Random Heegaard splittings

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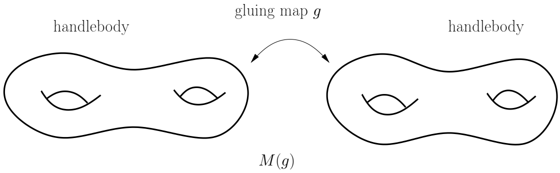
Heegaard splitting: decomposition of a closed 3-manifold into two handlebodies.

Handlebody: regular neighbourhood of a graph in $\mathbb{R}^3$.



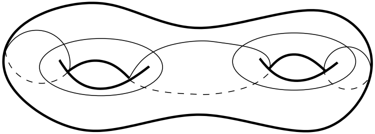
Every closed 3-manifold has a Heegaard splitting.

Isotopic gluing maps give homeomorphic 3-manifolds.

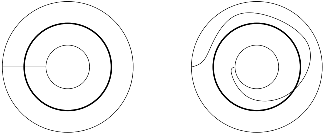
Fix a homeomorphism between the two handlebodies.

The gluing map is an element of the mapping class group of the surface S:

$G = \text{MCG}(S) = \{$ surface homeomorphisms $\}/$ isotopy.



The mapping class group is finitely generated by Dehn twists.



Random Heegaard splitting: $M(w_n)$, where w_n is a random word of length n in the mapping class group G.

Random word: Fix generating set $A = \{a_1, \ldots a_k\}$.
Choose s_i from A with uniform probability.

$$w_n = s_1 s_2 s_3 \ldots s_{n-1} s_n$$

Equivalently: w_n is the nearest neighbour random walk of length n on the Cayley graph for G with respect to the generating set A.

More generally: choose a probability distribution μ on G, choose the s_i independently, distributed according to μ.

(μ finite support, generates a complete subgroup)

[Dunfield-W. Thurston] The probability that $M(w_n)$ is a rational homology sphere tends to 1 as $n \to \infty$.

[Dunfield-D. Thurston] The probability a random tunnel number one manifold fibers over the circle tends to 0 as $n \to \infty$.

[Dunfield-W. Thurston] Conjecture: The probability that $M(w_n)$ is hyperbolic tends to 1 as $n \to \infty$, and hyperbolic volume grows linearly in n.

[M]: $\mathbb{P}(M(w_n) \text{ is hyperbolic}) \to 1$ as $n \to \infty$.

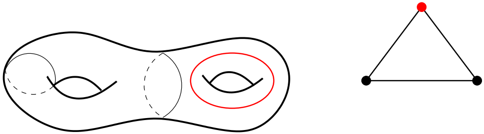
Alternative model for random 3-manifolds:

[Cannon-Floyd-Parry] twisted face pairings.

The mapping class group acts on the complex of curves $\mathcal{C}(S)$.

The complex of curves is a simplicial complex.

- vertices: isotopy classes of essential simple closed curves.
- simplices: spanned by disjoint simple closed curves.



Finite dimensional, but not locally finite.

G acts by simplicial isometries on $\mathcal{C}(S)$.

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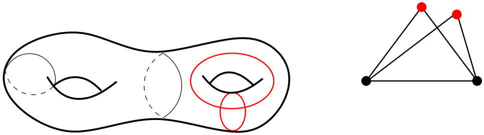

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G acts by simplicial isometries on $\mathcal{C}(S)$.

Disc set: collection of simple closed curves in the boundary surface which bound discs in the handlebody.

A Heegaard splitting $M(g)$ has two handlebodies, with disc sets Δ and $g\Delta$.

Splitting distance: minimum distance in $\mathcal{C}(S)$ between Δ and $g\Delta$.

[T. Kobayashi; Hempel] If the splitting distance is more than two, then M is irreducible, trivial JacoSJ decompostion, not Seifert fibered.

[Perelman] Geometrization $\Rightarrow$ M is hyperbolic, if splitting distance > 2.

[M] Splitting distance of $M(w_n)$ grows linearly in n.

[Brock-Souto] volume grows linearly in n.

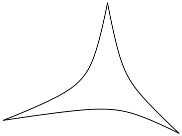
[Hartshorn] Essential surface of genus g implies distance at most $2g$.

[Scharlemann-Tomova] High distance implies the manifold has Heegaard genus g.

[Lustig-Moriah] High distance splittings generic in terms of Lebesgue measure on PML.

[Masur-Minsky] the complex of curves is δ-hyperbolic.

Recall a metric space is δ-hyperbolic if every geodesic triangle is δ-thin, i.e. any side is contained in a δ-neighbourhood of the other two.



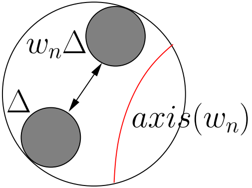
Examples: hyperbolic space, trees, the complex of curves $\mathcal{C}(S)$.

Isometries of δ-hyperbolic spaces are

- elliptic, fix a point in the interior (periodic, reducible)
- parabolic (none of these)
- hyperbolic (pseudo-Anosov)

[Rivin, M]: The probability that w_n is pseudo-Anosov tends to 1 as $n \longrightarrow \infty$.

[M]: Translation length of w_n on $\mathcal{C}(S)$ grows linearly in n.



[Masur-Minsky]: The disc set is quasiconvex.

Gromov boundary: $\{$ set of quasi-geodesic rays $\}/\sim$
Two rays are equivalent if they stay a bounded distance apart.

Pick basepoint x_0 in $\mathcal{C}(S)$.

[Kaimanovich-Masur, Klarreich] $\{w_n x_0\}$ converges to the Gromov boundary with probability one.

This gives a hitting measure or harmonic measure ν on Gromov boundary.

[Gadre] Harmonic measure mutually singular with respect to Lebesgue measure.

[Kerckhoff] Limit set of Δ has harmonic measure zero.

Need to understand joint distribution of endpoints of axes.

For g pseudo-Anosov, let

$$(\lambda^+(g), \lambda^-(g)) \in \partial\mathcal{C}(S) \times \partial\mathcal{C}(S)$$

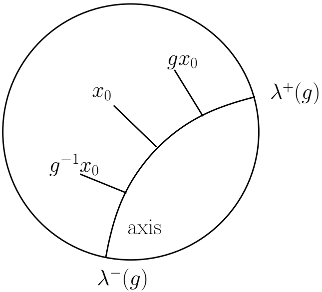
be the endpoints of the axis of g.

Let L_n be the distribution of endpoints of axes of pseudo-Anosov random words of length n.

Claim: $L_n \to \nu \times \widetilde{\nu}$ as $n \to \infty$.

$\widetilde{\nu}$ reflected harmonic measure from random walks generated by $\widetilde{\mu}(g) = \mu(g^{-1})$.

If the translation length of g is bigger than $K(\delta)$, then $\lambda^+(g)$ is close to gx_0, and $\lambda^-(g)$ is close to $g^{-1}x_0$, in the visual metric based at x_0.

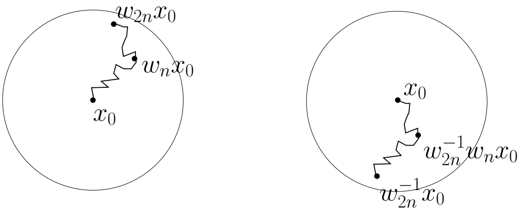


So $L_n \sim (w_n, w_n^{-1})$.

Visual metric: $e^{-\frac{1}{4}(x \cdot y)}$, where $(x \cdot y)$ Gromov product, roughly distance geodesics from x_0 to x fellow travels with geodesic from x_0 to y.

In visual metric based at x_0:

$w_n x_0$ and $w_{2n} x_0$ are close.
$w_{2n}^{-1} x_0$ and $w_{2n}^{-1} w_n x_0$ are close.



So $(w_{2n}, w_{2n}^{-1}) \sim (w_n, w_{2n}^{-1} w_n)$.

As $w_{2n} = s_1 \dots s_n s_{n+1} \dots s_{2n}$, then

$$w_n = s_1 \dots s_n \quad \text{and} \quad w_{2n}^{-1} w_n = s_{2n}^{-1} \dots s_{n+1}^{-1}$$

are independent.

Happy Birthday Bus!