martingale (as often assumed in theory), then the number of crosses will realize to either zero or infinity almost surely.
- For a finite interval width, the product of the number of upcrosses and the width converges to the local time of the process as the width shrinks down to zero.
- As a consequence, calls on local time can also be hedged by semi-static option trading.

Part III: Robust and Semi-Robust Hedges

- Let S_t denote the spot price of some asset which can be monitored continuously over a fixed time interval $[0, T]$.
- Let $M_T \equiv \max_{t \in [0,T]} S_t$ be the continuously-monitored maximum of this asset price over $[0, T]$.
- Let $D_T \equiv M_T - S_T$ be the terminal drawdown or just "drawdown" for brevity.
- A call on the drawdown with payoff $(D_T - K_d)^+$ provides insurance for the call buyer against large drawdown realizations, with the maximum loss limited to the initial premium.
- Assuming only frictionless markets and no arbitrage, a new model-free exact hedge for a drawdown call is presented on the next page.

Robust Hedge of Drawdown Call

- Let $B_t(T) > 0$ be the price at $t \in [0,T]$ of a default-free bond paying \$1 at T. No arbitrage implies the existence of a probability measure $\mathbb{Q}$ associated with B.
- Let $P_t(K,T) = B_t(T)E_t^{\mathbb{Q}}(K - S_T)^+$ be the standard put value at time $t \in [0,T]$.
- Let τ_H be the first passage time of the process S to a barrier $H > S_0$. Let $UIBP_t(K_u,T;H) = B_t(T)E_t^{\mathbb{Q}}1(M_T > H, S_T < K_u)$ be the value at time $t \in [0,\tau_H]$ of an up-and-in binary put with strike K_u, maturity $T \geq t$, and barrier $H \geq S_0$.
- In frictionless markets, the drawdown call value $C_t^d(K_d,T) \equiv B_t(T)E_t^{\mathbb{Q}}(D_T - K_d)^+ =$

$$P_t(M_t - K_d, T) + \int_{M_t}^{\infty} UIBP_t(H - K_d, T; H)dH, \qquad t \in [0,T], K_d \geq 0.$$

- In words, a drawdown call with strike K_d is robustly replicated by keeping a put struck K_d dollars below M_t and also holding dH up-and-in binary puts struck K_d dollars below H for each in-barrier $H > M_t$. If $M_T = M_t$, then the put provides the desired payoff, while if $M_T > M_t$, then the up-and-in binary puts which knock in at each rise in M are used to roll up the put strike.

Robust Hedge of Call on the Maximum

- Let $BC_t(K_b, T) = B_t(T) E_t^{\mathbb{Q}} 1(S_T > K_b)$ denote the value at time $t \in [0, T]$ of a European binary call with strike $K_b \in \mathbb{R}$ and maturity $T \geq 0$.
- Let $OTPE_t(T; H) = UIBP_t(H, T; H) + BC_t(H, T)$ denote the value at $t \in [0, T]$ of a one touch with payment at its expiry T and with barrier H.
- In frictionless markets, the max call value $C_t^m(K_m, T) \equiv B_t(T) E_t^{\mathbb{Q}}(M_T - K_m)^+ =$

$$(M_t - K_m)^+ B_t(T) + \int_{M_t \vee K_m}^{\infty} OTPE_t(T; H) dH, \qquad t \in [0, T], K_m \geq 0.$$

- In words, a call on the maximum is robustly replicated by keeping its intrinsic value in bonds and keeping its volatility value in dH one touches for each barrier $H > (M_t \vee K_m)$.
- When the maximum increases above K_m, the one touches which knock in become the bonds which provide the payoff.

Hedging Barrier Options with Standard Options

- In many markets, barrier options do not trade liquidly and hence it is of interest to find alternative hedges that just use standard options.
- There are three known approaches for hedging barrier options in terms of standard options:
  1. Model-free: generates upper and lower bounds. See Brown, Hobson, and Rogers (2001).
  2. Semi-robust: provides exact replication but eliminates the possibility of jumps over the barrier and asymmetries after the barrier crossing time (including risk-neutral drift). See Carr and Lee (2005).
  3. Model-based: exact replication given that risk-neutral dynamics are known and Markovian. See Andersen, Andreasen, and Eliezer (2002).
- The first two approaches value barrier options relative to co-terminal options, while the third approach uses all maturities up to that of the barrier option.

Semi-Static Hedging of Barriers with Vanillas

- Although no approach dominates any other on all dimensions, we will focus on the second approach.
- Hence, we will require that the price of the underlying asset be a martingale under $\mathbb{Q}$. We think of the underlying as a forward price denoted by F_t.
- We also rule out up jumps in the path of the running maximum of F. Hence, at each $t \in [0, T]$, we allow the possibility of down jumps in F_t and we allow the possibility of up jumps of limited size in F_t, but we give zero probability to up jumps in F_t which are sufficiently large so that $M_t \equiv \max_{s \in [0,t]} F_s$ could increase by a jump.
- We also place alternative assumptions on the symmetry of the risk-neutral process for F.

Arithmetic Put Call Symmetry

- We say that Arithmetic Put Call Symmetry (APCS) holds at a particular time $t \geq 0$ for a particular maturity $T \geq t$ if a put maturing at T has the same market price as the co-terminal call struck the same distance away from F_t:

$$P_t(K_p, T) = C_t(K_c, T),$$

  for all strikes K_p, K_c satisfying: $K_p = F_t + \triangle K, K_c = F_t - \triangle K$, where $\triangle K \in \mathbb{R}$.

- APCS implies that options of the same moneyness have the same value.
- In addition to assuming that M never increases by a jump, we now assume that APCS holds at all times t for which the running maximum increases.
- All of our requirements are met by the class of Ocone martingales:

$$dF_t = a_t dW_t, \qquad t \in [0, T]$$

  where the absolute volatility process a evolves independently of W.

Drawdown Call Under APCS

- We now define: **Assumption set A1:** *The price process F is a $\mathbb{Q}$ martingale whose running maximum is continuous and for which APCS holds at all times τ when $dM_\tau > 0$.*
- Under frictionless markets &**A1**, no arbitrage implies that for $K_d \geq 0, t \in [0,T]$:

$$C^d(K_d, T) = P_t(M_t - K_d, T) + C_t(M_t + K_d, T), \qquad t \in [0,T], K_d \geq 0.$$

- In words, a drawdown call is replicated by alway holding a strangle centered at the running maximum M_t, and whose width is the strike K_d of the drawdown call.
- The strategy is self-financing because the cash outflow required to move the put strike up when the running maximum increases infinitessimally is financed by the cash inflow received from moving the call strike up (given that APCS is in fact holding at such times).

Call on the Maximum Under APCS

- In frictionless markets and under assumption set **A1**, no arbitrage implies:

$$C_t^m(K_m,T) = (M_t - K_m)^+ B_t(T) + 2C_t(M_t \vee K_m, T), \quad K_m \geq 0, t \in [0,T].$$

- In words, a call on the maximum is replicated by keeping its intrinsic value in bonds and alway holding two standard calls struck at the larger of the running maximum M_t and the call strike K_m.
- The strategy is self-financing because each standard call's strike derivative is one half when the running maximum is above K_m and increases infinitessimally (given that APCS is in fact holding at such times).
- Hence, the cash generated by moving up the two call strikes is exactly the intrinsic value of the call on the maximum.

Geometric Put Call Symmetry

- To prevent the possibility of negative prices, one can use Geometric Put Call Symmetry (GPCS) instead of APCS.
- We say that GPCS holds at a particular time $t \geq 0$ for a particular maturity $T \geq t$ if the time t market prices of puts and calls of maturity T are such that:

$$\frac{P_t(K_p, T)}{\sqrt{K_p}} = \frac{C_t(K_c, T)}{\sqrt{K_c}},$$

  for all strikes K_p, K_c satisfying $K_c = F_t \times u, K_p = F_t/u$, where $u > 0$.
- A sufficient condition on the dynamics of F for engendering GPCS is that:

$$dF_t = F_t \sqrt{V_t} dW_t, \qquad t \in [0, T], \tag{1}$$

  where W is a $\mathbb{Q}$ standard Brownian motion and the stochastic instantaneous variance process V evolves independently of W.

Drawdown Call Under GPCS

- **Assumption set A2:** *The price process F is a $\mathbb{Q}$ martingale whose running maximum is continuous and for which GPCS holds at all times τ when $dM_\tau > 0$.*
- Under frictionless markets & **A2**, no arbitrage $\Rightarrow$ for $K_d \in \left[0, \frac{F_0}{2}\right), t \in [0,T], C_t^d(K_d,T) =$

$$P_t(M_t - K_d, T) + \frac{M_t - K_d}{M_t - 2K_d} C_t\left(\frac{M_t^2}{M_t - K_d}, T\right) + \int_{\frac{M_t^2}{M_t - K_d}}^{\infty} N^c(K_c, K_d) C_t(K_c, T)\, dK_c,$$

  for $t \in [0,T]$, where $N^c(K_c, K_d) \equiv \frac{(\sqrt{K_c - 4K_d})^3 + K_c^{3/2} - 6K_d\sqrt{K_c}}{2K_c(\sqrt{K_c - 4K_d})^3}$.
- Suppose again that a drawdown call is sold at $t = 0$. The hedge involves buying a ratioed strangle and calls.
- As new maxima are achieved, the center of the ratioed strangle increases, with the calls that are sold financing the change.
- All calls held at expiry finish out-of-the-money, while the put struck at $M_T - K_d$ furnishes the drawdown call payoff.

Call on the Maximum under GPCS

- Under frictionless markets and $\mathbf{A2}$, no arbitrage $\Rightarrow$ for $K_m \geq 0, t \in [0,T]$:

$$C_t^m(K_m,T) = (M_t - K_m)^+ B_t(T) + 2C_t(M_t \vee K_m, T) + \int\limits_{M_t \vee K_m}^{\infty} \frac{1}{H} C_t(H,T)dH.$$

- Comparing this result with the corresponding one under APCS, we see that the hedge has an additional holding in $\frac{dH}{H}$ calls for all strikes $H > M_t \vee K_m$.
- It can be shown that for any fixed $t \in [0,T]$, holding this call position static to maturity would create the payoff:

$$f(F_T) = \left\{ F_T \ln\left(\frac{F_T}{M_t \vee K_m}\right) - [F_T - (M_t \vee K_m)] \right\} 1(F_T > (M_t \vee K_m)). \qquad (2)$$

- Interestingly, this is also the position in standard calls used to synthesize the payoff for a corridor gamma variance swap, whose floating payoff is:

$$\int_t^T \frac{F_u}{F_t} 1[F_u > (M_t \vee K_m)] \left(\frac{dF_u}{F_u}\right)^2.$$

Path Independent Options on Trading Gains

- A dynamic trading strategy is *binary* when the shareholdings can only be ± 1.
- The running P&L π_t of a binary trading strategy is defined by $\pi_t \equiv \int_0^t c_s dF_s, \qquad t \in [0,T]$, where $c_s = \pm 1$ and F has zero risk-neurral drift.
- Let $C_t^{\pi}(k,T)$ denote the arbitrage-free value at time $t \in [0,T]$ of a call written on the terminal P&L from a binary trading strategy.
- The payoff at T is $C_T^{\pi}(k,T) = (\pi_T - k)^+$, where $k \in \mathbb{R}$ is the strike price.
- We henceforth refer to this call as a passport for brevity.

Semi-Robust Hedge of Passport Call

- **Assumption set A3:** *Interest rates and dividends are zero. The price process F defining the reference binary strategy is a $\mathbb{Q}$ martingale for which APCS holds at all times τ when parity changes.*
- Under frictionless markets and **A3**, no arbitrage $\Rightarrow$ for $k \in \mathbb{R}, t \in [0,T]$:

$$C_t^\pi(k,T) = 1(c_t = 1)C_t(F_t^- - \pi_t^- + k, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - k, T),$$

  where F^- and π^- indicate the forward price and P&L at the time of the last trade at or before t.
- Define the *parity* at t of the gains process π as c_t. Then the parity of the standard option held at t (i.e. call or put) matches the parity at t of the underlying gains process.
- Define the moneyness of the passport call as $\pi_t - k$, the moneyness of the standard call as $F_t - K$, and the moneyness of the standard put as $K - F_t$. Then the passport call always has the same value as the standard option with the same parity and with the same moneyness as the passport at the last switch time.

Semi-Robust Hedge of Passport Put

- Now let $P_t^{\pi}(k,T)$ denote the arbitrage-free value at time $t \in [0,T]$ of a *put* option on the gains from a binary trading strategy.
- At its expiry T, the payoff is $P_t^{\pi}(k,T) = (k - \pi_T)^+$, where $k \in \mathbb{R}$ is the put strike.
- One can show that the passport put always has the same value as a standard option with the same moneyness at the last switch time. The parity of the standard option held at t is now the *opposite* of the parity at t of the gains process.
- Hence, under frictionless markets & $\mathbf{A3}$, no arbitrage $\Rightarrow$ for $k \in \mathbb{R}, t \in [0,T]$:

$$P_t^{\pi}(k,T) = 1(c_t = -1)C_t(F_t^- + \pi_t^- - k, T) + 1(c_t = 1)P_t(F_t^- - \pi_t^- + k, T).$$

- Since puts and calls on gains can be semi-statically hedged with standard options, it follows that any path-independent payoff on gains can also be semi-statically hedged with standard options.

Barrier Options on Gains Processes

- **Assumption set A4:** *Interest rates and dividends are zero. The price process F defining the reference binary strategy is an Ocone martingale under $\mathbb{Q}$.*
- Hence, APCS holds at all times $t \in [0,T]$. Assumption set **A4** implies that the gains from a binary trading strategy in F is also an Ocone martingale.
- Let $M_T^\pi \equiv_{t\in[0,T]}^{\max} \pi_t$ be the maximum profit that a binary trading strategy earned over $[0,T]$.
- Let $UIP_t^\pi(k_u,T;h) \equiv E_t^{\mathbb{Q}} 1(M_T^\pi > h)(k_u - \pi_T)^+$ be the arbitrage-free value at time $t \geq 0$ of an up-and-in put on the gains from a binary trading strategy, with strike $k_u \in \mathbb{R}$, maturity $T \geq t$, and in-barrier $h \geq 0$.
- Let τ_h^π be the first passage time of the gains process to the barrier h.
- Under frictionless markets and **A4**, no arbitrage $\Rightarrow$ for $k,h \in \mathbb{R}, t \geq [0,\tau_h)$

  $$UIP_t^\pi(k,T;h) = 1(c_t = 1)C_t(F_t^- - \pi_t^- + 2h - k, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - 2h + k, T).$$

Drawdown Call on Trading Gains

- Define the drawdown on the gains from a binary trading strategy:

$$D_T^\pi \equiv M_T^\pi - \pi_T.$$

- Let $C_t^{d\pi}(K_d, T)$ be the arbitrage-free value of a call on the drawdown of a gains process paying $(D_T^\pi - K_d)^+$ at T.
- Under frictionless markets & **A4**, no arbitrage $\Rightarrow$ for $K_d \geq 0, t \geq [0, T], C_t^{d\pi}(K_d, T) =$

$$[1(c_t = -1)C_t(F_t^- + \pi_t^- - M_t^\pi + K_d, T) + 1(c_t = 1)P_t(F_t^- - \pi_t^- + M_t^\pi - K_d, T)$$
$$+1(c_t = 1)C_t(F_t^- - \pi_t^- + M_t^\pi + K_d, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - M_t^\pi - K_d, T)].$$

Call on the Maximum of Trading Gains

- Let $C_t^{m\pi}(K_m,T)$ be the arbitrage-free value of a call on the maximum of a gains process paying $(M_T^\pi - K_m)^+$ at T.
- Under frictionless markets & **A4**, no arbitrage $\Rightarrow$ for $K_m \geq 0, t \geq [0,T], C_t^{m\pi}(K_m,T) =$

$$(M_t^\pi - K_m)^+ B_t(T)$$
$$+21(c_t = 1)C_t(F_t^- - \pi_t^- + (M_t^\pi \vee K_m),T) + 21(c_t = -1)P_t(F_t^- + \pi_t^- - (M_t^\pi \vee K_m),T).$$

- In words, a call on the maximum of the gains from a binary trading strategy is replicated by keeping its intrinsic value in bonds and alway holding two standard options of the same parity as the underlying gains process.
- The strike held is such that at the last time the parity changed, the standard option acquired has the same moneyness as a call on gains struck at the larger of the running maximum M_t and the call strike K_m.

Max Call on Trading Gains (Con'd)

- Recall that under frictionless markets & **A4**, no arbitrage $\Rightarrow$ for $K_m \geq 0, t \geq [0,T], C_t^{m\pi}(K_m,T) = (M_t^{\pi} - K_m)^+ B_t(T)$

  $$+21(c_t = 1)C_t(F_t^- - \pi_t^- + (M_t^{\pi} \vee K_m), T) + 21(c_t = -1)P_t(F_t^- + \pi_t^- - (M_t^{\pi} \vee K_m), T).$$

- The strategy is self-financing when $M^{\pi} < K_m$, because the trade just involves changing the parity of the standard option held while preserving moneyness.

- The strategy is also self-financing when $M^{\pi} > K_m$ and the maximum increases, because either $c_t = 1$, in which case the infinitessimal rollup of the strikes of the 2 ATM calls held finances the bond position, or else $c_t = -1$, in which case the infinitessimal rolldown of the strikes of the 2 ATM puts held finances the bond position.

- In either case, one can keep the intrinsic value of the call on the maximum in bonds.

Call on Crosses

- Suppose that we specify a spatial interval $(F_0, F_0 + w)$ with some width $w > 0$.
- The number of crosses of this interval is the sum of the upcrosses, downcrosses, and partial crosses arising at expiry.
- Consider a call written on the product of the number of crosses and the width w.
- This call arises as a call on the maximum of the gains from the *contrarian strategy*

$$c_t = \begin{cases} 1 \text{ if } F_t \leq F_0, \\ 1 \text{ if } F_t \in (F_0, F_0 + w) \text{ and } F_t^- = F_0 \\ -1 \text{ if } F_t \in (F_0, F_0 + w) \text{ and } F_t^- = F_0 + w \\ -1 \text{ if } F_t \geq F_0 + w. \end{cases}, \qquad t \in [0, T].$$

- In words, the contrarian investor is long the risky asset if F is below the spatial interval $(F_0, F_0 + w)$ and short the risky asset if F is above this interval. When F is inside the interval, the investor just keeps the position held when this interval was last entered.

Semi-Robust Hedge of a Call on Crosses

- Let $C^c_t(K_m, T)$ denote the arbitrage-free value of a call option at time $t \in [0, T]$ written on the product of w and the sum of the crosses. Let $K_m \geq 0$ be the call strike and let $k_m \equiv \frac{K_m}{w}$. Thus, the payoff of the call at expiry is:

$$C^c_T(K_m, T) = w(n^c_T + f_T - k_m)^+ = (M^\gamma_T - K_m)^+,$$

  where f_t is the fraction of the cross completed by t.

- Under frictionless markets and **A4**, no arbitrage $\Rightarrow$ for $t \in [0, T), k_m \geq 0$:

$$\begin{aligned} C^{m\gamma}_t(k_m w, T) &= w(n^c_t + f_t - k_m)^+ B_t(T) \\ &+21(F^-_t = F_0) C_t(F_0 + w f_t + w[k_m - (n^c_t + f_t)]^+), T) \\ &+21(F^-_t = F_0 + w) P_t(F_0 + w - w f_t - w[k_m - (n^c_t + f_t)]^+), T). \end{aligned}$$

Hedge of Call on Crosses (Con'd)

- Recall that for $t \in [0,T), k_m \geq 0$:

$$\begin{aligned} C_t^{m\gamma}(k_m w, T) &= w(n_t^c + f_t - k_m)^+ B_t(T) \\ &\quad + 21(F_t^- = F_0) C_t(F_0 + w f_t + w[k_m - (n_t^c + f_t)]^+), T) \\ &\quad + 21(F_t^- = F_0 + w) P_t(F_0 + w - w f_t - w[k_m - (n_t^c + f_t)]^+), T). \end{aligned}$$

- Suppose that at $t = 0$, an investor sells a call on the product of w and the number of crosses of the interval $(F_0, F_0 + w)$. Since $n_0^c = 0$, $f_0 = 0$, and $F_0^- = F_0$, the initial hedge requires buying 2 calls struck $K_m = w k_m$ dollars above F_0.
- As each cross completes, the investor switches the polarity of the option held.
- While the number of crosses $n_t^c + f_t < k_m$, the completion of each upcross involves selling 2 calls & buying 2 puts. As $F = F_0 + w$ at the completion of each upcross, the 2 puts purchased are struck $2w$ dollars closer to F_0 than the 2 calls sold.
- While $n_t^c < k_m$, the completion of each downcross involves selling 2 puts and buying 2 calls. As $F = F_0$ at the completion of each downcross, the 2 puts sold are struck the same distance away from F_0 as the 2 calls bought.

Hedge of Call on Crosses (Con'd)

- In this regime, each round trip causes the strikes held to be nearer to F_0 by $2w$. One can say that both strikes move in towards F_0 by w on each cross, so long as one remembers that only one parity is held at a time.
- If the number of crosses at expiry $n_T^c + f_T$ is less than k_m, then the call on crosses expires worthless as does its hedge.
- If the number of crosses n^c exceeds k_m prior to T, then at the first time that $n_t^c \geq k_m$, the hedger enters a different regime, where the only strikes held creep through the interval $(F_0, F_0 + w)$. The parity of the options held changes only when another cross completes. When the trading strategy is long (i.e. $F_t^- = F_0$), the hedger holds 2 calls struck wf_t dollars above F_0. When the trading strategy is short (i.e. $F_t^- = F_0 + w$), the hedger holds 2 puts struck wf_t dollars below $F_0 + w$. As the strikes of the 2 calls move up or the strikes of the 2 puts move down, cash is generated and used to purchase bonds. The total cash generated allows the investor to buy $w(n_T^c + f_T - k_m)^+$ bonds, which provides the desired payoff at expiry.

Initial Value of Call on Crosses

- Setting $t = 0$ in the valuation formula:

$$C_0^{m\gamma}(k_m w, T) = 2C_0(F_0 + wk_m, T).$$

- Hence, the initial fair value of a call paying w times $(n_T^c + f_T - k_m)^+$ at T is simply twice the value of a standard call struck $K_m = wk_m$ dollars above F_0.
- Notice that once K_m is fixed, the initial value of the call on crosses is independent of the width w.
- As we send $w \downarrow 0$, the initial value is unchanged, but the hedging strategy changes.
- As we send $w \downarrow 0$, the underlying product of w and the number of crosses converges to the local time of F at F_0.
- We consider calls on local time shortly.

Upcrosses

- Assuming only continuity of the underlying asset. one can replicate a claim paying the number of upcrosses (both complete and partial) of a given interval (L,H).
- For example, if the interval is (L=100,H=110) and the underlying completes 3 upcrosses and finishes at 107, then the payoff on the claim paying the number of upcrosses is $3.70.
- To formally define the payoff of an upcrosser, let $\tau_0 \equiv 0$ and for $i = 1, 2, \ldots,$ recursively define the stopping times σ_n and τ_n by:

$$\sigma_i \equiv \inf\{i \geq \tau_{i-1} : F_t \leq L\} \qquad \tau_i \equiv \inf\{t \geq \sigma_i : F_t \geq H\}, \tag{3}$$

where we adopt the usual convention that the infimum of the empty set is infinity. If we adopt the dual convention that the maximum of the empty set is zero, then the number of completed upcrosses by time t is:

$$n_t^u \equiv \max\{i : \tau_i \leq t\}, \qquad t \in [0,T].$$

Upcrossers

- At any time t, it will be useful to know whether or not F has been at or below L since the last upcross, if any. Accordingly, we also define:

$$n_t^d \equiv \max\{i : \sigma_i \leq t\}, \qquad t \in [0, T],$$

  but stress that it is not necessarily the number of completed downcrosses.

- If $n_t^d = n_t^u$, then at time t, the first requirement for the next upcross has not been met, while if $n_t^d > n_t^u$, then at time t, it has.

- Let $V_t^u(L, H, T)$ denote the arbitrage-free value of an upcrosser at time $t \in [0, T]$.

$$V_T^u(L, H, T) = n_T^u + 1(n_T^d > n_T^u, F_T < H)\frac{(F_T - L)^+}{H - L}.$$

  The last term gives credit for a partially completed upcross, if any.

Semi-Robust Hedge of Upcrossers

- We assume nothing about riskfree rates and dividends. When F can not skip over H or L, semi-static trading in European options replicates the payoff to an upcrosser perfectly. As a result, no arbitrage implies:

$$
\begin{aligned}
V_t^u(L,H,T) &= \frac{1}{H-L}\left[1(F_t \leq L) + 1(n_t^d > n_t^u, F_t \in (L,H))\right] C_t(L,T) \\
&+ \frac{1}{H-L}\left[1(F_t \geq H) + 1(n_t^d = n_t^u, F_t \in (L,H))\right] P_t(L,T), \quad t \in [0,T].
\end{aligned}
$$

- First suppose that $F_0 \leq L$, so that the writer of the upcrosser uses the sale proceeds to buy $\frac{1}{H-L}$ calls of strike L. Since $F_0 \leq L$, the calls are out-of-the-money (OTM). If the forward price never hits H before maturity and finishes below L, then the investor has no liability and the call finishes OTM.

- If the forward price never hits H before maturity but finishes between L and H, then the payoff from the calls covers the partially completed upcross.

Semi-Robust Hedge of Upcrossers (Con'd)

- If the forward price does touch H before maturity, then at the first time before maturity that it does so, the investor sells the $\frac{1}{H-L}$ calls and buys $\frac{1}{H-L}$ puts. By put call parity:

$$C_t(L,T) - P_t(L,T) = B_t(F_t - L),$$

this conversion results in enough money to buy $\frac{F_{\tau_1}-L}{H-L}$ bonds. Since $F_{\tau_1} = H$ under the skipfree assumption, one bond is purchased which just covers the increase in the intrinsic value of the investor's terminal liability due to the completed upcross.

- If the forward price never returns to L before maturity, then these $\frac{1}{H-L}$ puts expire worthless, but the liability of one dollar is covered.

- If after the first upcross, the forward price touches L before maturity, then at the first time that it does so, the investors sells the $\frac{1}{H-L}$ puts and buys $\frac{1}{H-L}$ calls. Since $F_{\sigma_1} = L$, PCP implies that this reversal is self financing. After this trade, the investor is left holding an OTM call with the forward price at L. As this was the investor's initial position, we are done when $F_0 \leq L$.

Semi-Robust Hedge of Upcrossers (Con'd)

- Now, consider the trading strategy when $F_0 > L$. The writer of the upcrosser can use the proceeds from the sale to buy $\frac{1}{H-L}$ OTM puts struck at L. If the forward price never hits or touches L, then no liability is due and the puts expire worthless.
- If the forward price does hit L, then at the first time prior to maturity that it does so, the investor sells the $\frac{1}{H-L}$ puts and buys $\frac{1}{H-L}$ calls. Afterwards, the investor is in the same position as when $F_0 \leq L$. Hence the investor can follow the strategy described above from then on.
- Thus, from then on the investor always holds $\frac{1}{H-L}$ calls while below L and $\frac{1}{H-L}$ puts while above H. What is held between L and H is exactly what was held when the corridor was last entered. Thus, if the corridor was last entered from below, the investor holds $\frac{1}{H-L}$ calls and if the corridor was last entered from above, the investor holds $\frac{1}{H-L}$ puts instead.
- We conclude that we have constructed a simple trading strategy in European options which replicates the payoff.

Calls on Upcrosses under GPCS

- We set $L = F_0$ and $H = F_0 u$ with $u > 1$.
- Let $C_t^{nu}(k,T)$ denote the arbitrage-free value at time $t \in [0,T]$ of a call on the number of upcrosses of the interval $(F_0, F_0 u)$. For simplicity, we assume that the call strike is a positive integer k.
- To value this call, we assume that the underlying is a forward price which never jumps over F_0 or $F_0 u$. We furthermore assume that GPCS holds whenever $F = F_0$ or $F = F_0 u$.
- A sufficient condition on the dynamics of F for engendering these dynamics is:

$$dF_t = F_t \sqrt{V_t} dW_t, \qquad t \in [0,T],$$

  where W is a $\mathbb{Q}$ standard Brownian motion and the stochastic instantaneous variance process V evolves independently of W.

Calls on Upcrosses under GPCS

- **Assumption set A5:** *The price process F is a $\mathbb{Q}$ martingale which never jumps over F_0 or F_0u and for which GPCS holds at all times τ when $F = F_0$ or $F = F_0u$.*
- Recall that GPCS holds at a given forward price G for options of maturity T if:

$$C_t(K_c, T) = \frac{K_c}{G} P_t(K_p, T), \qquad t \in [0, T],$$

  where the geometric mean of the two strikes is G, i.e. $\sqrt{K_c K_p} = G$.
- Under frictionless markets & **A5**, no arbitrage $\Rightarrow$ for positive integer k, $t \in [0, T]$:

$$\begin{aligned} C_t^{nu}(k, T) &= (n_t^u - k)^+ B_t(T) \\ &+ \frac{1}{F_0(u-1)} \left[1(F \leq F_0) + 1(F_t \in (F_0, F_0 u), n_t^d > n_t^u)\right] u^{-(k-n_t^u)^+} C_t(F_0 u^{2(k-n_t^u)^+}, T) \\ &+ \frac{1}{F_0(u-1)} \left[1(F \leq F_0) + 1(F_t \in (F_0, F_0 u), n_t^d = n_t^u)\right] u^{(k-n_t^u)^+} P_t(F_0 u^{-2(k-n_t^u)^+}, T), \end{aligned}$$

  where n_t^u is the number of upcrosses completed over $[0, t]$ and $n_t^d \equiv \max\{i : \sigma_i \leq t\}$.

Hedging Calls on Upcrosses under GPCS

- At $t = 0$, $n_t^u = 0$ and $F_0^- = F_0$ and so $C_0^{nu}(k,T) = \frac{1}{F_0(u-1)} u^{-k} C_0(F_0 u^{2k}, T)$.
- As with calls on crosses, trading splits into two regimes. In regime I, the hedger flips back and forth between calls stuck above $F_0 u$ and puts struck below this level. At each time τ that a new upcross completes, we have $F_\tau = F_0 u$, and $(n_\tau^u)^- = n_\tau^u - 1$. The hedger sells his holding in $\frac{1}{F_0(u-1)} u^{-(k-n_\tau^u+1)}$ calls struck at $F_0 u^{2(k-n_\tau^u+1)}$ and buys a position in $\frac{1}{F_0(u-1)} u^{k-n_\tau^u}$ puts struck at $F_0 u^{-2(k-n_\tau^u)}$.
- Notice that the geometric mean of the two strikes is $F_0 u$.
- Notice that the ratio of the call strike to this mean is $\frac{F_0 u^{2(k-n_\tau^u+1)}}{F_0 u} = u^{2(k-n_\tau^u)+1}$.
- Finally, notice that the ratio of the number of puts bought to the number of calls sold is also:

$$\frac{\frac{1}{F_0(u-1)} u^{k-n_\tau^u}}{\frac{1}{F_0(u-1)} u^{-(k-n_\tau^u+1)}} = u^{2(k-n_\tau^u)+1}.$$

- As a result, GPCS implies that this trade is self-financing.

Hedging Calls on Upcrosses under GPCS

- While still in the first regime, at each time σ that F returns to F_0 with $F^- = F_0 u$, we have $F_\sigma = F_0$, and $(n_\sigma^u)^- = n_\sigma^u$. The hedger sells his holding in $\frac{1}{F_0(u-1)} u^{(k-n_\sigma^u)}$ puts struck at $F_0 u^{-2(k-n_\sigma^u)}$ dollars and buys a position in $\frac{1}{F_0(u-1)} u^{-(k-n_\sigma^u)}$ calls struck at $F_0 u^{2(k-n_\sigma^u)}$ dollars.
- Notice that the geometric mean of the two strikes is F_0.
- Notice that the ratio of the call strike to this mean is:

$$\frac{F_0 u^{2(k-n_\sigma^u)}}{F_0} = u^{2(k-n_t^u)}.$$

- Finally, notice that the ratio of the number of puts sold to the number of calls bought is also: $\frac{\frac{1}{F_0(u-1)} u^{k-n_\sigma^u}}{\frac{1}{F_0(u-1)} u^{-(k-n_\sigma^u)}} = u^{2(k-n_t^u)}$.
- As a result, GPCS implies that this trade is self-financing.

Hedging Calls on Upcrosses under GPCS

- Define a round trip as an upcross followed by a downcross.
- At the end of each such round trip, the forward price is at F_0 and the strikes being traded are each a factor of u^2 closer to F_0 than at the beginning of the round trip.
- When a round trip ends with the number of upcrosses equal to k, the hedger is holding $\frac{1}{F_0(u-1)}$ puts struck at F_0 as the forward price returns to F_0. This ends regime I.
- Regime II begins with the sale of these puts which from PCP generates exactly enough cash to buy $\frac{1}{F_0(u-1)}$ calls struck at F_0.
- After this point in time, the hedger can create the number of upcrosses beyond k, by exploiting put call parity as indicated when replicating upcrossers. **QED**

A New Paradox

- Suppose that a stock price process is continuous, and for simplicity suppose that we have zero interest rates and dividends.
- Suppose that an investor initially sells an at-the-money (ATM) straddle and pockets a positive premium.
- The investor tries a hedging strategy which appears to be costless. This trading strategy is simply to be short 2 units of whichever option is presently OTM.
- Hence if the stock price rises initially, then since $S_{0+} > S_0$, the investor buys one call and sells one put, both struck at S_0.
- Conversely, if the stock price falls initially, then since $S_{0+} < S_0$, the investor buys one put and sells one call, both struck at S_0.
- If the stock price returns to S_0, the investor returns to a short ATM straddle position.
- Whenever a stock move does not involve leaving or returning to S_0, no trade is required.

A New Paradox

- At expiry, the investor is holding an ATM or OTM option position and hence no liability arises.
- As the strategy is static when S is comfortably away from S_0, the only issue determining whether or not this trading strategy is an arbitrage opportunity is the determination of whether or not this strategy is self-financing when options are bought and sold around S_0.
- In fact, if the continuous underlying price process also has bounded variation, then the strategy is both replicating and self-financing.
- Suppose instead that the underlying price process is continuous over time and that all sample paths have unbounded variation. Then the above option trading strategy need not be self-financing whenever the stock price leaves or returns to S_0. Losses accumulate according to the *local time* of the stock price at its initial level. No arbitrage requires that the initial straddle premium is just the initial risk-neutral expectation of the local time at expiry.

Calls on Local Time

- While at-the-money options tell us the mean local time at F_0, it is of interest to know whether or not the implied volatility smile of a fixed maturity can tell us the whole risk-neutral distribution of local time at the options' expiry.
- We show that the smile does have this information content so long as the underlying price is an Ocone martingale under $\mathbb{Q}$.
- We also explicitly show how to value a call option on local time using just one option price.
- Furthermore, we indicate the hedging strategy for the sale of a call on local time, which just involves changing the strike and parity of an option held whenever it goes in-the-money.

Calls on Local Time

- Let $C_t^l(K_m, T)$ be the value at t of a call on local time paying $(L_T^F - K_m)^+$ at T.
- When the underlying forward price F is an Ocone martingale, no arbitrage implies:

$$
\begin{aligned}
C_t^l(K_m, T) &= (L_t^F - K_m)^+ B_t(T) \\
&\quad + 21(F_t > F_0) P_t(F_0 - (K_m - L_t^F)^+, T) + 21(F_t \leq F_0) C_t(F_0 + (K_m - L_t^F)^+, T).
\end{aligned}
$$

- Suppose that an investor sells a call on local time at time 0 and that the investor wishes to replicate the payoff via semi-static option trading.
- Since $L_0^F = 0$, the replicating strategy starts by holding 2 calls struck K_m dollars above F_0. As the hedger moves through calendar time with $L_t^F \leq K_m$, the hedger always holds two units of whichever standard option is out-of-the-money.
- The strike of the option held is $K_m - L_t^F \geq 0$ dollars away from F_0. If $F_t = F_0$ with $L_t^F \leq K_m$, then the hedger holds 2 puts with strike $K_m - L_t^F \geq 0$ dollars below F_0.

Calls on Local Time

- The reason that the strikes move in towards F_0 is that all option trades are made just as F moves away from F_0 in such a way that the 2 options held become ITM. Each such trade involves a disposition of the two ITM options and an acquisition of two OTM options with the same price. As a result, the strike of the two acquired options must necessarily be closer to F_0. This strategy is self-financing by design.
- If the running local time L never reaches K_m before T, the hedger just ends up holding 2 worthless OTM options.
- If L reaches K_m before T, then at the first passage time of L to K_m, the strike of the 2 options held first reaches F_0. Between this first passage time and expiry, the hedger holds 2 puts with strike F_0 if $F_t > F_0$, and 2 calls with strike F_0 otherwise.
- As this strategy is just the reverse of the one describing the paradox, it generates a positive cash flow each time that F crosses F_0, whose magnitude at T accumulates to $L_T^F - K_m$.

Initial Value of Call on Local Time

- Evaluating the valuation formula at $t = 0$ implies:

$$C_0^l(K_m, T) = 2C_0(F_0 + K_m, T).$$

- Hence at initiation, a call on local time with strike K_m has twice the value of a standard call struck K_m dollars out-of-the-money.
- What could be simpler?

Summary

- We showed that the problem of pricing and hedging a claim paying the product of call payoffs on the maximum and the drawdown reduces to the problem of pricing and hedging simple standard and barrier options.
- We then gave alternative sufficient conditions under which the hedge just involves holdings in standard options.
- We also showed that these continuity and sufficient conditions permit all path-independent and some path-dependent options on trading gains to be semi-statically hedged using standard options.
- We showed that a call option on the maximum of a particular contrarian trading strategy is equivalent to a call written on the total number of crosses of a given spatial interval.
- By shrinking the width of this interval down to zero, we can find the semi-static hedge of options on local time.

Future Research

- For future research, one can experiment with the allowed dynamics. One can treat the two symmetry conditions explored here as special cases arising from a more general family of symmetries.
- One can also explore adding the possibility of default to the regimes which assume a continuous underlying asset price process.
- And since we are looking back at what has been done, it is worth noting that the payoff $(M_T - S_0 - K_m)^+(M_T - S_T - K_d)^+$ looks the same when the flow of time is reversed.