Period three actions on the three-sphere.

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In the first part of this abstract, I give a general overview of the field of mathematics in which I work, and my results. In the second section, I give a brief indication of some of the technical aspects of my thesis.

1 Background and results.

The field of mathematics in which I work is geometric topology, which involves the study of surfaces, and their higher dimensional analogs, known as manifolds. In my research, I have worked mainly with surfaces and three-manifolds, and I now describe what these are, and why people might be interested in them.

A surface is an object that looks "two-dimensional" locally. The Euclidean plane from geometry is a good example, but we can also build spaces that on a small scale look like the plane, but globally are very different.

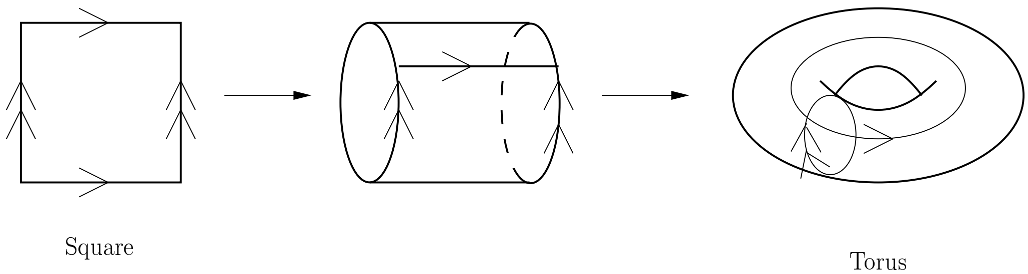

Figure 1: Gluing the sides of a square gives you a torus.

If you take a square and glue opposite edges together, you get a surface called a torus, which is the same as the surface of a donut; not the solid donut, just its surface. If you pick a point in the torus, you can find a small region around it which looks the same as a small region around any point in the plane, so locally the plane and the torus look the same, but globally they are very different. For example, the plane has infinite area, but the torus has finite area.

Three-manifolds are spaces that locally look like ordinary three-dimensional Euclidean space, but again they may be very different globally. These spaces are often harder to visualize, as they usually do not fit nicely inside three-dimensional space, just as the two-sphere does not fit inside the Euclidean plane. The universe we live in looks locally three-dimensional, so the universe is a three-manifold.[1]

This is of course one of the reasons why people are interested in the properties of manifolds, as the universe itself is a manifold! In fact, modern theories of

[1]Due to the time coordinate, the universe is actually a 4-manifold with a Minkowski metric, which in general may not be just three-manifold $\times$ time, but apparently current cosmological models indicate that it makes sense to think of the universe as a three-manifold $\times$ time, at least for times away from the initial big bang.

physics make extensive use of manifolds of many different dimensions. Perhaps more prosaically, solution sets of equations can often be thought of as manifolds, so they can crop up in places not normally considered “geometric”.

Surfaces are reasonably well understood, in the sense in which we have a complete classification of surfaces. The (orientable) surfaces consist of the two-sphere, the torus, which you can think of as a “two-sphere with a handle”, and then the higher genus surfaces, which are just two-spheres with more handles. There is a similar list of non-orientable surfaces, including the Klein bottle, for example.

One approach in attempting to classify surfaces is to notice that we can tile the plane with squares, shown below:

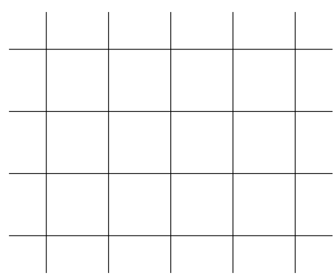

Figure 2: Tiling the plane with squares.

This tiling has some special symmetries. If you pick the tiling up, and move it one unit to the right or left, you can put it back down on top of itself. Similarly you can also move the tiling up or down by a unit. It is possible to interpret these symmetries as instructions for gluing up the sides of a single square in the tiling, and so we can think of this tiling as corresponding to a particular surface, the torus.

Apart from the two-sphere, all the other surfaces come from special tilings of the plane.[2] The key similarity between the two-sphere and the plane, is that they are both **simply connected**. Simply connected means that you can shrink every loop you draw in the space down to a point without leaving the space. For example in two dimensions, the plane is simply connected, but the torus isn't. Neither of the loops on the torus with arrows on them in Figure 1 can be shrunk down to points without leaving the surface.

In two dimensions, we understand the simply connected surfaces, and their tilings, so we can attempt to apply the same strategy to three-manifolds. An example of a simply connected three-manifold with finite volume is the three-sphere. The three-sphere is the analog of the two-sphere, the surface of a solid ball, but one dimension up. One way to think about the three-sphere, is to note that we can make a two-sphere by gluing two discs together along their boundaries. This is illustrated below:

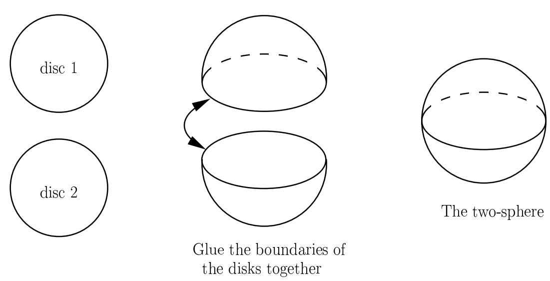

Figure 3: Making spheres from disks.

We can make a three-sphere in a similar way. The analog of a disc, one

[2]The torus comes from the Euclidean plane, to get the other surfaces we need to put a different metric on the plane, and think of it as the hyperbolic plane.

dimension up, is a solid ball. So take two solid balls, and glue their boundaries together. This space does not fit nicely in three-dimensional Euclidean space, so is not always easy to visualize.

Here is another way to make the three-sphere. If you are familiar with thinking of the two-sphere as the Euclidean plane, together with a point at infinity (by using stereographic projection), then you can also think of the three-sphere as Euclidean three-space, together with a point at infinity.

The three-sphere is a three-manifold of finite volume which is simply connected. To apply our strategy to understand three-manifolds, we need to know if there are any other (finite volume) simply connected three-manifolds. In fact, this is a famous unsolved problem, known as the Poincaré Conjecture. This problem is considered to be of sufficient importance that the Clay Institute for Mathematics has offered a prize of a million dollars for a solution. People have been trying to prove this without success for the past hundred years!

Another open problem concerning the three-sphere is the Spherical Space-form Conjecture, which sounds very imposing, but is really just about the different tilings of the three-sphere. There are some standard tilings, which are well understood (called spherical spaces, because in some sense they have the same curvature as the three-sphere), and the conjecture states that these are the **only** possible tilings. In fact, this has been shown to be true in certain special cases, for example, if the number of tiles is a power of two. In my thesis, I show that the result is true for tilings with three tiles, and it is possible the techniques I use will extend to powers of three as well.

2 Technical aspects.

The main technical tool I use is called a **sweepout**. A sweepout is a one-parameter family of surfaces that "fill-up" a manifold. We normally think of the parameter as "time". The simplest example comes from thinking of the three-sphere as a union of two balls. The two-sphere boundary of the balls is the surface at time zero, say. Then as we go forward in time, the two-sphere shrinks down to a point in one of the balls, as we go back in time, the two-sphere shrinks down to a point in the other ball. The picture below shows this happening one dimension down:

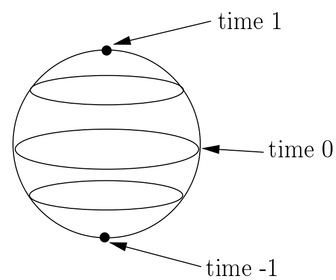

Figure 4: A sweepout of a two-sphere by circles (one-spheres).

If you are familiar with height functions or Morse theory, you can think of this sweepout as a Morse function on the three-sphere with only two critical points. Alternatively, if you are familiar with Heegaard splittings, you can think of it as coming from the genus zero splitting of the three-sphere.

A "tiling" is given by a map from the three-sphere to itself which is periodic, i.e. when you apply the map three times you are back where you started. The map must also have no fixed points, in order to get a quotient which is a manifold. The images of this sweepout under the period three map gives rise to three families of surfaces. When they start out, the two-spheres in each family are small and disjoint. As time passes, they grow larger, and intersect each other in double curves and triple points, and then shrink down to points again.

The basic idea is to define a complexity for the sweepout by counting the maximum number of triple points that occur, and then show that a "mini-max" sweepout is standard. A "mini-max" sweepout is one which has lowest complexity, i.e. it minimizes the maximum number of triple points.

In fact, we need a more general definition of sweepout, in which the sweepout spheres are allowed to break up into finitely many spheres during the sweepout, and the complexity we use is actually pairs: (number of triple points, number of simple closed double curves), ordered lexicographically.

At the beginning and end of the sweepout, complexity is $(0, 0)$, so the main argument consists of taking a piece of the sweepout which contains a local maximum of complexity, and showing that we can either change the sweepout to reduce complexity, or else the tiling standard. The reduction consists of a precise definition of ways of simplifying the sweepout, and then a combinatorial argument showing that the configurations that arise at the local maximum can either be reduced, or contain an invariant unknotted curve, which shows the tiling is standard.

My thesis is available from my website at:

`http://www.math.ucsb.edu/~maher/research`.