

# On a class of additive functionals of two-dimensional Brownian motion and random walk

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*Abstract:* Sample path properties of a class of additive functionals of two-dimensional Brownian motion and random walk are studied.

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# 1. Introduction and main results

Let  $\{W(t), t \geq 0\}$  be a standard Wiener process (Brownian motion) with local time  $L(x, t)$  and for  $\alpha < 3/2$  define

$$Y_\alpha(t) := \int_0^t \frac{ds}{W^\alpha(s)} = \int_0^\infty \frac{L(x, t) - L(-x, t)}{x^\alpha} dx.$$

Here  $z^\alpha = |z|^\alpha \operatorname{sgn}(z)$  and the first integral is defined as Cauchy principal value for  $1 \leq \alpha < 3/2$ . In our previous works [CsCsFS, 99a and 99b], we were interested in the almost sure path properties of  $Y_\alpha(t)$  and its corresponding counterparts for the case of simple symmetric random walk. The general case  $0 < \alpha < 3/2$  was studied in [CsCsFS, 99a], and the case  $\alpha = 1$  in [CsCsFS, 99b]. Moreover, in [CsCsFS, 99a], for  $\alpha > 3/2$ , we investigated the properties of the modified process

$$Y_\alpha^*(t) := \int_0^t \frac{1}{W^\alpha(s)} \mathbf{1}_{\{|W(s)| \geq 1\}} ds.$$

In this note we discuss the corresponding two-dimensional problem.

Let  $\{\mathbf{W}(t) := (W_1(t), W_2(t)), t \geq 0\}$  be a two-dimensional Wiener process, where  $W_1(t)$  and  $W_2(t)$  are two independent one-dimensional Wiener processes, with  $W_1(0) = W_2(0) = 0$ . Put

$$R(t) := \|\mathbf{W}(t)\| = \sqrt{W_1^2(t) + W_2^2(t)}.$$

It is well-known that  $\{R(t), t > 0\}$  is a two-dimensional Bessel process. We are interested in the additive functional

$$(1.1) \quad Z_\alpha(t) := \int_0^t \frac{ds}{R^\alpha(s)},$$

the critical case being  $\alpha = 2$  (instead of  $3/2$ ). It can be seen that the integral in (1.1) converges for  $\alpha < 2$ , but diverges for  $\alpha \geq 2$  almost surely. In the latter case we define the modified process

$$(1.2) \quad Z_\alpha^*(t) := \int_0^t \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s) \geq 1\}} ds.$$

Considering the random walk counterpart, let  $\{\mathbf{S}_n\}_{n=1}^\infty$  be a simple symmetric random walk on the integer lattice  $\mathbf{Z}^2$ , i.e.  $\mathbf{S}_n = \sum_{k=1}^n \mathbf{X}_k$ , where the random variables  $\mathbf{X}_i$ ,  $i = 1, 2, \dots$  are i.i.d., with

$$\mathbf{P}(\mathbf{X}_1 = (0, 1)) = \mathbf{P}(\mathbf{X}_1 = (0, -1)) = \mathbf{P}(\mathbf{X}_1 = (1, 0)) = \mathbf{P}(\mathbf{X}_1 = (-1, 0)) = \frac{1}{4}.$$

We also propose to study the discrete process

$$(1.3) \quad U_\alpha(n) := \sum_{k=1}^n \frac{1}{\|\mathbf{S}_k\|^\alpha} \mathbf{1}_{\{\mathbf{S}_k \neq \mathbf{0}\}}.$$

Define

$$\xi(\mathbf{x}, n) := \#\{k; 0 < k \leq n, \mathbf{S}_k = \mathbf{x}\},$$

for any lattice point  $\mathbf{x}$  of  $\mathbf{Z}^2$ . This is the local time process of  $\{\mathbf{S}_n\}_{n=1}^\infty$ . Let furthermore  $\{\rho_n, n \geq 0\}$  denote the consecutive return times of the walk to zero, that is

$$\rho_0 := 0, \quad \rho_n := \min\{k > \rho_{n-1}, \mathbf{S}_k = \mathbf{0}\}.$$

Our first result concerns the case  $0 < \alpha < 2$ , for which we show that the processes  $Z_\alpha(\cdot)$  and  $U_\alpha(\cdot)$  are close to each other.

**Theorem 1.1.** *Let  $0 < \alpha < 2$ . Then on a probability space one can define a two-dimensional Wiener process  $\mathbf{W}(\cdot)$  with the additive functional  $Z_\alpha(\cdot)$  as in (1.1) and a two-dimensional simple symmetric random walk  $\mathbf{S}$  with the additive functional  $U_\alpha(\cdot)$  as in (1.3) such that, as  $n \rightarrow \infty$ , for any  $\varepsilon > 0$*

$$(1.4) \quad |2^{-\alpha/2}U_\alpha(n) - Z_\alpha(n)| = \begin{cases} \mathcal{O}(n^{1-\alpha+\varepsilon}), & \text{a.s.} & \text{if } 0 < \alpha \leq 1, \\ \mathcal{O}(\log^{2+\varepsilon} n), & \text{a.s.} & \text{if } 1 < \alpha < 2. \end{cases}$$

We note that by self-similarity,  $Z_\alpha(t)$  is of order  $t^{1-\alpha/2}$ , so the rates in (1.4) are useful.

In the case when  $\alpha > 2$ , the corresponding processes,  $Z_\alpha^*$  and  $U_\alpha$  are certainly not close to each other. Instead, as in the one-dimensional case, we show separately, that both processes, suitably centered, are close to certain iterated processes.

**Theorem 1.2.** *Let  $\alpha > 2$ . There exists a probability space where one can define*

- (i) *a two-dimensional simple symmetric random walk  $\{\mathbf{S}_n\}_0^\infty$  with its local time process  $\xi(\mathbf{x}, n)$ , and with the corresponding additive functional process  $\{U_\alpha(n), n = 1, 2, \dots\}$  as in (1.3);*
- (ii) *a process  $\{\xi^{(1)}(\mathbf{0}, n), n = 0, 1, 2, \dots\} \stackrel{\mathcal{D}}{=} \{\xi(\mathbf{0}, n), n = 0, 1, 2, \dots\}$ ;*
- (iii) *a standard Wiener process  $\{W(t), t \geq 0\}$ , independent of  $\{\xi^{(1)}(\mathbf{0}, n), n = 0, 1, 2, \dots\}$ ;*

such that, for some  $\varepsilon > 0$ , as  $n \rightarrow \infty$ ,

$$U_\alpha(n) - \bar{f}_\alpha^D \xi(\mathbf{0}, n) = \sigma_\alpha^D W(\xi^{(1)}(\mathbf{0}, n)) + \mathcal{O}(\log^{1/2-\varepsilon} n), \quad \text{a.s.},$$

where

$$\bar{f}_\alpha^D := \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{1}{\|\mathbf{x}\|^\alpha}, \quad \sigma_\alpha^D := \sqrt{\text{Var}(U_\alpha(\rho_1))}.$$

Moreover, the processes  $\xi(\mathbf{0}, n)$  and  $\xi^{(1)}(\mathbf{0}, n)$  can be chosen such that for some  $\delta > 0$ ,

$$\xi(\mathbf{0}, n) = \xi^{(1)}(\mathbf{0}, n) + \mathcal{O}(\log^{1-\delta} n), \quad \text{a.s.}$$

We would like to prove a similar result for the corresponding problem of a two-dimensional Wiener process. Formulating a claim in this case needs some careful consideration as points are polar for the two-dimensional Wiener process, so that it has no local time. However its norm  $R(t)$  does have a local time which could perfectly fit in our theorem. For technical reasons we introduce a slightly different definition as our modified local time for  $R(t)$ . This idea is not new at all, similar but more general concepts were already discussed e.g. in Burdzy *et al.* [BPY, 90]. Define two increasing sequences of stopping times  $\{\sigma_n, n \geq 0\}$  and  $\{\theta_n, n \geq 1\}$  by

$$(1.5) \quad \sigma_0 := \inf\{t > 0; R(t) = 1\},$$

$$(1.6) \quad \theta_n := \inf\{t > \sigma_{n-1}; R(t) = 2\}, \quad n \geq 1,$$

$$(1.7) \quad \sigma_n := \inf\{t > \theta_n; R(t) = 1\}, \quad n \geq 1,$$

$$(1.8) \quad \tau_i := \sigma_i - \sigma_{i-1}, \quad i = 1, 2, \dots$$

Let furthermore

$$(1.9) \quad \eta(t) = \max\{n; \sigma_n - \sigma_0 \leq t\}$$

**Theorem 1.3.** *Let  $\alpha > 2$ . There exists a probability space where one can define*

- (i) *a two-dimensional Wiener process  $\{\mathbf{W}(t), t \geq 0\}$  with its corresponding additive functional process  $\{Z_\alpha^*(t), t \geq 0\}$  as in (1.2), and with  $\{\eta(t), t > 0\}$  as in (1.9);*
- (ii) *a process  $\{\eta^{(1)}(t), t > 0\} \stackrel{\mathcal{D}}{=} \{\eta(t), t > 0\}$ ;*
- (iii) *a standard Wiener process  $\{W(t), t \geq 0\}$ , independent of  $\{\eta^{(1)}(t), t \geq 0\}$ ;*

such that for some  $\varepsilon > 0$ ,

$$(1.10) \quad Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta^{(1)}(t)) + \mathcal{O}(\log^{1/2-\varepsilon} t), \quad \text{a.s.},$$

where

$$\bar{f}_\alpha^W := \frac{2}{(\alpha - 2)^2}, \quad (\sigma_\alpha^W)^2 := \mathbf{E} \left( \int_1^\infty \frac{1}{x^\alpha} \ell(x, \tau_1) dx \right)^2 - (\bar{f}_\alpha^W)^2,$$

and  $\ell(x, t)$  is the local time at time  $t$  and position  $x$  of  $\{R(t), t \geq 0\}$ . Furthermore,  $\eta$  and  $\eta^{(1)}$  can be chosen such that for some  $\delta > 0$ ,

$$(1.11) \quad \eta(t) = \eta^{(1)}(t) + \mathcal{O}(\log^{1-\delta} t), \quad \text{a.s.}$$

The rest of the paper is organized as follows. Section 2 is devoted to a collection of known results which will be used later on. Theorems 1.1 – 1.3 are proved in Sections 3–5 respectively. In Section 6 we present some further weak and strong convergence results, as consequences of our theorems.

## 2. Preliminary results

In this section we list some results which will be used in the proofs of the theorems.

Let  $\{\mathbf{S}_n, n \geq 0\}$  be a simple symmetric random walk on the plane, with  $\mathbf{S}_0 = \mathbf{0}$ , and let  $\{\rho_n, n \geq 0\}$  denote, as before, the consecutive return times to zero.

For any real valued function  $f(\cdot)$  on  $\mathbf{Z}^2$  we define the additive functional

$$A(f, n) = \sum_{k=0}^n f(\mathbf{S}_k).$$

We assume that for some  $\delta > 0$ ,

$$(2.1) \quad \mathbf{E} \left( \left| \sum_{k=1}^{\rho_1} f(\mathbf{S}_k) \right|^{2+\delta} \right) < \infty.$$

It is easy to compute the expectation of  $\sum_{k=1}^{\rho_1} f(\mathbf{S}_k)$ . Indeed, by Spitzer [S, 64],  $\mathbf{E}(\xi(\mathbf{x}, \rho_1)) = 1$  for any  $\mathbf{x} \in \mathbf{Z}^2$ , and hence we have

$$\mathbf{E} \left( \sum_{k=1}^{\rho_1} f(\mathbf{S}_k) \right) = \mathbf{E} \sum_{\mathbf{x} \in \mathbf{Z}^2} \xi(\mathbf{x}, \rho_1) f(\mathbf{x}) = \sum_{\mathbf{x} \in \mathbf{Z}^2} f(\mathbf{x}) =: \bar{f}.$$

Now we recall a few known results that will be of use later on.

**Theorem A** (Csáki and Földes [CsF, 99]). *Let  $(X_i, \tau_i)_{i=1}^\infty$  be a sequence of i.i.d. vectors, such that  $\tau_i \geq 0$  and*

$$(2.2) \quad \mathbf{P}(|X_i| > x) < \frac{c}{x^\beta}, \quad \mathbf{P}(\tau_i > x) \leq \frac{1}{h(x)}$$

*for  $x$  large enough, where  $\beta > 2$ ,  $c > 0$  and  $h(x)$  is slowly varying at infinity, increasing with  $\lim_{x \rightarrow \infty} h(x) = +\infty$ . Then on an appropriate probability space one can construct two independent copies  $(X_i^{(1)}, \tau_i^{(1)})_{i=1}^\infty$  and  $(X_i^{(2)}, \tau_i^{(2)})_{i=1}^\infty$  together with  $(X_i, \tau_i)_{i=1}^\infty$  such that*

$$\begin{aligned} (S_n, \rho_n)_{n=1}^\infty &\stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)})_{n=1}^\infty, \quad j = 1, 2, \\ \sup_{k \leq n} |S_k - S_k^{(2)}| &= \mathcal{O}(n^{1/\beta^*}), \quad \text{a.s.}, \\ \sup_{k \leq n} |\rho_k - \rho_k^{(1)}| &= \mathcal{O}(h^*(n^\gamma)), \quad \text{a.s.}, \end{aligned}$$

*as  $n \rightarrow \infty$ , where  $S_k^{(j)} = \sum_{i=1}^k X_i^{(j)}$ ,  $\rho_k^{(j)} = \sum_{i=1}^k \tau_i^{(j)}$ ,  $S_k = \sum_{i=1}^k X_i$ ,  $\rho_k = \sum_{i=1}^k \tau_i$ ,  $\gamma < 1$ ,  $\beta^* > 2$ , and  $h^*(\cdot)$  is the inverse of  $h(\cdot)$ .*

The next result is an application of Theorem A for simple symmetric random walk in  $\mathbf{Z}^2$ .

**Theorem B** (Csáki and Földes [CsF, 99]). *For a simple symmetric random walk  $\{\mathbf{S}_n\}_0^\infty$  in  $\mathbf{Z}^2$  and any real valued function  $f(\cdot)$  for which (2.1) holds, there exists a probability space where we can redefine  $\{\mathbf{S}_n\}_0^\infty$  together with its local time process  $\xi(\mathbf{0}, n)$  and with the corresponding additive functional  $A(f, n)$  in such a way, that on the same probability space there exist*

(i) *a standard Wiener process  $\{W(t), t \geq 0\}$ ,*

(ii) *and a process*

$$\{\xi^{(1)}(\mathbf{0}, n), n = 0, 1, 2, \dots\} \stackrel{\mathcal{D}}{=} \{\xi(\mathbf{0}, n), n = 0, 1, 2, \dots\}$$

*such that  $\{W(t), t \geq 0\}$  and  $\{\xi^{(1)}(\mathbf{0}, n), n = 0, 1, 2, \dots\}$  are independent and, as  $t \rightarrow \infty$ , we have*

$$A(f, n) - \bar{f}\xi(\mathbf{0}, n) = \sigma W(\xi^{(1)}(\mathbf{0}, n)) + \mathcal{O}(\log^\kappa n), \quad \text{a.s.},$$

*and*

$$|\xi^{(1)}(\mathbf{0}, n) - \xi(\mathbf{0}, n)| = \mathcal{O}(\log^{1-\delta} n), \quad \text{a.s.},$$

*where  $\sigma = \sqrt{\text{Var}(\sum_{k=1}^{\rho_1} f(\mathbf{S}_k))}$ ,  $\kappa < 1/2$ , and  $\delta > 0$ .*

Concerning the moments of the local time of  $\xi(\mathbf{x}, \rho_1)$  we quote the following result.

**Theorem C** (Révész [R, 90]). *For any integer  $m \geq 1$ ,*

$$|\mathbf{E}(\xi(\mathbf{x}, \rho_1) - 1)^m| \leq q + m!p^{1-m}q,$$

where

$$p := \mathbf{P}(\mathbf{0} \rightsquigarrow \mathbf{x}) = \mathbf{P}(\{\mathbf{S}_n\} \text{ hits } \mathbf{x} \text{ before returning to } \mathbf{0}),$$

and  $q = 1 - p$ .

The order of magnitude of  $p$  is well-known, namely we have

**Theorem D** (Spitzer [S, 64], pp. 117, 124, 125, and Révész, [R, 90] p. 219). *As  $\|\mathbf{x}\|$  goes to infinity,*

$$\mathbf{P}(\mathbf{0} \rightsquigarrow \mathbf{x}) = \frac{\pi + o(1)}{4 \log \|\mathbf{x}\|}.$$

The next theorem concerns the asymptotics of  $\xi(\mathbf{0}, n)$ .

**Theorem E** (Dvoretzky and Erdős [DE, 51], Erdős and Taylor [ET, 60]).

$$\lim_{n \rightarrow \infty} \mathbf{P}(\xi(\mathbf{0}, n) < x \log n) = 1 - e^{-\pi x}.$$

We also have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\xi(\mathbf{0}, n)}{\log n \log_3 n} &= \frac{1}{\pi}, \quad \text{a.s.} \\ \limsup_{n \rightarrow \infty} \frac{\sup_{\mathbf{x}} \xi(\mathbf{x}, n)}{(\log n)^2} &\leq \frac{1}{\pi}, \quad \text{a.s.} \end{aligned}$$

Furthermore, as  $n$  goes to infinity,

$$\mathbf{P}(\rho_1 > n) = \mathbf{P}(\xi(\mathbf{0}, n) = 0) = \frac{\pi}{\log n} + \mathcal{O}((\log n)^{-2}).$$

Now we collect certain results for the two-dimensional Bessel process  $\{R(t), t \geq 0\}$  (with  $R(0) = 0$ ). Let  $\{\sigma_n, n \geq 0\}$  and  $\{\theta_n, n \geq 1\}$  be as in (1.3)–(1.5). It is a consequence of the strong Markov property that  $\{\sigma_i - \sigma_{i-1}\}_1^\infty$  is an i.i.d. sequence of random variables, for which we quote the following result.

**Theorem F** (Csáki et al. [CsFRS, 98]). *We have,*

$$\mathbf{P}(\sigma_1 - \sigma_0 > x) = \frac{2 \log 2}{\log x} + \mathcal{O}(\log^{-2} x),$$

as  $x \rightarrow \infty$ .

We also need the following iterated logarithm law.

**Theorem G** (Csáki et al. [CsFRS, 98]). *Let  $\{\eta(t), t > 0\}$  be as in (1.9). Then*

$$(2.3) \quad \limsup_{t \rightarrow \infty} \frac{\eta(t)}{\log t \log_3 t} = \frac{1}{2 \log 2}, \quad \text{a.s.}$$

Let us recall now the following theorem concerning the increments of  $n \mapsto \xi(\mathbf{0}, n)$ .

**Theorem H** (Csáki, Földes and Révész [CsFR, 98]). *Let  $a_n = \exp((\log n)^K)$  and  $b_n = \exp((\log n)^b)$  for some constants  $K > 0$  and  $b > 0$ . Then for any  $\varepsilon > 0$ ,*

$$\sup_{a \leq a_n} (\xi(\mathbf{0}, a + b_n) - \xi(\mathbf{0}, a)) = \mathcal{O}(\log^{b+\varepsilon} n), \quad \text{a.s.}$$

Finally, we will need the following strong approximation theorem for the two-dimensional random walk and a Wiener process;

**Theorem J** (Révész [R, 90]). *On a rich enough probability space one can define a Wiener process  $\mathbf{W}(\cdot) \in \mathbf{R}^2$  and a simple symmetric random walk  $\mathbf{S}_n \in \mathbf{Z}^2$  such that*

$$\|\sqrt{2}\mathbf{S}_n - \mathbf{W}(n)\| = \mathcal{O}(\log n), \quad \text{a.s.}$$

### 3. Proof of Theorem 1.1

We begin this section with a few lemmas and facts which will be needed in the proof.

**Fact 1.** *For  $x \geq 1, y \geq 1$*

$$(3.1) \quad |x^\alpha - y^\alpha| \leq \alpha|x - y|(x^{\alpha-1} + y^{\alpha-1}) \quad \text{if } \alpha > 1$$



and

$$(3.2) \quad |x^\alpha - y^\alpha| \leq |x - y| \quad \text{if } 0 < \alpha \leq 1.$$

Fact 1 is a simple consequence of the mean value theorem.

**Fact 2.** For  $x \geq 1$ ,  $y \geq 1$ , and  $\alpha \geq 0$  we have

$$(3.3) \quad \frac{1}{xy^\alpha} + \frac{1}{yx^\alpha} \leq 2 \left( \frac{1}{x^{\alpha+1}} + \frac{1}{y^{\alpha+1}} \right).$$

Fact 2 can be checked by elementary computation.

Recall that  $\{R(t), t \geq 0\}$  is a two-dimensional Bessel process and  $\{\ell(x, t), x \geq 0, t \geq 0\}$  is its local time. First we give a rough estimate for  $Z_\alpha(t)$  for large  $t$ .

**Lemma 3.1.** For  $0 < \alpha < 2$ , and any  $\varepsilon > 0$ , as  $t \rightarrow \infty$ ,

$$(3.4) \quad Z_\alpha(t) := \int_0^t \frac{ds}{R^\alpha(s)} = \mathcal{O}\left(t^{1-\alpha/2+\varepsilon}\right), \quad \text{a.s.}$$

**Proof:** By the well-known scale change property of  $R(\cdot)$

$$(3.5) \quad R(ut) \stackrel{\mathcal{D}}{=} \sqrt{t}R(u)$$

which implies by simple substitution that

$$(3.6) \quad Z_\alpha(t) \stackrel{\mathcal{D}}{=} t^{1-\alpha/2} \int_0^1 \frac{du}{R^\alpha(u)}.$$

Observe furthermore that  $R^2(u)/u$  is exponentially distributed, thus we have

$$(3.7) \quad \mathbf{E} \left( \int_0^1 \frac{1}{R^\alpha(u)} du \right) = c_1 \int_0^\infty \frac{1}{z^{\alpha/2}} e^{-z} dz = c_2,$$

where  $c_1$  and  $c_2$  are finite constants. Hence by Markov's inequality, we get for any  $A > 0$  that

$$(3.8) \quad \mathbf{P} \left( Z_\alpha(t) > t^{1-\alpha/2} A \right) \leq \frac{c_2}{A}.$$

For any given  $\varepsilon > 0$ , let  $t_k = k^\beta$  such that  $\beta\varepsilon > 1$  should hold. From (3.8) we get that

$$(3.9) \quad \mathbf{P} \left( Z_\alpha(t_{k+1}) > t_k^{1-\alpha/2+\varepsilon} \right) \leq \frac{2c_2}{k^{\beta\varepsilon}}$$

for  $k$  large enough, as  $t_{k+1}/t_k \rightarrow 1$ . Using Borel–Cantelli lemma and the monotonicity of  $Z_\alpha(t)$  we obtain that

$$(3.10) \quad Z_\alpha(t) = \mathcal{O}\left(t^{1-\alpha/2+\varepsilon}\right), \quad \text{a.s.},$$

completing the proof of Lemma 3.1. □

**Lemma 3.2.** *For any  $\gamma > 2$ , and any  $0 \leq \alpha < 2$ , as  $t \rightarrow \infty$ ,*

$$(3.11) \quad Q(t) := \int_0^1 \frac{\ell(x, t)}{x^\alpha} dx = \mathcal{O}(\log^\gamma t), \quad \text{a.s.}$$

**Proof:** Using the notations introduced in (1.5)–(1.8) we can decompose the local time  $\ell(x, t)$  of  $R(t)$  as follows:

$$(3.12) \quad \begin{aligned} \ell(x, t) &= \ell(x, \sigma_0) + \sum_{i=1}^{\eta(t)} (\ell(x, \sigma_i) - \ell(x, \sigma_{i-1})) + \ell(x, t) - \ell(x, \sigma_{\eta(t)}) \\ &\leq \ell(x, \sigma_0) + \sum_{i=1}^{\eta(t)+1} (\ell(x, \sigma_i) - \ell(x, \sigma_{i-1})). \end{aligned}$$

It follows from Borodin and Salminen ([BS, 96], Formula 2.3.1 on p. 297) that

$$(3.13) \quad \mathbf{E}_1 \left( e^{-\gamma \ell(x, \sigma_1 - \sigma_0)} \right) = \begin{cases} \frac{1+2\gamma x |\log x|}{1+2\gamma x |\log x/2|} & \text{if } 0 < x \leq 1 \\ \frac{1}{(1+2\gamma x |\log x|)(1+2\gamma x |\log x/2|)} & \text{if } 1 \leq x < 2 \\ \frac{1+2\gamma r |\log r/2|}{1+2\gamma r |\log r|} & \text{if } x \geq 2, \end{cases}$$

where  $\mathbf{E}_1$  denotes expectation under the condition  $R(0) = 1$ . Hence for  $0 < x \leq 1$

$$(3.14) \quad \mathbf{E}(\ell(x, \sigma_i) - \ell(x, \sigma_{i-1})) = \mathbf{E}_1(\ell(x, \sigma_i - \sigma_{i-1})) = 2x \log 2.$$

Similarly using again the same formula of Borodin and Salminen [BS, 96], we obtain for  $0 < x \leq 1$

$$(3.15) \quad \mathbf{E} \left( e^{-\gamma \ell(x, \sigma_0)} \right) = \frac{1}{1 + 2\gamma x |\log x|}.$$

implying that

$$(3.16) \quad \mathbf{E}(\ell(x, \sigma_0)) = 2x \log(1/x).$$

Based on (3.14), (3.16) and Theorem G, we get that for any  $\delta > 0$ , as  $t \rightarrow \infty$ ,

$$(3.17) \quad \mathbf{E}(Q(t)) \leq \int_0^1 \frac{x \log(1/x)}{x^\alpha} dx + (\log^{1+\delta} t) \int_0^1 \frac{x \log 2}{x^\alpha} dx = \mathcal{O}(\log^{1+\delta} t)$$

for all  $0 \leq \alpha < 2$  and any  $\delta > 0$ . Based on (3.17), we can finish the proof in the same way as that of Lemma 3.1, using the monotonicity of  $Q(t)$  in  $t$ , the subsequence  $t_k = e^k$  and Borel–Cantelli lemma. Namely, by (3.17) and Markov’s inequality we have

$$(3.18) \quad \mathbf{P}(Q(t_{k+1}) > (\log t_k)^\gamma) \leq c_3 \frac{(\log t_{k+1})^{1+\delta}}{(\log t_k)^\gamma} \leq c_4 \frac{(k+1)^{1+\delta}}{k^\gamma},$$

whose sum in  $k$  being convergent, whenever  $2 + \delta < \gamma$ . This can be achieved by selecting  $\delta$  small enough for any given  $\gamma > 2$ .  $\square$

Lemma 3.1 and Lemma 3.2 imply

**Lemma 3.3.** *For  $0 \leq \alpha < 2$ , and any  $\varepsilon > 0$*

$$(3.19) \quad Z_\alpha^*(t) := \int_0^t \frac{\mathbf{1}_{\{R(s) \geq 1\}}}{R^\alpha(s)} ds = \mathcal{O}(t^{1-\alpha/2+\varepsilon}), \quad \text{a.s.}$$

**Lemma 3.4.** *As  $n \rightarrow \infty$ , we have almost surely*

$$(3.20) \quad U_{2\alpha}(n) := \sum_{k=1}^n \frac{\mathbf{1}_{\{\|\mathbf{S}_k\| \neq 0\}}}{\|\mathbf{S}_k\|^{2\alpha}} = \begin{cases} \mathcal{O}(n^{1-\alpha+\varepsilon}) & \text{if } \alpha < 1 \\ \mathcal{O}(\log^{3+\varepsilon} n) & \text{if } \alpha = 1 \\ \mathcal{O}(\log^2 n) & \text{if } \alpha > 1. \end{cases}$$

**Proof:** Let

$$(3.21) \quad M_n = \max_{k \leq n} \|\mathbf{S}_k\|.$$

Then clearly

$$(3.22) \quad M_n = \mathcal{O}(n \log \log n)^{1/2}.$$

By Theorem E

$$(3.23) \quad \sum_{k=1}^n \frac{\mathbf{1}_{\{\|\mathbf{S}_k\| \neq 0\}}}{\|\mathbf{S}_k\|^{2\alpha}} = \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{\xi(\mathbf{x}, n)}{\|\mathbf{x}\|^{2\alpha}} = \mathcal{O}(\log^2 n) \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}, \|\mathbf{x}\| \leq M_n} \frac{1}{\|\mathbf{x}\|^{2\alpha}}.$$

Due to the following equiconvergence relation

$$(3.24) \quad \sum_{\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}, \|\mathbf{x}\| \leq N} \frac{1}{\|\mathbf{x}\|^{2\alpha}} \sim \int_1^N r^{1-2\alpha} dr$$

and the fact that

$$(3.25) \quad \int_1^N r^{1-2\alpha} dr = \begin{cases} \mathcal{O}(N^{2-2\alpha}) & \text{if } \alpha < 1 \\ \mathcal{O}(\log N) & \text{if } \alpha = 1 \\ \mathcal{O}(1) & \text{if } \alpha > 1. \end{cases}$$

we have our statement combining (3.23)–(3.25).  $\square$

**Proof of Theorem 1.1:** We start with the following decomposition:

$$(3.26) \quad \begin{aligned} \left| 2^{-\alpha/2} U_\alpha(n) - Z_\alpha^*(n) \right| &= \left| \sum_{k=1}^n \int_{k-1}^k \left( \frac{\mathbf{1}_{\{\mathbf{s}_k \neq \mathbf{0}\}}}{\|\mathbf{S}_k^*\|^\alpha} - \frac{\mathbf{1}_{\{R(s) \geq 1\}}}{(R(s))^\alpha} \right) ds \right| \\ &\leq \left| \sum_{k=1}^n \int_{k-1}^k \frac{(R(s))^\alpha - \|\mathbf{S}_k^*\|^\alpha}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \right| \\ &\quad + \left| \sum_{k=1}^n \int_{k-1}^k \frac{\mathbf{1}_{\{\|\mathbf{s}_k\| = \mathbf{0}, R(s) \geq 1\}}}{(R(s))^\alpha} ds \right| + \left| \sum_{k=1}^n \int_{k-1}^k \frac{\mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) < 1\}}}{\|\mathbf{S}_k^*\|^\alpha} ds \right| \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where  $\mathbf{S}_k^* = \sqrt{2} \mathbf{S}_k$ .

First observe that

$$(3.27) \quad I_2 \leq \sum_{k=1}^n \mathbf{1}_{\{\|\mathbf{s}_k\| = \mathbf{0}\}} = \xi(\mathbf{0}, n) = \mathcal{O}(\log^{1+\varepsilon} n), \quad \text{a.s.}$$

for any  $\varepsilon > 0$ , by Theorem E. Similarly we have that for any  $\varepsilon > 0$

$$(3.28) \quad I_3 \leq \int_0^n \mathbf{1}_{\{R(s) < 1\}} ds = \int_0^1 \ell(x, n) dx = \mathcal{O}(\log^{2+\varepsilon} n), \quad \text{a.s.}$$

by Lemma 3.2 (applied in the special case  $\alpha = 0$ ).

The estimation of  $I_1$  is more delicate and we have to consider two cases. First we consider the case  $0 < \alpha \leq 1$ . According to (5.2) we have

$$(3.29) \quad \begin{aligned} I_1 &\leq \sum_{k=1}^n \int_{k-1}^k \frac{|(R(s))^\alpha - \|\mathbf{S}_k^*\|^\alpha|}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\ &\leq \sum_{k=1}^n \int_{k-1}^k \frac{|R(s) - \|\mathbf{S}_k^*\||}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\ &\leq \sum_{k=1}^n \int_{k-1}^k \frac{|R(s) - \|\mathbf{S}_k^*\||}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\ &\leq \mathcal{O}(\log n) \sum_{k=1}^n \int_{k-1}^k \frac{\mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}}}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} ds, \end{aligned}$$

where in the last line we used Theorem J. Apply now (3.3) in the special case of  $\alpha = 1$  to get from (3.29) that

$$\begin{aligned}
I_1 &\leq \mathcal{O}(\log n) \sum_{k=1}^n \int_{k-1}^k \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} \left( \frac{1}{\|\mathbf{S}_k^*\|^{2\alpha}} + \frac{1}{(R(s))^{2\alpha}} \right) ds \\
(3.30) \quad &\leq \mathcal{O}(\log n) \left( \sum_{k=1}^n \frac{\mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}\}}}{\|\mathbf{S}_k^*\|^{2\alpha}} + \int_0^n \frac{\mathbf{1}_{\{R(s) \geq 1\}}}{(R(s))^{2\alpha}} ds \right).
\end{aligned}$$

Apply now Lemmas 3.3 and 3.4 to get our statement for  $0 < \alpha \leq 1$ .

Consider now the case  $1 < \alpha < 2$ . Then based on (3.1), using again Theorem J and (3.3), we get that

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^n \int_{k-1}^k \frac{|(R(s))^\alpha - \|\mathbf{S}_k^*\|^\alpha|}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\
&\leq \sum_{k=1}^n \int_{k-1}^k \frac{2|R(s) - \|\mathbf{S}_k^*\| | ((R(s))^{\alpha-1} + \|\mathbf{S}_k^*\|^{\alpha-1})}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\
&\leq \mathcal{O}(\log n) \sum_{k=1}^n \int_{k-1}^k \frac{1}{\|\mathbf{S}_k^*\|^\alpha (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\
&\quad + \mathcal{O}(\log n) \sum_{k=1}^n \int_{k-1}^k \frac{1}{\|\mathbf{S}_k^*\| (R(s))^\alpha} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}, R(s) \geq 1\}} ds \\
(3.31) \quad &\leq \mathcal{O}(\log n) \sum_{k=1}^n \frac{1}{\|\mathbf{S}_k^*\|^{\alpha+1}} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}\}} + \mathcal{O}(\log n) \int_0^n \frac{1}{(R(s))^{\alpha+1}} \mathbf{1}_{\{R(s) \geq 1\}} ds.
\end{aligned}$$

Observe now that as  $\alpha > 1$ , in the last line above both the sum and the integral are exactly the ones which are discussed in Theorems 1.2 and 1.3, except a constant factor in the first case (as  $\alpha + 1 > 2$ ). Hence we can apply those theorems to conclude that

$$(3.32) \quad \sum_{k=1}^n \frac{1}{\|\mathbf{S}_k^*\|^{\alpha+1}} \mathbf{1}_{\{\|\mathbf{s}_k\| \neq \mathbf{0}\}} = \mathcal{O}(\log^{1+\varepsilon} n),$$

and similarly

$$(3.33) \quad \int_0^n \frac{1}{(R(s))^{\alpha+1}} \mathbf{1}_{\{R(s) \geq 1\}} ds = \mathcal{O}(\log^{1+\varepsilon} n)$$

for any  $\varepsilon > 0$ , proving our Theorem 1.1. □

## 4. Proof of Theorem 1.2

In view of Theorem B in Section 2, to prove Theorem 1.2, it suffices to show that, when  $\alpha > 2$ , the function  $\mathbf{x} \mapsto f(\mathbf{x}) = \mathbf{1}_{\{\mathbf{x} \neq \mathbf{0}\}} / \|\mathbf{x}\|^\alpha$  satisfies (2.1) with an appropriately chosen  $\delta > 0$ .

Write  $\|Y\|_p := (\mathbf{E}(|Y|^p))^{1/p}$  for any random variable  $Y$  and  $p \geq 1$ . Fix an integer  $m \geq 1$ . It follows from Theorems C and D that for any  $\mathbf{x} \in \mathbf{Z}^2 - \{\mathbf{0}\}$ ,

$$\begin{aligned}
 \|\xi(\mathbf{x}, \rho_1)\|_m &\leq 1 + \|\xi(\mathbf{x}, \rho_1) - 1\|_m \\
 &\leq 1 + C_m p^{(1-m)/m} \\
 &\leq C_m (\log \|\mathbf{x}\| + 1)^{(m-1)/m} \\
 (4.1) \qquad &\leq C_m (\log \|\mathbf{x}\| + 1),
 \end{aligned}$$

where  $C_m$  denotes a (finite and positive) constant depending only on  $m$ , whose value varies from line to line.

Let  $f(\mathbf{x}) = \mathbf{1}_{\{\mathbf{x} \neq \mathbf{0}\}} / \|\mathbf{x}\|^\alpha$ . To check (2.1), observe that

$$\sum_{k=1}^{\rho_1} f(\mathbf{S}_k) = \sum_{x \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{\xi(\mathbf{x}, \rho_1)}{\|\mathbf{x}\|^\alpha}.$$

Hence for any integer  $m \geq 1$ , by the triangle inequality and (4.1),

$$\left\| \sum_{k=1}^{\rho_1} f(\mathbf{S}_k) \right\|_m \leq \sum_{x \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{\|\xi(\mathbf{x}, \rho_1)\|_m}{\|\mathbf{x}\|^\alpha} \leq C_m \sum_{x \in \mathbf{Z}^2 - \{\mathbf{0}\}} \frac{1 + \log \|\mathbf{x}\|}{\|\mathbf{x}\|^\alpha},$$

which is finite whenever  $\alpha > 2$ . Thus any moments of order  $m \geq 1$  of  $\sum_{k=1}^{\rho_1} f(S_k)$  exist, proving Theorem 1.2.  $\square$

## 5. Proof of Theorem 1.3

The proof of this theorem is based on Theorem A. Denote by  $\ell(x, t)$  the local time (at time  $t$  and position  $x$ ) of the Bessel process  $\{R(t), t \geq 0\}$ . Let  $\{\sigma_n, n \geq 0\}$  and  $\{\theta_n, n \geq 1\}$  be as in (1.5)–(1.7). Both  $\{\theta_i - \sigma_{i-1}\}_1^\infty$  and  $\{\sigma_i - \theta_i\}_1^\infty$  are i.i.d. sequences. The Laplace transform of  $\ell(x, \sigma_1) - \ell(x, \sigma_0)$  is given by (3.13), from which we can obtain for  $x \geq 1$ ,

$$(5.1) \qquad \mathbf{E}(\ell(x, \sigma_1) - \ell(x, \sigma_0)) = 2x \log x,$$

$$(5.2) \qquad \mathbf{E}\left((\ell(x, \sigma_1) - \ell(x, \sigma_0))^3\right) \leq Cx^3(\log x)^2,$$

where  $C > 0$  is a constant.

Define, for  $i \geq 1$ ,

$$(5.3) \quad \tau_i := \sigma_i - \sigma_{i-1},$$

$$(5.4) \quad X_i := \int_{\sigma_{i-1}}^{\sigma_i} \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s) \geq 1\}} ds.$$

We show that the sequence  $(X_i, \tau_i)_{i=1}^\infty$  of i.i.d. vectors satisfies the conditions of Theorem A. Indeed, by (5.2),

$$\begin{aligned} \|X_i\|_3 &= \left( \mathbf{E}_1 \left( \int_{\sigma_0}^{\sigma_1} \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s) \geq 1\}} ds \right)^3 \right)^{1/3} \\ &= \left( \mathbf{E} \left( \int_1^\infty \frac{\ell(x, \tau_1)}{x^\alpha} dx \right)^3 \right)^{1/3} \\ &\leq \int_1^\infty \frac{\|\ell(x, \tau_1)\|_3}{x^\alpha} dx \\ &\leq C \int_1^\infty \frac{(x^3 (\log x)^2)^{1/3}}{x^\alpha} dx, \end{aligned}$$

which is finite, whenever  $\alpha > 2$ . Using Chebyshev's inequality, this leads to

$$(5.5) \quad \mathbf{P}(|X_i| > x) \leq \frac{c}{x^3},$$

for some constant  $c > 0$  and all  $x > 0$ . Thus, the first condition of (2.2) is satisfied by  $X_i$  with  $\beta = 3$ . It is clear that the second condition in (2.2) is satisfied by  $\tau_i$  with  $h(x) = C \log x$ , since by Theorem F,  $\mathbf{P}(\tau_i > x) \leq C/\log x$ .

Consequently, the sequence of vectors  $(X_i, \tau_i)_{i=1}^\infty$  defined in (5.3)–(5.4) satisfies the conditions of Theorem A. Let

$$\begin{aligned} S_i &:= \int_{\sigma_0}^{\sigma_i} \frac{1}{R^\alpha(s)} \mathbf{1}_{\{R(s) \geq 1\}} ds = Z_\alpha^*(\sigma_i), \quad i \geq 1, \\ \rho_n &:= \sigma_n - \sigma_0, \quad n \geq 0. \end{aligned}$$

According to Theorem A, on an appropriate probability space one can construct two independent copies, say  $(X_i^{(1)}, \tau_i^{(1)})_{i=1}^\infty$  and  $(X_i^{(2)}, \tau_i^{(2)})_{i=1}^\infty$ , of  $(X_i, \tau_i)_{i=1}^\infty$ , such that

$$(5.6) \quad (S_n, \rho_n)_{n=1}^\infty \stackrel{\mathcal{D}}{=} (S_n^{(1)}, \rho_n^{(j)})_{n=1}^\infty, \quad j = 1, 2,$$

$$(5.7) \quad \sup_{k \leq n} |S_k - S_k^{(2)}| = \mathcal{O}\left(n^{1/\beta^*}\right), \quad \text{a.s.},$$

$$(5.8) \quad \sup_{k \leq n} |\rho_k - \rho_k^{(1)}| = \mathcal{O}\left(e^{cn^\gamma}\right), \quad \text{a.s.}$$

where  $S_k^{(j)} = \sum_{i=1}^k X_i^{(j)}$ ,  $\rho_k^{(j)} = \sum_{i=1}^k \tau_i^{(j)}$ ,  $\gamma < 1$  and  $2 < \beta^* < 3$ ,

Apply now the Komlós–Major–Tusnády theorem [KMT, 75] (see also Csörgő and Révész [CsR, 81], p. 108, Theorem 2.6.6) to get that, there exists a Wiener process  $\{W(t), t \geq 0\}$  such that for any  $2 < \beta^* < 3$ , when  $N$  goes to infinity,

$$(5.9) \quad S_N^{(2)} - N\mathbf{E}(X_1) - \sigma_\alpha^W W(N) = O(N^{1/\beta^*}), \quad \text{a.s.},$$

where  $\sigma_\alpha^W = \sqrt{\text{Var}(X_1)}$ . Since  $W$  is constructed from  $\{S_n^{(2)}, n \geq 1\}$ , it can be chosen to be independent of  $(X_i^{(1)}, \tau_i^{(1)})_{i=1}^\infty$ .

By (5.1),

$$\mathbf{E}(X_1) = \mathbf{E} \int_1^\infty \frac{\ell(x, \sigma_1 - \sigma_0)}{x^\alpha} dx = \int_1^\infty \frac{2 \log x}{x^{\alpha-1}} dx = \frac{2}{(\alpha - 2)^2} =: \bar{f}_\alpha^W.$$

It follows from (5.9) and (5.7) that

$$S_N - N\bar{f}_\alpha^W - \sigma_\alpha^W W(N) = O(N^{1/\beta^*}), \quad \text{a.s.}$$

Now let  $\eta(t)$  be as in (1.9). We get

$$S_{\eta(t)} - \bar{f}_\alpha^W \eta(t) - \sigma_\alpha^W W(\eta(t)) = \mathcal{O}\left((\eta(t))^{1/\beta^*}\right), \quad \text{a.s.},$$

which, in view of (2.3), yields that, for any small enough  $\varepsilon > 0$ ,

$$(5.10) \quad S_{\eta(t)} - \bar{f}_\alpha^W \eta(t) - \sigma_\alpha^W W(\eta(t)) = \mathcal{O}(\log^{1/2-\varepsilon} t), \quad \text{a.s.}$$

Now we want to get an almost sure upper bound for  $|Z_\alpha^*(t) - S_{\eta(t)}|$ . Since  $\eta(t) < t \leq \eta(t) + 1$ , we have

$$|Z_\alpha^*(t) - S_{\eta(t)}| \leq X_{\eta(t)+1}.$$

On the other hand, a routine Borel–Cantelli argument, using (5.5), yields that for a small enough  $\delta > 0$ ,  $X_k = \mathcal{O}(k^{1/2-\delta})$  almost surely (when  $k$  goes to infinity). Therefore, by (2.3),  $X_{\eta(t)+1} = \mathcal{O}(\log^{1/2-\delta^*} t)$ , for any  $0 < \delta^* < \delta$ . Accordingly,

$$Z_\alpha^*(t) - S_{\eta(t)} = \mathcal{O}(\log^{1/2-\delta^*} t), \quad \text{a.s.}$$

Going back to (5.10), we arrive at

$$(5.11) \quad Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta(t)) + \mathcal{O}(\log^{1/2-\gamma^*} t), \quad \text{a.s.},$$

with a small enough  $\gamma^* > 0$ .



What remains to show is that on the right hand side of (5.11) we can replace  $W(\eta(t))$  by  $W(\eta^{(1)}(t))$ , at the price of a possible small increase in the order of the error term in the approximation. For this purpose we need two preliminary estimates. The first one (Lemma 5.1) is analogous to Theorem H. Its proof, which goes along the same lines as that of Theorem H, is omitted.

**Lemma 5.1.** *Let  $a_t = \exp((\log t)^K)$  and  $b_t = \exp((\log t)^b)$ , for  $K > 0$  and  $b > 0$ . Then for any  $\varepsilon > 0$ ,*

$$\sup_{a \leq a_t} (\eta(a + b_t) - \eta(a)) = \mathcal{O}(\log^{b+\varepsilon} t), \quad \text{a.s.}$$

**Lemma 5.2.** *There exists  $\nu < 1$  such that, as  $t \rightarrow \infty$ ,*

$$|\eta^{(1)}(t) - \eta(t)| = \mathcal{O}(\log^\nu t), \quad \text{a.s.}$$

**Proof:** Since  $\eta(t) = \eta^{(1)}(\rho_{\eta(t)}^{(1)})$ ,  $\eta^{(1)}(t) = \eta(\rho_{\eta^{(1)}(t)})$ ,  $t \geq \rho_{\eta(t)}$  and  $t \geq \rho_{\eta^{(1)}(t)}^{(1)}$ , we have

$$(5.12) \quad \eta^{(1)}(\rho_{\eta(t)}) - \eta^{(1)}(\rho_{\eta^{(1)}(t)}^{(1)}) \leq \eta^{(1)}(t) - \eta(t) \leq \eta(\rho_{\eta^{(1)}(t)}) - \eta(\rho_{\eta^{(1)}(t)}^{(1)}).$$

By (2.3), for any  $\varepsilon > 0$  and all large  $t$ ,  $\eta(t) < (\log t)^{1+\varepsilon}$ . It follows from (5.8) that for some  $\gamma < 1$  and all large  $t$ ,

$$|\rho_{\eta(t)} - \rho_{\eta^{(1)}(t)}^{(1)}| \leq \sup_{i \leq (\log t)^{1+\varepsilon}} |\rho_i - \rho_i^{(1)}| \leq \exp(c(\log t)^{(1+\varepsilon)\gamma}), \quad \text{a.s.}$$

Now apply Lemma 5.1 to  $b_t = \exp(c(\log t)^{(1+\varepsilon)\gamma})$  and  $a_t = t$  (noting that  $\rho_{\eta^{(1)}(t)}^{(1)} \leq t$ ) to see that

$$\eta^{(1)}(\rho_{\eta(t)}) - \eta^{(1)}(\rho_{\eta^{(1)}(t)}^{(1)}) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \quad \text{a.s.}$$

A similar argument shows that

$$\eta(\rho_{\eta^{(1)}(t)}) - \eta(\rho_{\eta^{(1)}(t)}^{(1)}) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \quad \text{a.s.}$$

In view of (5.12), we have

$$\eta^{(1)}(t) - \eta(t) = \mathcal{O}(\log^{(1+2\varepsilon)\gamma} t), \quad \text{a.s.}$$

Choosing  $\varepsilon > 0$  so small that  $(1 + 2\varepsilon)\gamma < 1$ , gives the lemma.  $\square$

We now go back to the proof of Theorem 1.3. Note that Lemma 5.2 yields (1.11). To check (1.10), we can control the increments of  $W$  by using Theorem 1.2.1 of Csörgő and Révész [CsR, 81]: for any  $\varepsilon > 0$  and any  $\nu > 0$ ,

$$\sup_{0 \leq u, v \leq (\log t)^{1+\varepsilon}, |u-v| \leq (\log t)^\nu} |W(u) - W(v)| = \mathcal{O}\left((\log t)^{\nu/2} (\log \log t)^{1/2}\right), \quad \text{a.s.}$$

Since  $\eta(t) = \mathcal{O}(\log^{1+\varepsilon} t)$  (cf. Theorem G), it follows from Lemma 5.2 and (5.11) that for any  $1 > \nu^* > \nu$ ,

$$Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta^{(1)}(t)) + \mathcal{O}(\log^{\nu^*/2} t) + \mathcal{O}(\log^{1/2-\gamma^*} t), \quad \text{a.s.}$$

Take

$$\kappa := \max\left(\frac{1}{2} - \gamma^*, \frac{\nu^*}{2}\right) < \frac{1}{2}$$

to get

$$Z_\alpha(t) - \bar{f}_\alpha^W \eta(t) = \sigma_\alpha^W W(\eta^{(1)}(t)) + \mathcal{O}(\log^\kappa t), \quad \text{a.s.},$$

as  $t \rightarrow \infty$ . Since  $W$  was chosen to be independent of  $(X_i^{(1)}, \tau_i^{(1)})_{i=1}^\infty$ , it is a fortiori independent of  $\{\eta^{(1)}(t), t > 0\}$ . This completes the proof of Theorem 1.2.  $\square$

## 6. Further results

Our theorems can be used to deduce weak and strong laws for the additive functionals studied in this paper. Considering  $Z_\alpha(t)$ , defined by (1.1), by scale change we have (cf. (3.6))

$$(6.1) \quad Z_\alpha(t) \stackrel{\mathcal{D}}{=} t^{1-\alpha/2} Z_\alpha(1).$$

Hence, Theorem 1.1 implies for  $0 < \alpha < 2$ ,

$$(6.2) \quad 2^{-\alpha/2} n^{\alpha/2-1} U_\alpha(n) \stackrel{\mathcal{D}}{\rightarrow} Z_\alpha(1), \quad n \rightarrow \infty,$$

where  $\stackrel{\mathcal{D}}{\rightarrow}$  denotes convergence in distribution. It follows also that the LIL for the process  $Z_\alpha(\cdot)$  is inherited by  $U_\alpha(\cdot)$ . Now we study limsup and liminf properties of  $Z_\alpha$ . It is obvious from (6.1) that first we have to study the upper and lower tail behaviour of  $Z_\alpha(1)$ . A direct approach however does not seem available, instead, we apply Corollary XI.1.12 of Revuz and Yor [RY, 99] telling us that

$$(6.3) \quad Z_\alpha(1) := \int_0^1 \frac{ds}{R^\alpha(s)} \stackrel{\mathcal{D}}{=} \left(\frac{2}{2-\alpha}\right)^\alpha \left(\int_0^1 R^\beta(t) dt\right)^{-(1-\alpha/2)},$$

where  $\beta = 2\alpha/(2 - \alpha)$ . For  $\beta \geq 1$  Azencott [A, 80] shows that there exists a finite constant  $c(\beta)$  such that

$$(6.4) \quad \log \mathbf{P} \left( \int_0^1 R^\beta(t) dt > u \right) \sim -c(\beta)u^{2/\beta}, \quad u \rightarrow \infty$$

which combined with (6.3) yields

$$(6.5) \quad \log \mathbf{P}(Z_\alpha(1) < x) \sim -c_1(\alpha)x^{-2/\alpha}, \quad x \rightarrow 0,$$

where

$$(6.6) \quad c_1(\alpha) = \frac{4c(\beta)}{(2 - \alpha)^2}.$$

Now the usual argument gives the following liminf result for  $2/3 \leq \alpha < 2$ :

$$(6.7) \quad \liminf_{t \rightarrow \infty} t^{-(1-\alpha/2)}(\log \log t)^{\alpha/2} Z_\alpha(t) = K(\alpha), \quad \text{a.s.},$$

where

$$(6.8) \quad K(\alpha) = (c_1(\alpha))^{\alpha/2}.$$

Hence, by Theorem 1.1 we have also for  $2/3 \leq \alpha < 2$

$$(6.9) \quad \liminf_{n \rightarrow \infty} n^{-(1-\alpha/2)}(\log \log n)^{\alpha/2} 2^{-\alpha/2} U_\alpha(n) = K(\alpha), \quad \text{a.s.}$$

Concerning the limsup results, we can use the following small deviation theorem of Borovkov and Mogulskii [BM, 91]: for  $\beta \geq 1$

$$(6.10) \quad \log \mathbf{P} \left( \int_0^1 R^\beta(t) dt < u \right) \sim -\tilde{c}(\beta)u^{-2/\beta}, \quad u \rightarrow 0,$$

where

$$(6.11) \quad \tilde{c}(\beta) = \left( \frac{\beta}{2 + \beta} \right)^{-2/\beta} \left( \cos \frac{\pi}{2 + \beta} \right)^{2/\beta} \lambda(\beta)$$

and  $\lambda(\beta)$  is the smallest positive solution of the eigenvalue problem

$$(6.12) \quad \frac{1}{2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \varphi(u, v) - (u^2 + v^2)^{\beta/2} \varphi(u, v) = -\lambda \varphi(u, v).$$

We note that in [BM, 91] the analogue result is proved for one-dimensional Wiener process, but as they remark, their result holds also true for higher dimensional case.

Based on (6.10), similarly to the liminf case, one can easily obtain

$$(6.13) \quad \limsup_{t \rightarrow \infty} \frac{Z_\alpha(t)}{t^{1-\alpha/2}(\log \log t)^{\alpha/2}} = \tilde{K}(\alpha), \quad \text{a.s.}$$

with

$$(6.14) \quad \tilde{K}(\alpha) = (\tilde{c}(\beta))^{-\alpha/2} \frac{2^\alpha}{(2-\alpha)^\alpha}$$

and consequently

$$(6.15) \quad \limsup_{n \rightarrow \infty} \frac{2^{-\alpha/2} U_\alpha(n)}{n^{1-\alpha/2}(\log \log n)^{\alpha/2}} = \tilde{K}(\alpha), \quad \text{a.s.}$$

Turning now to the case  $\alpha > 2$ , we consider first some limiting distribution results. Let  $E$  be a random variable with density function  $e^{-|x|}/2$ ,  $x \in \mathbf{R}$ . Then  $E$  is a bilateral exponential random variable and  $|E|$  has an exponential distribution, i.e. it has the density function  $e^{-x}$ ,  $x \in [0, \infty)$ . By Theorem E,

$$(6.16) \quad \frac{\pi \xi(\mathbf{0}, n)}{\log n} \xrightarrow{\mathcal{D}} |E|, \quad n \rightarrow \infty$$

and similarly, from Theorem F one can see that

$$(6.17) \quad \frac{(2 \log 2) \eta(t)}{\log t} \xrightarrow{\mathcal{D}} |E|, \quad t \rightarrow \infty.$$

Moreover, as easily seen,

$$(6.18) \quad \frac{\sqrt{2\pi} W(\xi^{(1)}(\mathbf{0}, n))}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} E, \quad n \rightarrow \infty$$

and

$$(6.19) \quad \frac{2\sqrt{\log 2} W(\eta^{(1)}(t))}{\sqrt{\log t}} \xrightarrow{\mathcal{D}} E, \quad t \rightarrow \infty,$$

where  $W(\xi^{(1)}(\mathbf{0}, \cdot))$  and  $W(\eta^{(1)}(\cdot))$ , resp. are the processes of Theorems 1.2 and 1.3, resp.

Now our Theorems 1.2 and 1.3 imply the following weak convergence results:

$$(6.20) \quad \frac{\pi U_\alpha(n)}{\bar{f}_\alpha^D \log n} \xrightarrow{\mathcal{D}} |E|, \quad n \rightarrow \infty,$$

$$(6.21) \quad \frac{U_\alpha(n) - \bar{f}_\alpha^D \xi(\mathbf{0}, n)}{\sigma_\alpha^D \sqrt{\log n}} \sqrt{2\pi} \xrightarrow{\mathcal{D}} E, \quad n \rightarrow \infty,$$

$$(6.22) \quad \frac{(2 \log 2) Z_\alpha^*(t)}{\bar{f}_\alpha^W \log t} \xrightarrow{\mathcal{D}} |E|, \quad t \rightarrow \infty,$$

$$(6.23) \quad \frac{Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t)}{\sigma_\alpha^W \sqrt{\log t}} 2\sqrt{\log 2} \xrightarrow{\mathcal{D}} E, \quad t \rightarrow \infty.$$

Concerning strong limit results, our Theorems 1.2 and 1.3 combined with Theorems E and G yield

$$(6.24) \quad \limsup_{n \rightarrow \infty} \frac{U_\alpha(n)}{\log n \log_3 n} = \frac{\bar{f}_\alpha^D}{\pi}, \quad \text{a.s.}$$

and

$$(6.25) \quad \limsup_{t \rightarrow \infty} \frac{Z_\alpha^*(t)}{\log t \log_3 t} = \frac{\bar{f}_\alpha^W}{2 \log 2}, \quad \text{a.s.}$$

Moreover, the following LIL is true (cf. [MR], [CsFR,98], [CsF,99])

$$(6.26) \quad \limsup_{n \rightarrow \infty} \frac{W(\xi^{(1)}(\mathbf{0}, n))}{\sqrt{\log n \log_3 n}} = \frac{1}{\sqrt{2\pi}}, \quad \text{a.s.}$$

Similarly, one can see that

$$(6.27) \quad \limsup_{t \rightarrow \infty} \frac{W(\eta^{(1)}(t))}{\sqrt{\log t \log_3 t}} = \frac{1}{2\sqrt{\log 2}}, \quad \text{a.s.}$$

Consequently, we have the following LIL's:

$$(6.28) \quad \limsup_{n \rightarrow \infty} \frac{U_\alpha(n) - \bar{f}_\alpha^D \xi(\mathbf{0}, n)}{\sqrt{\log n \log_3 n}} = \frac{\sigma_\alpha^D}{\sqrt{2\pi}}, \quad \text{a.s.}$$

and

$$(6.29) \quad \limsup_{t \rightarrow \infty} \frac{Z_\alpha^*(t) - \bar{f}_\alpha^W \eta(t)}{\sqrt{\log t \log_3 t}} = \frac{\sigma_\alpha^W}{2\sqrt{\log 2}}, \quad \text{a.s.}$$

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(xxxx Where do we use [CsCsFR, 89], [CsCsFR, 92] and [CsF, 98]?)

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