Some of my favorite results with Endre Csáki and Pál Révész

A survey

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Dedicated to Endre Csáki and Pál Révész

on the occasion of their 70-th birthday

Abstract: Some topics of our twenty some years of joint work is discussed. Just to name a few; joint behavior of the maximum of the Wiener process and its location, global and local almost sure limit theorems, strong approximation of the planar local time difference, a general Strassen type theorem, maximal local time on subsets.

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1. Introduction.

Endre Csáki and Pál Révész are two of the most influential probabilists of our time. Their books and papers are source of inspiration for generations of mathematicians. I have been privileged to know them and learn from them for more than three decades. I wrote my first joint paper ever with Pál Révész ([**?**]) in 1974, the first joint paper with Endre Csáki ([**?**]) in 1976 . They wrote their first joint paper ([**?**]) in 1979. The first time the three of us had a paper together was in 1983, when we wrote a paper with Miklós Csörgő ([**?**]) a long time friend and collaborator of Pál Révész. Since that time various subgroups of the four of us wrote many papers together. I cherish this collaboration and friendship as the highlight of my professional life. It would be almost impossible to discuss all of these papers here, and fortunately most of them were discussed in two wonderful survey papers written by Miklós Csörgő ([**?**]), ([**?**]) in the occasion of Endre Csáki and Pál Révész 65th birthday. For a broad survey of their great achievements since 1979 see Miklós Csörgő's paper in this volume.

In this note I try to accomplish a much smaller goal, namely to give a short overview of the few papers which were written solely by the three of us, Endre Csáki, Pál Révész and myself.(c.f. ([**?**],... ,[**?**]). These papers, written between 1987 and 2004, are very loosely connected and most of them are very close to my heart. A section is devoted to each in the sequel in their chronological order, and titled accordingly.

2. On the maximum of a Wiener Process and Its Location

I will describe the results of this paper rather briefly as it is very nicely summarized in the "Bible" as young probabilists often refer to Pál Révész's book ([**?**]). The joint behavior of two stochastic processes is always fascinating. The first ever integral test for the joint behavior of two random processes was given by Endre Csáki in 1978 ([**?**]) for the joint behavior of the maximum and the minimum of the Wiener process. While the behavior of maximum and the location of the maximum separately, has been well understood, we were interested in their joint behavior. Let $M(t) = \max_{0\le s\le t} |W(s)|$ where $\{W(t), t \ge 0\}$ is a Wiener process, and define $\nu(t)$ to be the location of the maximum of the absolute value of $W(.)$, i.e. $|W(\nu(t))| = M(t)$. While the laws of the iterated logarithm easily imply that with probability one if t is big enough

$$\nu(t) \ge (1-\epsilon)\frac{\pi^2}{16}\frac{t}{(\log\log t)^2} \tag{2.1}$$

we wanted to know whether the lower bound in (**??**) can be attained. Our next question was, that if $\nu(t)$ is almost as small as the above theorem permits then is it possible for $M(t)$ to be as big as the law of the iterated logarithm permits. The answer to both of these questions turned out to be negative. We also wanted to know how small can $\nu(t)$ be if $M(t)$ is as small as possible. These questions were separately answered in the paper. I will only quote the main comprehensive result containing the answers to the above questions and also the law and other law of iterated logarithm.

Theorem 2.1. *Let*

$$a(t) = \frac{(\log\log t)^2}{t}\nu(t), \qquad b(t) = \left(\frac{\log\log t}{t}\right)^{1/2} M(t) \tag{2.2}$$

$$K = \left\{(x,y) : x \geq \frac{\pi^2}{4},\ x^{1/2}\left(1-\left(1-\frac{\pi^2}{4x}\right)^{1/2}\right)^{1/2} \leq y \leq x^{1/2}\left(1+\left(1-\frac{\pi^2}{4x}\right)^{1/2}\right)^{1/2}\right\}$$

$$= \left\{(x,y) :\ x > 0,\ y > 0,\ \frac{y^2}{2x} + \frac{\pi^2}{8y^2} \leq 1\right\}. \tag{2.3}$$

Then the set of limit points of the net $(a(t), b(t))$ *(as* $t \to \infty$ *) is* K *with probability 1.*

One more interesting consequence of the above theorem is an exact lim inf result for $t - \varphi(t)$ where $\varphi(t)$ is the longest flat interval of $\{M(s),\ 0 \leq s \leq t\}$, i.e., $\varphi(t)$ is the largest positive number for which there exists a positive number α such that

$$0 < \alpha < \alpha + \varphi(t) < t \qquad \text{and} \qquad M(\alpha) = M(\alpha + \varphi(t)).$$

However to get the lim inf behavior of $\varphi(t)$ itself was a very interesting problem as well. We proved that

Theorem 2.2.

$$\liminf_{t\to\infty} \frac{\log\log t}{t}\varphi(t) = \beta$$

where β is the root of the equation

$$\sum_{k=1}^{\infty} \frac{\beta^k}{k!(2k-1)} = 1.$$

Let me remark here, that this result tells us that the lim inf of the longest flat interval of $M(t)$ is the same as that of $M^+(t)(= \sup_{s\leq t} W(s))$ which was established in the celebrated Csáki, Erdős and Révész paper ([**?**]).

I would like to mention now a few papers which are strongly related to this topic. Investigating how big could $\nu(t)$ and $M(t)$ be simultaneously Chen ([?]) proved that the set of limit points of

$$\left(\frac{\nu(t)}{t}, \frac{M(t)}{2t\log\log t}\right)$$

as $t \to \infty$ is almost surely

$$K^* = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1, x \ge y^2\}.$$

Some of the above results were extended for d-dimensional Brownian motion by Zao and Lin ([?]). The lim inf of the difference of the location of the maximum and the minimum of the Brownian motion (not the reflected one) was elegantly treated by Zhan Shi ([?]), and a joint integral test for the location of the maximum and the minimum is given in a forthcoming paper of Randjiou ([?]).

3. On almost sure local and global central limit theorems

At the beginning of the nineties there was a lot of activity and interest in the field of almost sure central limit theorems. Lacey and Philipp ([?]) proved the following nice theorem:
Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables with $E(X_i) = 0$, $E(X_i^2) = \sigma^2 < \infty$, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\{S_k \le x\sigma k^{1/2}\}}{k} = \Phi(x) \qquad \text{a.s.} \tag{3.1}$$

where $\Phi(x)$ is the standard normal distribution function.
However about 40 years earlier Chung and Erdös ([?]) proved the following beautiful result. Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables with $E(X_i) = 0$. Assume that every integer a is a possible value of S_k for all sufficiently large k. Then

$$\lim_{n\to\infty} \frac{1}{\log M_n} \sum_{k=1}^{n} \frac{I\{S_k = a\}}{M_k} = 1 \qquad \text{a.s.} \tag{3.2}$$

where $M_k = \sum_{i=1}^{k} P(S_i = a)$. If we spell out this theorem in the case when $E(X_i^2) = \sigma^2 < \infty$, and $a = 0$ we have $M_k \sim \sigma^{-1}(2k/\pi)^{1/2}$, hence we get that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\{S_k = 0\}}{k^{1/2}} = \frac{2\varphi(0)}{\sigma}. \qquad \text{a.s.} \tag{3.3}$$

where $\varphi(.)$ is the standard normal density function.

We will refer to (**??**) and (**??**) as global and local almost sure central limit theorem respectively. Our goal was to understand the connection between these two theorems by stating and proving a general result which contains both of them. Namely we considered the limit behavior of the logarithmic average

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{I\{a_k \le S_k < b_k\}}{k\mathbf{P}(a_k \le S_k < b_k)} \tag{3.4}$$

with $-\infty \le a_k \le 0 \le b_k \le \infty$, where the terms in the sum above are defined to be 1 if their denominator happens to be 0. More precisely let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, and let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of real numbers and put

$$p_k = \mathbf{P}(a_k \le S(k) < b_k) \tag{3.5}$$

and

$$\alpha_k = \begin{cases} \frac{I\{a_k \le S_k < b_k\}}{p_k} & \text{if } \; p_k \ne 0 \\ 1 & \text{if } \; p_k = 0. \end{cases}$$

So we need to investigate

$$\mu_n = \sum_{k=1}^{n}\frac{\alpha_k}{k}. \tag{3.6}$$

We considered three different cases, continuous, lattice valued and general type of distributions. By being lattice valued, we will mean taking the values $\{h+j\}_{j=-\infty}^{\infty}$ for some $0 \le h < 1$ with maximal span 1 (that is to say g.c.d.$\{j : \mathbf{P}(X_1 = h+j) > 0\} = 1$.) We made two type of assumptions as follows:

$$\text{Condition A}: \qquad \sum_{\{1\le k\le n, p_k\ne 0\}}\frac{1}{k^2 p_k} = O(\log n), \;\text{ as } n\to\infty. \tag{3.7}$$

$$\text{Condition B}: \qquad \sum_{\{1\le k\le n, p_k\ne 0\}}\frac{\log k}{k^{3/2} p_k} = O(\log n), \;\text{ as } n\to\infty. \tag{3.8}$$

where p_n was defined by (**??**).

Theorem 3.1. *Let $-\infty \le a_k \le 0 \le b_k \le \infty$ and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathbf{E}|X_i|^3 < \infty$ and $\mathbf{E}(X_i) = 0$. Let the random variables $X_i, i = 1, 2...$*

have bounded density or be lattice valued, satisfying condition A, or be arbitrary and satisfy condition B. Then

(3.9)
$$\lim_{n\to\infty} \frac{\mu_n}{\log n} = 1. \qquad \text{a.s.}$$

The starting point to prove the above theorem is the following extremely simple lemma.

Lemma 3.1. *Assume that* $\xi_1, \xi_2 ...$ *are random variables with* $\mathbf{E}(X_i) = 1$, $i = 1, 2, ...$ *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \mathbf{E}\left(\sum_1^n \frac{\xi_k}{k}\right) = 1.$$

If, furthermore $\xi_k \geq 0, k = 1, 2, ...,$ *and*

(3.10)
$$\text{Var}\left(\sum_1^n \frac{\xi_k}{k}\right) \leq C \log n$$

with some $C > 0$, *then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_1^n \frac{\xi_k}{k} = 1 \qquad \text{a.s.}$$

However to show that the condition in (**??**) holds is a very delicate job. In the years following this paper we had a few more results on this topic where refined applications of this covariance calculation were needed time and again. Very elegant general results were given by Berkes and Csáki ([**?**]), which is also a great source of literature on almost sure central limit theorems.

4. Random walk with alternating excursions.

This was a paper full of fun. Started like a game, ended as a paper. Here is the setup. We consider a simple symmetric random walk. Looking at the consecutive excursion we modify it in such a way, that, by flipping the excursions when necessary, we ensure that every second excursion should be positive, and the rest of them should be negative. More precisely, let $X_1, X_2, ...$ be i.i.d. r. v.-s with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$. Then $S_n = \sum_{i=1}^n X_i$, is an ordinary simple symmetric random walk, or briefly SSRW. Let

$$\rho_0 = 0, \quad \rho_k = \min\{i : i > \rho_{k-1},\ S_i = 0\}, \quad k = 1, 2, ...$$

$$S_n^* = S_n \quad if \quad 0 \leq n \leq \rho_1$$

and

$$S_n^* = (-1)^k X_1 |S_n| \quad if \quad \rho_k \le n \le \rho_{k+1}, \quad k = 1, 2, ...$$

We will call S_n^* a random walk with alternating excursions, or briefly RWAE.
We posed the following question: in what sense are these two types of walks different, and in what sense are they the same. Here are our answers in a nutshell.

Number of paths: In $2n$ steps the SSRW has 2^{2n} paths, while the RWAE has $2\binom{2n}{n}$ paths.

Distributions: are the same;

$$\mathbf{P}(S_n = k) = \mathbf{P}(S_n^* = k), \qquad n = 0, 1, 2...k = 0, \qquad \pm 1, \pm 2, ...$$

Joint distributions: usually do not match e.g.

$$\mathbf{P}(S_1 = 1, S_3 = 1) = 1/4 \quad \text{and} \quad \mathbf{P}(S_1^* = 1, S_3^* = 1) = 1/8$$

Maximum: Put $\nu_{2n} = \max_{0 \le i \le \rho_{2n}} S_i$ and $\nu_{2n}^* = \max_{0 \le i \le \rho_{2n}} S_i^*$. Then

$$\mathbf{P}(\nu_{2n} < k) = \left(1 - \frac{1}{2k}\right)^{2n} \quad \text{and} \quad \mathbf{P}(\nu_{2n}^* < k) = \left(1 - \frac{1}{k}\right)^{n},$$

but on the other hand

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\nu_{2n}}{n} < x\right) = \lim_{n\to\infty} \mathbf{P}\left(\frac{\nu_{2n}^*}{n} < x\right) = e^{-1/x}.$$

Local time: Let

$$\xi(k,n) = \#\{i;\ 0 < i \le n, \quad S_i = k\} \qquad \text{and} \qquad \xi^*(k,n) = \#\{i;\ 0 < i \le n, \quad S_i = k\}.$$

Then

$$\mathbf{P}(\xi(0,n) = j) = \mathbf{P}(\xi^*(0,n) = j),$$

but in general $\xi(i,n)$ and $\xi^*(i,n)$ have different distributions. However,

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{\xi^*(k,n)}{\xi(k,n)} = 1\right) = 1,$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\xi^*(k,n)}{\sqrt{n}}\right) = \lim_{n\to\infty} \mathbf{P}\left(\frac{\xi(k,n)}{\sqrt{n}}\right) = 2\Phi(x) - 1.$$

Arcsine law: Let $\mu_n = \#\{i : 1 \le i \le n, \quad S_i > 0\}$ and $\mu_n^* = \#\{i : 1 \le i \le n, \quad S_i^* > 0\}$. For $0 \le a \le b \le 1$ we have, for the excursion endpoints and even excursion endpoints respectively:

$$\lim_{k\to\infty} \mathbf{P}\left(a < \frac{\mu_{\rho_k}}{\rho_k} < b\right) = \int_a^b \frac{1}{\pi} \frac{1}{(x(1-x))^{1/2}}\, dx,$$

$$\lim_{k\to\infty} \mathbf{P}\left(a < \frac{\mu^*_{\rho_{2k}}}{\rho_{2k}} < b\right) = \int_a^b \frac{1}{\pi} \frac{1}{(x(1-x))^{1/2}}\, dx.$$

Using another representation of the RWAE, it turns out that we have
Lemma: *For any $\epsilon > 0$ we have as $n \to \infty$*

$$|S_n^* - S_n| = O(n^{1/4+\epsilon}) \qquad \text{a.s.}$$

As a consequence of this lemma one concludes that

- Donsker's theorem holds for the RWAE,
- Arcsine law holds for the RWAE (for each n not only for excursion endpoints).

Finally, one can prove the following strong approximation.

Theorem 4.1. *On an appropriate probability space one can define an $\{S_n^*\}_{n=1}^{\infty}$ and a standard Wiener process $\{W(t),\, t \ge 0\}$ such that for any $\epsilon > 0$, as $n \to \infty$,*

$$S_n^* - W(n) = o(n^{1/4+\epsilon}) \qquad \text{a.s..}$$

Conclusion: The deterministic changes we made, do not make an essential difference. So the obvious question is: how about changing the SSRW some other way. E.g., making approximately $\sqrt{n}$ deterministic changes according to some other rule. Is it possible to make more than $\sqrt{n}$ changes?

5. Strassen theorems for a class of iterated processes.

In the middle of the nineties the investigation of iterated processes became very popular. Just to mention a few of these papers, it started with the work of Burdzy ([?]) and it was followed Arcones ([?]), Hu and Shi ([?]), Csáki, Csörgő, Földes, and Révész ([?]) and Khoshnevisan and Lewis ([?]). In ([?]) we proved a very general Strassen theorem for the properly normalized vector $(W_1(yAW_2(xT)), W_2(xT))$ where $W_1(.), W_2(.)$ is a pair of independent Wiener processes and A is an operator on $C_0[0,1]$ satisfying certain conditions. I only quote a very simple case of this result which is needed here. Define $M_f(x) = \max_{0\le y\le x} f(y)$.

Let $\mathcal{S}$ be the Strassen class of functions, i.e., $\mathcal{S} \subset C[0,1]$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0) = 0 \qquad \text{and} \qquad \int_0^1 \dot{f}^2(x)dx \le 1. \tag{5.1}$$

The set of $\mathbf{R}^2$ valued, absolutely continuous functions

$$\{(g(y), h(x)),\ 0 \le y \le 1, 0 \le x \le 1\} \tag{5.2}$$

for which $g(0) = h(0) = 0$ and

$$\int_0^1 \dot{g}^2(y)dy + \int_0^1 \dot{h}^2(x)dx \le 1 \tag{5.3}$$

will be called *Strassen class* $\mathcal{S}^2$. In [?] we proved

Theorem 5.1. *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from zero. For $0 \le x \le 1$, $0 \le y \le 1$, the limit set of the vector*

$$\left(\frac{W_1(yM_{W_2}(xT))}{T^{1/4}(2\log\log T)^{3/4}}, \frac{W_2(xT)}{(2T\log\log T)^{1/2}}\right) \tag{5.4}$$

is $(g(yM_h(x)),\ h(x))$, where $(g,h) \in \mathcal{S}^2$.

We called the above theorem *composite* Strassen theorem because of the composite structure of $(g(yM_h(x)))$. In this paper, inspired by the work of Marcus and Rosen ([?]) and Bertoin ([?]), we wanted to explore whether we can prove a *direct* Strassen theorem where the class is described in terms of one function only. First we need a generalization of the usual Strassen class:

Let $\mathcal{A} \subset C[0,1]$ denote the set of functions $f(x), \quad 0 \le x \le 1, \quad f(0) = 0$ and absolutely continuous with respect to the Lebesgue measure. Define

$$\mathcal{D}^{(\gamma)} = \{f: \; f \in \mathcal{A}, \quad \int_0^1 |\dot{f}(x)|^\gamma \, dx \le 1\}. \tag{5.5}$$

Observe that $\mathcal{D}^2 = \mathcal{S}$.
We proved

Theorem 5.2. *Let W_1 and W_2 be two independent standard Wiener processes. Consider*

$$u_t(x) = \frac{W_1(\max_{0 \le s \le xt} W_2(s))}{2^{5/4} 3^{-3/4} t^{1/4} (\log \log t)^{3/4}}, \quad 0 \le x \le 1. \tag{5.6}$$

Then the set of functions $\{u_t(\cdot): \; 1 \le t < \infty\}$ is relatively compact in C[0,1] and the set of its limit points, as $t \to \infty$, is $\mathcal{D}^{(4/3)}$ a.s.

Clearly, this theorem states that the composite Strassen class in Theorem 5.1 and the direct Strassen class in Theorem 5.2 are equivalent. In fact the core of the proof of Theorem 5.2 is to show this equivalence which we established in a slightly more general form as follows. Put

$$\mathcal{F}^{(\beta)} = \left\{ f: \; f \in \mathcal{A}, \quad \int_0^1 |\dot{f}(x)|^{2\beta/(1+\beta)} \, dx \le \frac{\beta^{\beta/(1+\beta)}}{1+\beta} \right\} \tag{5.7}$$

and

$$\mathcal{G}^{(\beta)} = \left\{ f: \; f = g \circ h, \; g, h \in \mathcal{A}, \; h \text{ is nondecreasing and } \int_0^1 (|\dot{g}(x)|^2 + |\dot{h}(x)|^\beta) \, dx \le 1 \right\}.$$

Lemma 5.1. *For $\beta \ge 1$, the classes $\mathcal{F}^{(\beta)}$ and $\mathcal{G}^{(\beta)}$ are identical.*

From the many consequences of the above result I only want to mention the simplest one, namely that in Theorem 5.2 one can replace $\max_{0 \le s \le xt} W_2(s))$ with $L_2(xt)$, where $L_2(.)$ is the local time at zero of the process $W_2(.)$, which is based on the well-known equivalence in distribution of the maximum and the local time process, established by P. Lévy. This observation suggested that similar direct Strassen results can be proved for an iterated process where the inside process is replaced by the local time of a more general process like a Lévy process. In fact we went one step further in formulating a Strassen theorem for a class of stochastic processes satisfying two natural conditions. The first condition requires that the ordinary LIL should hold for certain linear combinations, while the second condition controls the increment behavior of the process.

Theorem 5.3. *Let $\{X(t), t \geq 0\}$ be a stochastic process with continuous sample paths and the following two properties:*
Property 1

$$\limsup_{t\to\infty} \frac{\sum_{i=1}^{d} c_i(X(it) - X((i-1)t))}{\chi(t)} = 1 \quad a.s., \tag{5.8}$$

$$\liminf_{t\to\infty} \frac{\sum_{i=1}^{d} c_i(X(it) - X((i-1)t))}{\chi(t)} = -1 \quad a.s. \tag{5.9}$$

with some $\chi(t) \nearrow \infty$, provided that $\sum_{i=1}^{d} |c_i|^q = 1,\ q > 1,\ d = 1, 2, ...$
Property 2 *For $0 < c \leq 1$*

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-cT} \sup_{0\leq s\leq cT} \frac{|X(t+s) - X(t)|}{\chi(T)} \leq A = A(c) \quad a.s. \tag{5.10}$$

where $\lim_{c\searrow 0} A(c) = 0$.

Let

$$\eta(x) = \eta_t(x) = \frac{X(xt)}{\chi(t)} \quad (0 \leq x \leq 1.) \tag{5.11}$$

Then the set

$$\{\eta_t(x), \quad 0 \leq x \leq 1\} \quad (t \to \infty) \tag{5.12}$$

is relatively compact in C[0,1] *and its set of limit points is $\mathcal{D}^{(p)}$ almost surely, where $1/p + 1/q = 1$.*

As an application of Theorem 5.3 we proved

Theorem 5.4. *Consider a symmetric Lévy process $\{Z(t),\ t \in R^+\}$ for which the conditions of Theorem B hold. Denote its local time process at zero by L_t. Let $Y(t) = W(L_t)$ where $W(\cdot)$ is a standard Wiener process, independent from $Z(\cdot)$ (and hence also from $L_\cdot$). The set of functions*

$$f_t(x) = \frac{Y(xt)}{G(t)}, \qquad 0 \leq x \leq 1, \tag{5.13}$$

with

$$G(t) = K_\beta \log\log t \left(\kappa\left(\frac{\log\log t}{t}\right)\right)^{1/2}, \tag{5.14}$$

$$K_\beta = \frac{2^{1/2}\beta}{(\beta+1)^{(\beta+1)/(2\beta)}(\beta-1)^{(\beta-1)/(2\beta)}} \tag{5.15}$$

is relatively compact in C[0,1] *and the set of its limit points, as* $t \to \infty$, *is* $\mathcal{D}^{(\frac{2\beta}{\beta+1})}$ *almost surely.*

This theorem was generalized in Eisenbaum and Földes ([**?**]) where, instead of the Lévy process and its local time, we considered symmetric stable processes and their additive functionals.

6. A strong invariance principle for the local time difference of a simple symmetric planar random walk.

This story started with Dobrushin's paper ([**?**]) in 1955. For us it began about 20 years ago with the couple of papers Csörgő and Révész ([**?**]) and Csáki and Földes ([**?**]), and resulted a long series of papers by various subgroups of us and others. For the interested reader a long list of references of the related papers is in Földes ([**?**]), where the whole topic was extensively discussed. The above mentioned two papers are dealing with the local time difference of Brownian motion and the simple symmetric walk on the line. Then the topic started to grow and the four of us together wrote two papers on the strong approximation of the Brownian local time by a Wiener process ([**?**]), and on the strong approximation of additive functionals ([**?**]). Our first venture into higher dimension was the paper in the title. All of these papers helped us to discover a general method of strong approximation for a pair of processes under very mild conditions. However this paper was a real eye opener. Let X_1, X_2,... be a sequence of i.i.d. r.v.-s with

$$P(X_1 = (0,1)) = P(X_1 = (0,-1)) = P(X_1 = (1,0)) = P(X_1 = (-1,0)) = \frac{1}{4}, \tag{6.1}$$

and let $S_0 = \mathbf{0}$, $S_n = X_1 + X_2 + ... + X_n$ $(n = 1,2,...)$ be a random walk on $\mathcal{Z}_2$ $(\mathbf{0} = (0,0))$. Its local time is defined by $\xi(\mathbf{a}, n) = \#\{k;\ 0 < k \le n,\ S_k = \mathbf{a}\}$, where $\mathbf{a} = (a_1, a_2)$ is a lattice point on the plane. Our main result was the following

Theorem 6.1. ([CsFR, 98]) *There is a probability space with*

- *a simple symmetric random walk process* S_n *with its two parameter local time* $\xi(\mathbf{a}, n)$, *a standard Wiener process* $\{W(t), t \ge 0\}$
- *and a process* $\{\xi^{(1)}(\mathbf{0}, n),\ n = 0,1,2,...\} \stackrel{\mathcal{D}}{=} \{\xi(\mathbf{0}, n),\ n = 0,1,2,...\}$
  such that for an arbitrary but fixed $\mathbf{a}$
- $\xi(\mathbf{a}, n) - \xi(\mathbf{0}, n) = \sigma_{\mathbf{a}} W(\xi^{(1)}(\mathbf{0}, n)) + O(\log n)^{\frac{2}{5}}$ a.s.

- $\xi(\mathbf{0},n) = \xi^{(1)}(\mathbf{0},n) + O(\log n)^{\frac{4}{5}}$ a.s., *as* $n \to \infty$,
- *where the processes* $\xi^{(1)}(\mathbf{0},n)$ *and* $\{W(t),\, t \geq 0\}$ *are independent*

and $\sigma_{\mathbf{a}}$ *is a constant, depending on* $\mathbf{a}$.

Let me mention here only one nice consequence of this theorem. Accordingly, the limit distribution of

$$\frac{\xi(\mathbf{a},n) - \xi(\mathbf{0},n)}{\sigma_a \sqrt{\log n}}$$

should be the same as the limit distribution of

$$\frac{W(\xi^{(1)}(n))}{\sqrt{\log n}}$$

where $\xi^{(1)}(n) = \xi^{(1)}(\mathbf{0},n)$. Namely, as $n \to \infty$,

$$\frac{W(\xi^{(1)}(n))}{\sqrt{\log n}} = \frac{W(\xi^{(1)}(n))}{\sqrt{\xi^{(1)}(n)}} \sqrt{\frac{\xi^{(1)}(n)}{\log n}} \xrightarrow{\mathcal{D}} U\sqrt{Z}.$$

Here U and Z are independent by the theorem and obviously U is a standard normal r.v. and we know from Erdős and Taylor ([**?**]) that Z is is an exponential r.v. with parameter π. From the various results which were motivated by this paper I will mention only the following

Theorem 6.2. ([**?**]) $(X_i, \tau_i)_{i=1}^{\infty}$ *i.i.d. with* $\tau_i \geq 0$ *and*

$$P(|X_i| > x) < \frac{c}{x^{\beta}}, \qquad P(\tau_i > x) \leq \frac{1}{h(x)}, \qquad x \geq 0$$

for x *large enough, where* $\beta > 2$, $c > 0$ *and* $h(x)$ *is slowly varying at infinity, increasing, and* $\lim_{x\to\infty} h(x) = +\infty$.

Then on an appropriate probability space one can construct

- *two independent copies* $\left(X_i^{(j)}, \tau_i^{(j)}\right)_{i=1}^{\infty}$ $j = 1,2$ *with* $(X_i, \tau_i)_{i=1}^{\infty}$ $j = 1,2$ *such that*
- $(S_n, \rho_n) \stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)})$, $j = 1,2$,
- $\sup_{k \leq n} |S_k - S_k^{(2)}| = O\left(n^{\frac{1}{\beta^*}}\right)$ a.s.

- $\sup_{k\le n}|\rho_k-\rho_k^{(1)}|=O\left(h^*(n^\alpha)\right)$ a.s.,

as $n\to\infty$, where

$$S_k^{(j)}=\sum_{i=1}^k X_i^{(j)},\quad \rho_k^{(j)}=\sum_{i=1}^k \tau_i^{(j)},\qquad S_k=\sum_{i=1}^k X_i,\quad \rho_k=\sum_{i=1}^k \tau_i,$$

$$\alpha<1,\quad \beta^*>2,$$

and $h^*(.)$ is the inverse of $h(.)$.

A similar, even simpler theorem holds when the tail of τ_i is regularly varying. It is still an open question whether the tail conditions in these theorems can be weakened.

7. Maximal Local Time of a d-dimensional Simple Random Walk on Subsets

This is the most recent one of our papers, and as such a favorite by definition. However I think it is dealing with simple but very interesting problems and left open a few intriguing questions.

Consider a simple symmetric random walk $\{\mathbf{S}_n\}_{n=1}^\infty$ starting at the origin $\mathbf{0}$ on the d-dimensional integer lattice $\mathcal{Z}_d$, i.e., $\mathbf{S}_0=\mathbf{0}$, $\mathbf{S}_n=\sum_{k=1}^n\mathbf{X}_k$, $n=1,2,\ldots$, where $\mathbf{X}_k$, $k=1,2,\ldots$ are i.i.d. random variables with distribution

$$\mathbf{P}(\mathbf{X}_1=\mathbf{e}_i)=\mathbf{P}(\mathbf{X}_1=-\mathbf{e}_i)=\frac{1}{2d},\qquad i=1,2,...,d,$$

and $\{\mathbf{e}_1,\mathbf{e}_2,...\mathbf{e}_d\}$ is a system of orthogonal unit vectors in $\mathcal{Z}_d$. We define the local time of the walk as

$$\xi^{(d)}(\mathbf{x},n):=\#\{k:\ 0<k\le n,\ \mathbf{S}_k=\mathbf{x}\}.$$

where $\mathbf{x}$ is any lattice point of $\mathcal{Z}_d$. The maximal local time is defined as

$$\xi^{(d)}(n):=\max_{\mathbf{x}\in\mathcal{Z}_d}\xi^{(d)}(\mathbf{x},n).$$

The nature of the two dimensional and that of the higher dimensional results are rather different, as usual, so we have to discuss them separately.

Two dimensions.

According to a well-known integral test of Erdős and Taylor ([**?**]) the local time of every particular lattice point of the plane is roughly $\log n$. On the other hand they also proved that

$$\frac{1}{4\pi} \leq \liminf_{n\to\infty} \frac{\xi^{(2)}(n)}{(\log n)^2} \leq \limsup_{n\to\infty} \frac{\xi^{(2)}(n)}{(\log n)^2} \leq \frac{1}{\pi} \qquad \text{a.s.}$$

They conjectured that the upper bound in the above theorem is the correct limit. This conjecture was not only confirmed but greatly generalized by Dembo, Peres, Rosen and Zeitouni ([**?**]) for aperiodic random walks with i.i.d. steps having finite moments of any order. The comparison of these two results suggest that taking the maximal local time over some appropriate subsets we might get orders in between $\log n$ an $(\log n)^2$. So our goal was to try to find the order of the maximal local time

$$\xi_A^{(2)}(n) := \max_{\mathbf{x}\in A} \xi^{(2)}(\mathbf{x}, n).$$

where A is a subset of $\mathcal{Z}_2$. According to a theorem of Auer ([**?**]), roughly speaking every point within a circle around the origin with radius

$$r_n = \exp((\log n)^{1/2}(\log\log n)^{-1/2-\epsilon}) \tag{7.2}$$

has the same local time. This suggests that getting higher order than $\log n$ the set A has to be rather big. We restricted our investigation to sets of two types: lines going through the origin and discs centered at the origin.

Let $B(r)$ denote the set of lattice points in the disc of radius r centered at the origin, i.e.,

$$B(r) := \{\mathbf{x} \in \mathcal{Z}_2 : ||\mathbf{x}|| \leq r\}.$$

Denote by $L = L(a_1, a_2)$ the lattice points $\mathbf{x} = (x_1, x_2)$ on the line $a_1x_1 + a_2x_2 = 0$, where a_1 and a_2 are integers, not both of them are zero.

Theorem 7.1. *For any line $L = L(a_1, a_2)$ such that a_1, a_2 are integers, not both of them are zero, we have*

$$\frac{1}{8\pi} \leq \liminf_{n\to\infty} \frac{\xi_L^{(2)}(n)}{(\log n)^2} \leq \limsup_{n\to\infty} \frac{\xi_L^{(2)}(n)}{(\log n)^2} \leq \frac{1}{2\pi} \qquad \text{a.s.}$$

Theorem 7.2. *Let* $r_n = n^\alpha, \quad 0 < \alpha \le 1/2$. *Then*

$$\frac{4\alpha^2}{\pi} \le \liminf_{n\to\infty} \frac{\xi^{(2)}_{B(r_n)}(n)}{(\log n)^2} \le \limsup_{n\to\infty} \frac{\xi^{(2)}_{B(r_n)}(n)}{(\log n)^2} \le \frac{2\alpha}{\pi} \qquad \text{a.s.}$$

The above two result show that the maximal local time on a line and on a disc (of radius $n^\alpha, \quad 0 < \alpha \le 1/2$.) has the same order of magnitude as on the whole plane. However the next result shows that, considering smaller but not very small discs (having a radius bigger than in (**??**)), we get all the orders from $\log n$ to $(\log n)^2$.

Theorem 7.3. *Let* $r_n = \exp((\log n)^\beta)$. *For any* $\epsilon > 0, \quad 1/2 \le \beta < 1$, *and large enough* n *we have*

$$\frac{4(1-\epsilon)}{\pi}(\log n)^{2\beta} \le \xi^{(2)}_{B(r_n)}(n) \le (\log n)^{2\beta+\epsilon} \qquad \text{a.s.}$$

Three and higher dimensions.
Just like in two dimensions, for a subset $A \subseteq \mathcal{Z}_d$ we define $\xi^{(d)}_A(n) := \max_{\mathbf{x}\in A} \xi^{(d)}(\mathbf{x}, n)$. It is known from the landmark paper of Erdős and Taylor ([**?**]) that, for $d \ge 3$, $\xi^{(d)}(\mathbf{0}, \infty)$, the total local time at $\mathbf{0}$ of the infinite path in $\mathcal{Z}_d$ has geometric distribution:

$$\mathbf{P}(\xi^{(d)}(\mathbf{0}, \infty) = k) = \gamma_d(1-\gamma_d)^k, \qquad k = 0, 1, 2, ...$$

where γ_d is the probability that the d-dimensional simple symmetric random walk never returns to its starting point. They also proved, that

$$\lim_{n\to\infty} \frac{\xi^{(d)}(n)}{\log n} = \lambda_d \qquad \text{a.s.,}$$

where $\lambda_d = -\frac{1}{\log(1-\gamma_d)}$. Thus we might ask again about the transition of the order of the maximal local time on different subsets. How does it change from being finite to attain the order $\log n$. Let $B(r)$ stand for the (discrete) ball, centered at the origin in the d-dimensional space with radius r, i.e.,

$B(r) := \{\mathbf{x} \in \mathcal{Z}_d : \ ||\mathbf{x}|| \le r\}$. Let furthermore $\mathbf{x} = (x_1, x_2, \ldots, x_d)$,

$$S_{d-1} := \{\mathbf{x} \in \mathcal{Z}_d : \ a_1x_1 + a_2x_2 + ... + a_dx_d = 0\}$$

and

$$S_{d-2} := \{\mathbf{x} \in \mathcal{Z}_d : \ a_1x_1 + a_2x_2 + ... + a_dx_d = 0, \quad b_1x_1 + b_2x_2 + ... + b_dx_d = 0\}$$

with integer coefficients $a_1, a_2, ...a_d, b_1, b_2, ...b_d$. Our results for subspaces are as follows.

Theorem 7.4. *Suppose that $a_1, a_2, ...a_d$ are integers, not all of them are zero, then*

$$\lim_{n\to\infty} \frac{\xi^{(d)}_{S_{d-1}}(n)}{\log n} = \frac{\lambda_d}{2} \qquad \text{a.s.}$$

Theorem 7.5. *Suppose that $a_1, a_2, ...a_d$ are integers, not all of them are zero and $b_1, b_2, ...b_d$ are also integers not all of them are zero. Assume also that the vectors $(a_1, a_2, \ldots, a_d)$ and $(b_1, b_2, \ldots, b_d)$ are not parallel. Then*

$$\lim_{n\to\infty} \frac{\xi^{(d)}_{S_{d-2}}(n)}{\log\log n} = \lambda_d \qquad \text{a.s.}$$

These results are roughly saying that taking the maximum on a subspace with one less dimension won't change the order but will change the constant. However, taking the maximum on a subspace of dimension $d-2$, the order will drop from $\log n$ to $\log\log n$.
Our last result explains how does the maximal local time, taken over a ball of radius r_n, grow in terms of r_n.

Theorem 7.6. *For any sequence $r_n \uparrow \infty$, such that*

$$\limsup_{n\to\infty} (\log r_n)/(\log n) \leq 1/2,$$

we have

$$\lim_{n\to\infty} \frac{\xi^{(d)}_{B(r_n)}(n)}{\log r_n} = 2\lambda_d \qquad \text{a.s.}$$

As it is seen from these results, there are many more natural but probably hard questions open in this area. I just want to mention a few of these. The theorems for the two-dimensional case might be sharpened. Results for other type of sets than subspaces and balls are missing. Furthermore one would like to see order transitions from, e.g, lines, subspaces to angular domains, cones and wedges. One would like to know how is the picture changing when the set A is moving away from the origin as times goes on. What is the role of the shape and the size of the set A in these results?

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