Random walks on groups with negative curvature

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F_2 free group on two generators

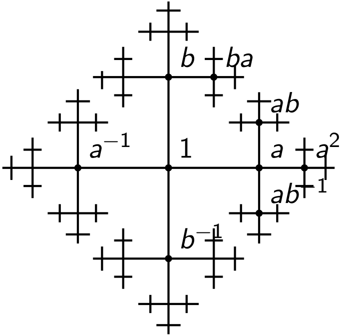
$$\begin{aligned}
\text{Set:}\ \ & \text{all finite strings of letters } \{a, a^{-1}, b, b^{-1}\}/\sim \\
\text{Equivalence relation:}\ \ & aa^{-1} \sim 1 \sim a^{-1}a \sim bb^{-1} \sim b^{-1}b \\
\text{Multiplication:}\ \ & \text{concatenation } (aba^{-1})(ab) = abaa^{-1}b = abb \\
\text{Identity:}\ \ & 1 = \varnothing \\
\text{Notation:}\ \ & F_2 = \langle a, b \mid \ \rangle
\end{aligned}$$

Simple random walk $\leftrightarrow$ random products of generators, e.g:

$$\begin{aligned}
w_{20} &= b^{-1}aa^{-1}baababbb^{-1}ab^{-1}a^{-1}bbb^{-1}b^{-1}b^{-1}a^{-1} \\
&= a^2babab^{-1}a^{-1}b^{-1}a^{-1}
\end{aligned}$$

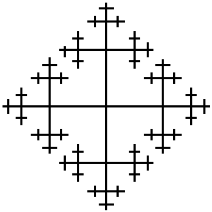
The *Cayley graph* of a finitely generated group is the graph with

- vertices: elements of the group
- edges: connect elements which differ by a generator



Simple random walk $\leftrightarrow$ nearest neighbour random walk on Cay(F_2)

The nearest neighbour random walk on the four-valent tree:



Useful properties:

- Transient

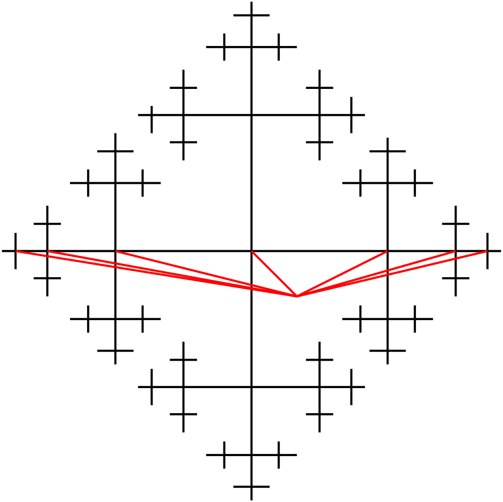
$$\mathbb{P}(\text{random walk hits } v_0 \text{ finitely often}) = 1.$$

- Convergence to ∂X.
- Linear progress, $\mathbb{E}(d(v_0, w_n)) \sim n$.

General setup: $G \curvearrowright X$

G	$\curvearrowright$	(X, d)	geodesic metric space
countable group	$\uparrow$		Gromov hyperbolic
	isometries		not locally compact

Cayley graph for an infinite generating set $A = \{a^{\pm n}, b^{\pm 1}\}$:



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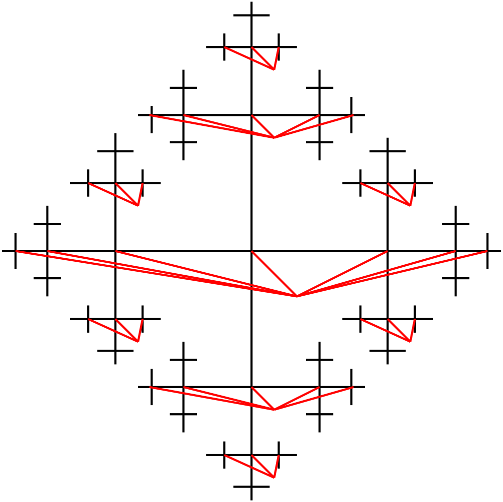

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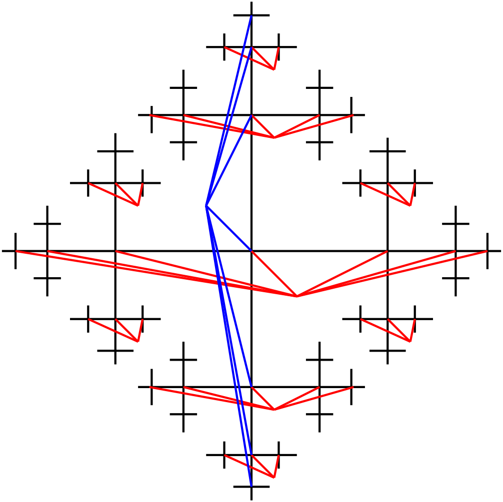

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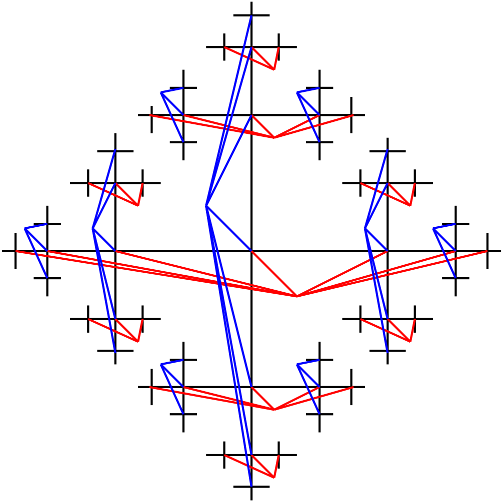


G countable group, μ probability distribution on G.

w_n sample path of length n:

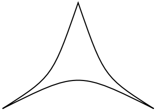
$$w_n = g_1 g_2 \dots g_n$$

Step space: $(G, \mu)^{\mathbb{N}} \ni (g_1, g_2, g_3, \dots)$
Path space: $(G^{\mathbb{N}}, \mathbb{P}) \ni (1, w_1, w_2, w_3, \dots)$

Example: simple random walk on F_2

$$G = F_2, \qquad \mu(g) = \begin{cases} \frac{1}{4} \text{ if } g \in \{a^{\pm 1}, b^{\pm 1}\} \\ 0 \text{ else} \end{cases}$$

[Gromov] A geodesic metric space (X, d) is δ-hyperbolic if every geodesic triangle is δ-thin, i.e. any side is contained in a δ-neighbourhood of the other two.



Examples: trees, hyperbolic space ...

Gromov hyperbolic group: Cayley graph has *thin triangles*.

Examples: free groups F_n, fundamental groups of closed hyperbolic manifolds.

Non-examples: groups containing $\mathbb{Z}^2$.

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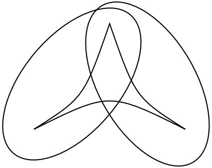

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Useful properties of (X, d) δ-hyperbolic spaces:

Gromov boundary ∂X: equivalence classes of geodesic rays.

Classification of isometries:

- elliptic: bounded orbits
- parabolic: unbounded orbits, $\tau(g) = 0$
- loxodromic: $\tau(g) > 0$, two fixed points in ∂X, invariant axis, north-south dynamics.

Translation length:

$$\tau(g) = \lim_{n\to\infty} \frac{1}{n} d_X(x_0, g^n x_0)$$

We say a group G is *weakly hyperbolic* if G acts by isometries on a Gromov hyperbolic space (X, d) and is *non-elementary* if G contains two independent loxodromic elements.

Examples:

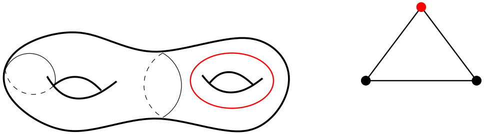
- Hyperbolic groups
- Relatively hyperbolic groups
- Mapping class groups of surfaces
- $\mathrm{Out}(F_n)$
- Right Angled Artin Groups (RAAGs)
- Acylindrical groups

Non-examples: Abelian groups, lattices in higher rank Lie groups.

The mapping class group of a surface S is $\mathrm{Diff}_+(S)/\text{isotopy}$

acts on the curve complex $\mathcal{C}(\Sigma)$.

- vertices: isotopy classes of simple closed curves.
- simplices: spanned by disjoint simple closed curves.



Finite dimensional, but not locally finite.

[Masur-Minsky] $\mathcal{C}(\Sigma)$ is δ-hyperbolic.

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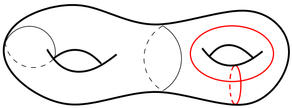

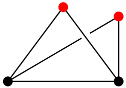

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Thm [M-Tiozzo]: Let G be a countable group acting by isometries on a separable Gromov hyperbolic space X, and let μ be a non-elementary probability distribution on G. Then almost every sample path $(w_n x_0)$ converges to a point in the Gromov boundary.

We say μ is *non-elementary* if $\langle \text{supp}(\mu) \rangle$ contains two independent loxodromic elements.

Orbit map: $G \to X$, $g \mapsto gx_0$.

separable: countable dense subset

Applications:

- Linear progress $\mathbb{P}(d_X(x_0, w_n x_0) \geq Ln) \to 1$ as $n \to \infty$. $(L > 0)$
- $\tau(w_n)$ grows linearly
- If $G \curvearrowright X$ is acylindrical then $(\partial X, \nu)$ is the Poisson boundary

Applications:

- [Rivin][Kowalski][M] Generic elements are loxodromic/pA in mapping class groups
- [Calegari-M] scl grows as n / log n in acylindrical groups
- [Lubotzky-M-Wu] Casson invariants for random homology spheres
- [Gekhtman-Taylor-Tiozzo] loxodromic are generic in Cayley graphs for G hyperbolic acting on X hyperbolic.
- [Haettel] Higher rank lattices have finite image in mapping class groups.
- [Hartnick-Sisto] Bounded cohomology of F_n
- [Gadre-M] generic pAs have trivalent singularities

Proof (locally finite case):

1. $X \cup \partial X$ compact, $\mathcal{P}(X \cup \partial X)$ compact, Cesaro averaging gives a μ-stationary probability measure on $X \cup \partial X$.

2. 3. 4. ...

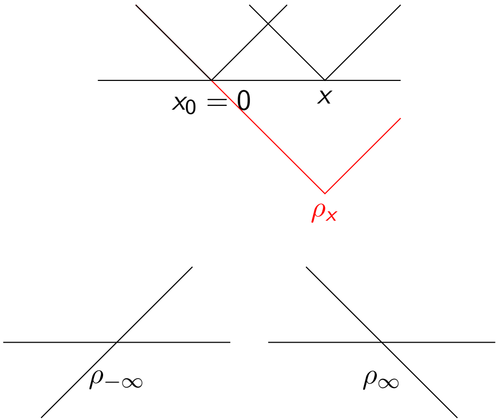
Problem: X not locally compact, $X \cup \partial X$ not necessarily compact.

Solution: Use *horofunction compactification* $\overline{X}^h$

Horofunctions: $X \hookrightarrow C(X)$.

$$\rho \colon x \mapsto d_X(x, \cdot) - d_X(x, x_0)$$

Example: $\overline{\mathbb{R}}^h = \mathbb{R} \cup \{\pm\infty\}$



Warning:

- $C(X)$ topology of uniform convergence on *compact* sets
- $\rho(X)$ not necessarily open in $\overline{X}^h$
- $\overline{X}^h \setminus \rho(X)$ not necessarily compact

$$\overline{X}^h = \overline{X}^h_{Finite} \sqcup \overline{X}^h_\infty$$

local minimum map $\phi \colon \overline{X}^h_\infty \to \partial X$