Random walks on the mapping class group

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- Random walks

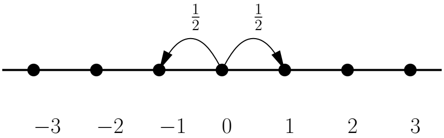
- Random walks on the mapping class group

  Theorem: A random walk on the mapping class group gives a pseudo-Anosov element with asymptotic probability one.

- Random Heegaard splittings

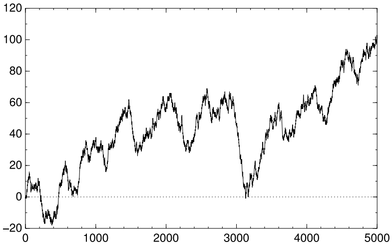
  Theorem: A random Heegaard splitting is hyperbolic with asymptotic probability one.

A random walk on $\mathbb{Z}$



At time $t = 0$ start at $w_0 = 0$

$$w_{t+1} = \begin{cases} w_t + 1 \text{ with probability } 1/2 \\ w_t - 1 \text{ with probability } 1/2 \end{cases}$$



The nearest neighbour random walk on a (finite valence) graph:

• Start at a particular vertex at time 0.
• At time n jump to one of your nearest neighbours, chosen with equal probability.

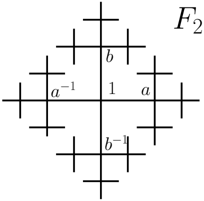
Random walks on groups:

Pick a (symmetric) generating set A.
The *Cayley graph* of a finitely generated group is the graph with

- vertices: elements of the group
- edges: connect elements which differ by a generator

The graph depends on the choice of generating set A, but any two choices give quasi-isometric graphs.

Example of a Cayley graph:



$F_2 = \langle a, b \mid \rangle$

Key example: the nearest neighbour random walk on a Cayley graph of the mapping class group.

- Start at the identity at time 0.
- At time *n* jump to one of your nearest neighbours, chosen with equal probability.

More generally: pick a probability distribution μ on G.
Consider the Markov chain with set G, and transition probabilities $p(x, y) = \mu(x^{-1}y)$.

Time 0: start at identity.
Time 1: distributed according to μ.
Time 2: distributed according to $\mu^2 =$ convolution of μ with itself.

$$\mu^2(x) = \sum_{y \in G} \mu(y)\mu(y^{-1}x)$$

Time n: distributed according to μ^n, n-fold convolution of μ with itself.

Path space: $(G^{\mathbb{Z}_+}, \mathbb{P})$, probability space.

$G^{\mathbb{Z}_+}$ infinite product of G's.

A sample path $\omega \in G^{\mathbb{Z}_+}$ is an infinite sequence of group elements corresponding to the locations of the random walk.

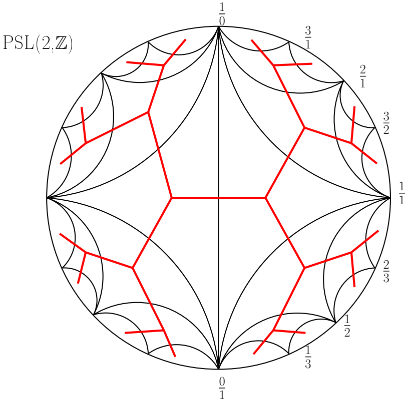
Projection $w_n : G^{\mathbb{Z}_+} \to G$ to the n-th factor is a random variable which gives the location of the sample path at time n.

The distribution of w_n is given by μ^n.

[Kolmogorov] This determines $\mathbb{P}$.

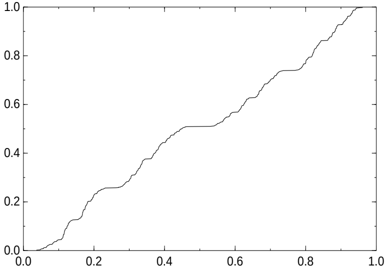
Key point: this enables us to talk about infinite length random walks.

Example: PSL(2,$\mathbb{Z}$)

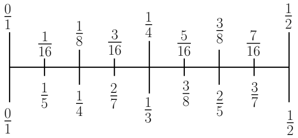


Sample paths converge to the boundary with probability one.
This gives a measure on the boundary, called *harmonic measure* ν.
$\nu(X) = \mathbb{P}$(sample paths which converge to points in X)

This harmonic measure on S^1 is *not* Lebesgue measure.



$$\frac{1}{a_1+\frac{1}{a_2+\dots}} \mapsto \overbrace{0.0\dots0}^{a_1}\overbrace{1\dots1}^{a_2}\dots$$



Convergence to the boundary works for:

matrix groups, e.g. SL(n,$\mathbb{Z}$) [Furstenberg]

- random matrices are irreducible [Rivin, Kowalski]

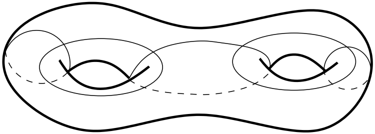
δ-hyperbolic groups [Kaimanovich-Woess]

- random elements are hyperbolic, translation length tends to infinity

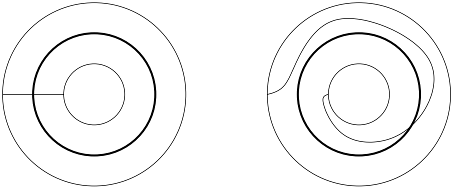
Mapping class groups, braid groups [Kaimanovich-Masur]

- random elements are pseudo-Anosov [M]

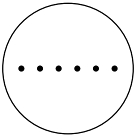
The mapping class group of a surface S is $\{$surface diffeomorphisms$\}$/isotopy.
$G = \mathrm{MCG}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$

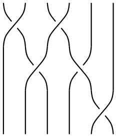


The mapping class group is finitely generated by Dehn twists.



The surface S may have boundary or punctures

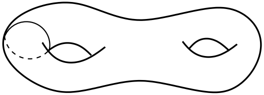




The mapping class group of the n-punctured disc is also known as the braid group.

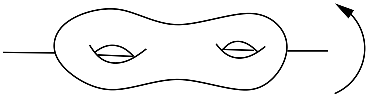
Thurston's classification of surface homeomorphisms

- Reducible:



The map fixes a disjoint collection of simple closed curves.

- Periodic:



Some power of the map is isotopic to the identity.

- Pseudo-Anosov:

Everything else...

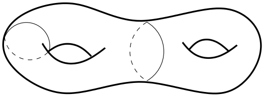
Useful facts about the mapping class group.

[Masur-Minksy] The mapping class group is weakly relative hyperbolic.

G finitely generated by A, gives word metric on G (same as Cayley graph metric).

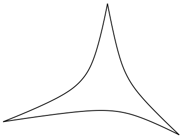
$\widehat{G} = G$ with word metric from an infinite generating set $A \cup \{H_i\}$.

In this case $H_i = \mathrm{stab}(\alpha_i)$, where α_i are representatives of simple closed curves under the action of G.



If $\widehat{G}$ is δ-hyperbolic then we say that G is weakly relatively hyperbolic (with respect to $\{H_i\}$).

Recall a metric space is δ-hyperbolic if every geodesic triangle is δ-thin, i.e. any side is contained in a δ-neighbourhood of the other two.

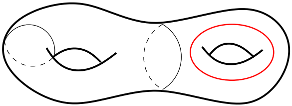


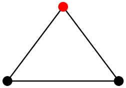
Examples: hyperbolic space, trees, the complex of curves $\mathcal{C}(S)$.

[Masur-Minksy] show that the relative space $\widehat{G}$ is quasi-isometric to the complex of curves.

The complex of curves is a simplicial complex.

- vertices: isotopy classes of simple closed curves.
- simplices: spanned by disjoint simple closed curves.





Finite dimensional, but not locally finite.

[Masur-Minsky] the complex of curves is δ-hyperbolic.

Isometries of δ-hyperbolic spaces are
- elliptic, fix a point in the interior (periodic, reducible)
- parabolic (none of these)
- hyperbolic (pseudo-Anosov)

Gromov boundary: $\{$ set of quasi-geodesic rays $\}/\sim$
Two rays are equivalent if they stay a bounded distance apart.

[Klarreich] The Gromov boundary of the complex of curves is $\mathcal{F}_{\mathsf{min}}$ the space of minimal foliations in PMF, Thurston's space of projective measured foliations.

PMF is a sphere of dimension $6g-5$, $g =$ genus of S.

pseudo-Anosov maps act on $\mathcal{C}(S) \cup \mathcal{F}_{\mathsf{min}}$ as translations along an axis with a unique pair of fixed points, the attracting and repelling fixed points.

[Kaimanovich-Masur, + Klarreich] A random walk on the mapping class group converges almost surely to a uniquely ergodic foliation in PMF, as long as the support of μ is a non-elementary subgroup. The resulting harmonic measure ν on $\mathcal{F}_{\text{min}}$ is non-atomic.

uniquely ergodic $\Rightarrow$ minimal

non-elementary: the subgroup contains a pair of pseudo-Anosov elements with distinct endpoints.

Recall $\nu(X) =$ proportion of sample paths which converge into X.

ν governs the long time behaviour of sample paths.

Theorem [Rivin, Kowalski]: The probability that $w_n(\omega)$ is pseudo-Anosov tends to 1 as $n \to \infty$.

Consider the action on homology, i.e. map from G to $Sp(2g, \mathbb{Z})$.

[Casson-Bleiler] If image of g is irreducible, no roots of unity as eigenvalues, characteristic polynomial not a power of a lower degree polynomial, then g is pseudo-Anosov.

Theorem [M]: The probability that the translation length of $w_n(\omega)$ on $\mathcal{C}(S)$ is at most K tends to zero as $n \to \infty$.

Requires support of μ generates a non-elementary subgroup not contained in a centralizer.

Translation length of g: $\lim \frac{1}{n} d_{\mathcal{C}(S)}(x, g^n x)$.

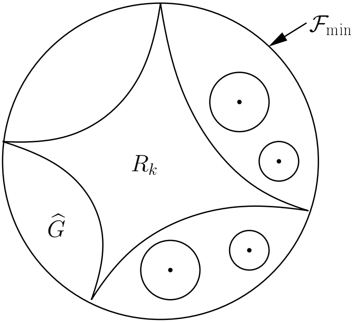
Sketch of proof.

Observation: if $X \subset G$ and limit set of X has (harmonic) measure zero in $\mathcal{F}_{\text{min}}$, then the random walk is transient on X. (A sample path hits X finitely many times almost surely.)

Let $R =$ elements of G of translation length at most K. Then $\nu(\overline{R}) = 1$.

Let $R_k =$ k-dense elements of R, i.e. $r \in R$ such that there is some other $r' \in R$ such that $d_G(r, r') \leqslant k$.

Claim: $\nu(\overline{R}_k) = 0$.



$$\mathbb{P}(w_n(\omega) \in R) = \mathbb{P}(w_n(\omega) \in R_k) + \mathbb{P}(w_n \in R \setminus R_k)$$

- $\mathbb{P}(w_n(\omega) \in R_k) \to 0$ as $n \to \infty$ by transience.
- $\mathbb{P}(w_n(\omega) \in R \setminus R_k) \leqslant 1/k$

True for all k implies $\mathbb{P}(w_n(\omega)) \to 0$ as $n \to \infty$.

More details:

$\overline{R}_k = \bigcup \overline{C(g)}$, where word length of g at most k.

$C(g) =$ centralizer of g, i.e. $h \in G$ such that $gh = hg$.

- g pseudo-Anosov: $C(g)$ virtually cyclic, limit set is fixed points.
- g reducible: centralizer bounded diameter in $\widehat{G}$, limit set empty.
- g periodic: $\overline{C(g)}$ lower dimensional sphere.

[Nielsen] a finite cyclic subgroup of G fixes a point in Teichmüller space = set of hyperbolic structures on S.

$\Rightarrow$ finite cyclic groups realized by covering translations.

So fixed set is lower dimensional Teichmüller space inside original one, so limit set is a lower dimensional PMF inside original one.

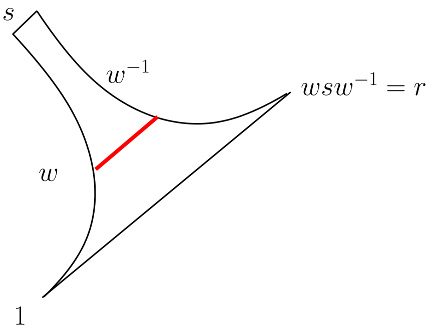
[distance reducing maps $G \to \mathcal{T}(S) \to \widehat{G}$]

Relative conjugacy bounds:

If a and b are conjugate in G then there is a conjugating word w such that $|\widehat{w}| \leqslant K(|\widehat{a}| + |\widehat{b}|)$.

[Masur-Minksy] Version for pseudo-Anosov elements using word length.

This implies if g is conjugate to a short word s, and w is a shortest conjugating word in the relative metric, then the path wsw^{-1} is a quasi-geodesic path, where the quasi-geodesic constants depend on the length of s.

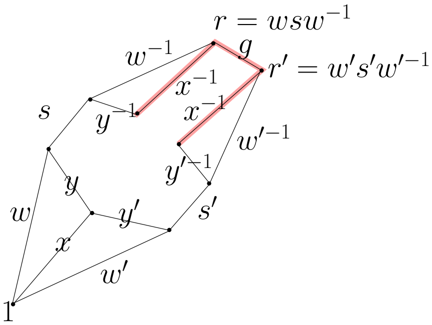


s has bounded length, so thin triangles implies if w very long, then a final segment of w fellow-travels with an initial segment of w^{-1}.

So red path is a short conjugate of s, so could have chosen a shorter conjugating word.

If $r \in R_k$, then there is g of word length at most k such that $rg = r' \in R_k$, so R_k is a finite union of $R \cap Rg$.

Claim: $\overline{R \cap Rg} = \overline{C(g)}$

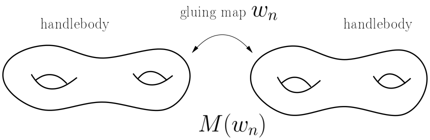


$r = wsw^{-1}$ and $r' = w's'w'^{-1}$, paths are quasi-geodesic, so fellow travel. Write $w = xy$, $w' = xy'$, for y, y' of bounded length.

$x^{-1}gx$ short group element, so conjugate by short z to g.

$x^{-1}gx = zgz^{-1} \Rightarrow g(xz) = (xz)g \Rightarrow x$ close to $C(g)$.

Random Heegaard splittings.



Theorem [M]: The probability that the splitting distance of $M(w_n)$ is at most K tends to zero as n tends to infinity.

Requires support of μ generates a subgroup which is dense in the boundary.

Given S as the boundary of a handlebody H, the disc set Δ is the collection of simple closed curves which bound discs in H.

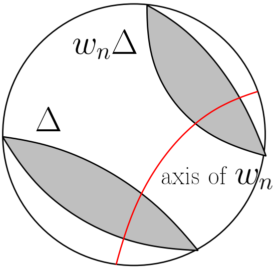
A Heegaard splitting has two handlebodies, with disc sets Δ and $w_n\Delta$.

Splitting distance: minimum distance between Δ and $w_n\Delta$ in $\mathcal{C}(S)$.

[T. Kobayashi;Hempel] If the splitting distance is more than two, then M is irreducible, atoroidal and not Seifert fibered.

[Perelmann] Geometrization $\Rightarrow$ M is hyperbolic.

Corollary: Probability $M(w_n)$ is hyperbolic tends to 1 as $n \to \infty$.



[Kerckhoff] Limit set of Δ has harmonic measure zero.

[Masur-Minsky] Disc set is quasi-convex.

Need to understand (joint) distribution of attracting and repelling endpoints.

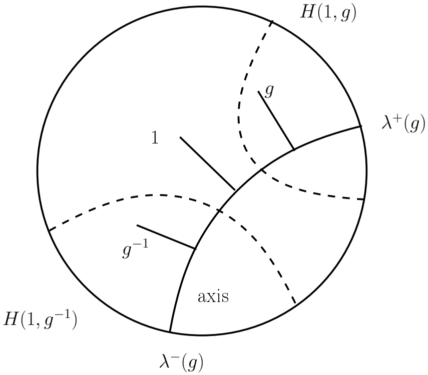
If g is pseudo-Anosov let $\lambda^+(g)$ be the attracting fixed point and let $\lambda^-(g)$ be the repelling fixed point.

Define $\lambda_n : G^{\mathbb{Z}_+} \to \mathcal{F}_{\min} \times \mathcal{F}_{\min} \cup \varnothing$
by $\omega \mapsto (\lambda^+(w_n(\omega)), \lambda^-(w_n(\omega)))$ if $w_n(\omega)$ is pseudo-Anosov.

Claim: $\lambda_n \to \nu \times \widetilde{\nu}$ as $n \to \infty$.

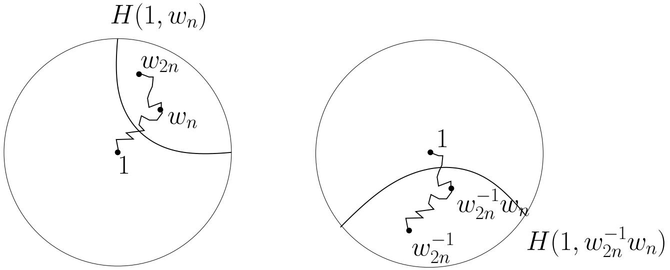
Reflected harmonic measure $\widetilde{\nu}$ is harmonic measure determined by the random walk generated by the reflected measure
$\widetilde{\mu}(g) = \mu(g^{-1})$.

Halfspace: $H(1, x) = \{y \in \widehat{G} \mid \widehat{d}(y, x) \leqslant \widehat{d}(y, 1)\}$.



If the translation length of g is bigger than $K(\delta)$, then $\lambda^+(g) \in H(1,g)$, and $\lambda^-(g) \in H(1,g^{-1})$.

So $\lambda_n \sim (w_n, w_n^{-1})$.



$\mathbb{P}(w_{2n}(\omega) \in H(1, w_n(\omega))) \to 1$ as $n \to \infty$.
$\mathbb{P}(w_{2n}^{-1}(\omega) \in H(1, w_{2n}^{-1}w_n(\omega))) \to 1$ as $n \to \infty$.

So $(w_{2n}, w_{2n}^{-1}) \sim (w_n, w_{2n}^{-1}w_n)$.

If $w_{2n} = s_1 \dots s_n s_{n+1} \dots s_{2n}$, then $w_n = s_1 \dots s_n$ and $w_{2n}^{-1}w_n = s_{2n}^{-1} \dots s_{n+1}^{-1}$, are independent.