Random Heegaard splittings

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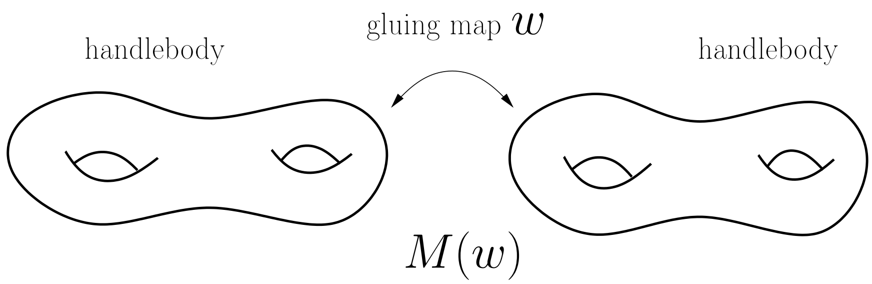
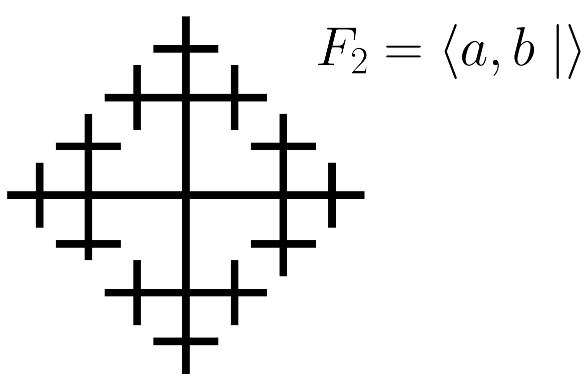
Every 3-manifold has a Heegaard splitting



The gluing map is an element of the mapping class group $G = \mathsf{MCG}(\Sigma) = \mathsf{Diff}^+(\Sigma)/\mathsf{Diff}_0(\Sigma)$

G is finitely generated, let Γ be a Cayley graph for G, consider the nearest neighbour random walk on Γ.



More generally: pick a probability distribution μ on G, and define a Markov chain with transition probabilities $p(x, y) = \mu(x^{-1}y)$.

Path space: $(\Omega, \mathbb{P})$, where $\Omega = G^{\mathbb{N}}$, $\mathbb{P}$ depends on μ

Require:
- subgroup generated by supp(μ) non-elementary
- μ has finite first moment

Conjecture [Dunfield-W. Thurston]: Let w_n be a random walk of length n on the mapping class group. Then
- $\mathbb{P}(M(w_n)$ is hyperbolic$) \to 1$ as $n \to \infty$
- Vol($M(w_n)$) grows linearly in n

[Kesten, Day et al.] A random walk on a non-amenable group has a linear rate of escape

The mapping class group is non-amenable, so $|w_n|_\Gamma$ grows linearly,
i.e. $\mathbb{P}(n/E \leqslant |w_n|_\Gamma \leqslant nE) \to 1$, as $n \to \infty$.

Complex of curves $\mathcal{C}(\Sigma)$:

- vertices: isotopy classes of simple closed curves
- simplices: spanned by disjoint collections of simple closed curves

Thm [M]: A random walk on the mapping class group makes linear progress in the complex of curves, i.e.
$\mathbb{P}(n/L \leqslant d_{\mathcal{C}}(x_0, w_n x_0) \leqslant Ln) \to 1$ as $n \to \infty$

[Brock-Souto] Hyperbolic volume coarsely equivalent to distance between disc sets in a (modified) pants complex

[Masur-Minsky] $\mathcal{C}(\Sigma)$ is δ-hyperbolic.

[Klarreich] $\partial\mathcal{C}(\Sigma) =$ minimal foliations $\subset \mathcal{PMF}$

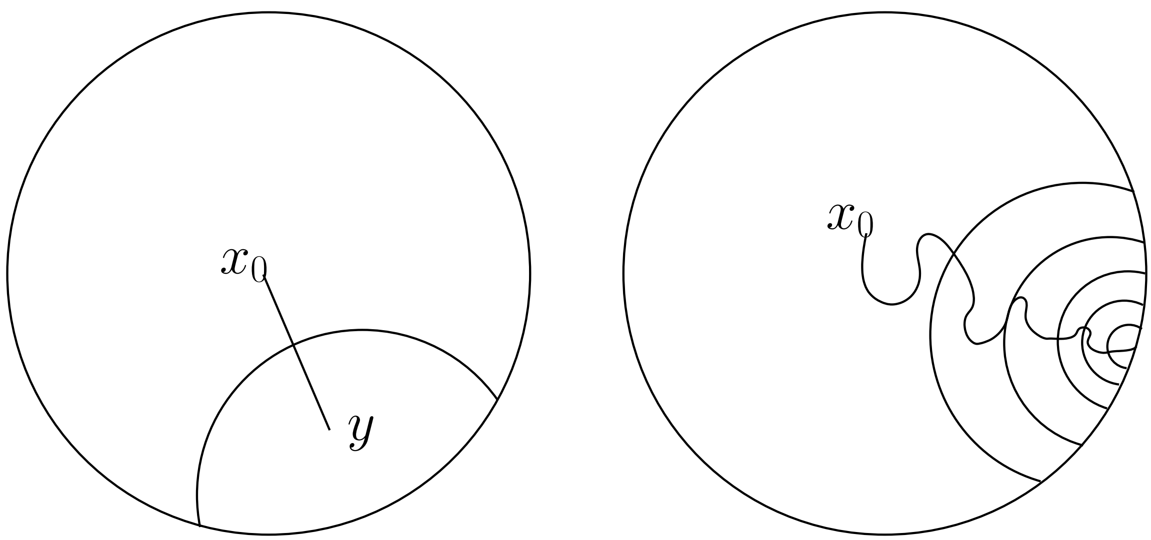
[Kaimanovich-Masur] $w_n x_0 \to F \in \mathcal{PMF}$ a.s.

This gives a measure ν on $\mathcal{PMF}$,
$\nu(X) =$ probability w_n converges to $F \in X$

Half space $H(x_0, y) = \{z \in \mathcal{C}(\Sigma) \mid d_{\mathcal{C}}(z, y) \leqslant d_{\mathcal{C}}(z, x_0)\}$

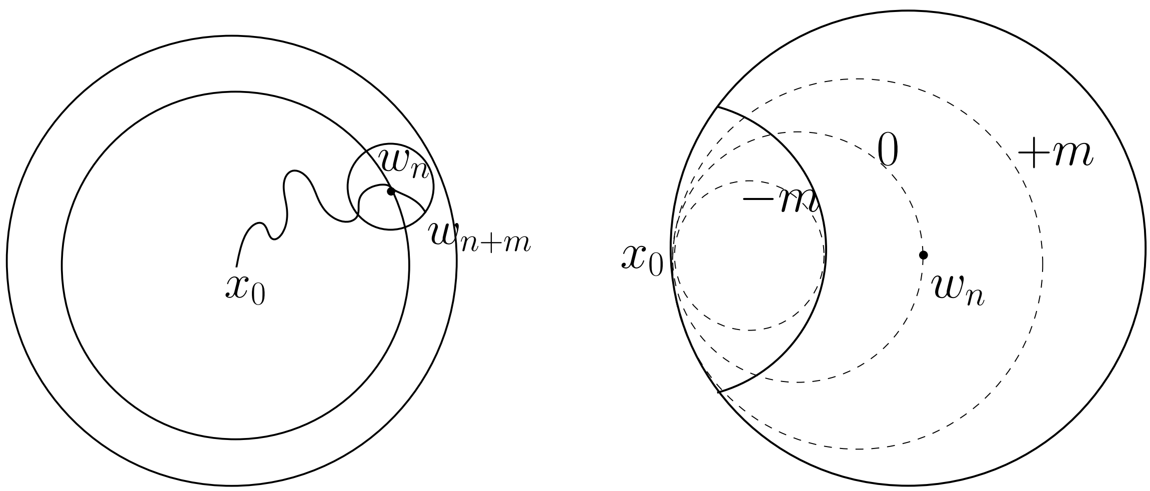
Lemma: $\nu(H(x_0, y)) \leqslant L^{|y|}$, for some constant $L < 1$ independent of y

Proof: $\nu(H(x_0, y)) \leqslant 1 - \epsilon$ for all $|y| \geqslant K$, for constants $\epsilon > 0$ and K



Nested half spaces, conditional probability.

Linear progress:



For large enough m, $-mL^m + (1 - L) \geqslant a > 0$

So $\mathbb{E}(w_{n+m}) \geqslant \mathbb{E}(w_n) + a$

use: Kingman's subadditive ergodic theorem