Heavy Points of a d-dimensional Simple Random Walk.

Endre Csáki[1]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary. E-mail address: csaki@renyi.hu

Antónia Földes[2]
City University of New York, 2800 Victory Blvd., Staten Island, New York 10314, U.S.A. E-mail address: afoldes@gc.cuny.edu

Pál Révész[1]
Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/107 A-1040 Vienna, Austria. E-mail address: reveszp@renyi.hu

Abstract: For a simple symmetric random walk in dimension $d \geq 3$, a uniform strong law of large numbers is proved for the number of sites with given local time up to time n.

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1. Introduction and main results

Consider a simple symmetric random walk $\{\mathbf{S}_n\}_{n=1}^{\infty}$ starting at the origin $\mathbf{0}$ on the d-dimensional integer lattice $\mathcal{Z}_d$, i.e. $\mathbf{S}_0 = \mathbf{0}$, $\mathbf{S}_n = \sum_{k=1}^{n} \mathbf{X}_k$, $n = 1, 2, \ldots$, where $\mathbf{X}_k$, $k = 1, 2, \ldots$ are i.i.d. random variables with distribution

$$\mathbf{P}(\mathbf{X}_1 = \mathbf{e}_i) = \mathbf{P}(\mathbf{X}_1 = -\mathbf{e}_i) = \frac{1}{2d}, \qquad i = 1, 2, ..., d$$

and $\{\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_d\}$ is a system of orthogonal unit vectors in $\mathcal{Z}_d$. Define the local time of the walk by

$$\xi(\mathbf{x}, n) := \#\{k : \ 0 < k \leq n, \ \ \mathbf{S}_k = \mathbf{x}\}, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $\mathbf{x}$ is any lattice point of $\mathcal{Z}_d$. The maximal local time of the walk is defined as

$$\xi(n) := \max_{\mathbf{x} \in \mathcal{Z}_d} \xi(\mathbf{x}, n). \tag{1.2}$$

Define also

$$\eta(n) := \max_{0 \leq k \leq n} \xi(\mathbf{S}_k, \infty). \tag{1.3}$$

Denote by $\gamma(n) = \gamma(n; d)$ the probability that in the first $n - 1$ steps the d-dimensional path does not return to the origin. Then

$$1 = \gamma(1) \geq \gamma(2) \geq ... \geq \gamma(n) \geq ... > 0. \tag{1.4}$$

It was proved in [2] that

Theorem A (Dvoretzky and Erdős [2]) *For* $d \geq 3$

$$\lim_{n \to \infty} \gamma(n) = \gamma = \gamma(\infty; d) > 0, \tag{1.5}$$

and

$$\gamma < \gamma(n) < \gamma + O(n^{1-d/2}), \tag{1.6}$$

or equivalently

$$\mathbf{P}(\xi(\mathbf{0}, n) = 0, \, \xi(\mathbf{0}, \infty) > 0) = O\left(n^{1-d/2}\right) \tag{1.7}$$

as $n \to \infty$.

So γ is the probability that the d-dimensional simple symmetric random walk never returns to its starting point.

Let $\xi(\mathbf{x}, \infty)$ be the total local time at $\mathbf{x}$ of the infinite path in $\mathcal{Z}_d$. Then (see Erdős and Taylor [3]) $\xi(\mathbf{0}, \infty)$ has geometric distribution:

$$\mathbf{P}(\xi(\mathbf{0},\infty)=k)=\gamma(1-\gamma)^k, \qquad k=0,1,2,... \tag{1.8}$$

Erdős and Taylor [3] proved the following strong law for the maximal local time:

Theorem B (Erdős and Taylor [3]) *For* $d \geq 3$

$$\lim_{n\to\infty}\frac{\xi(n)}{\log n}=\lambda \qquad \text{a.s.}, \tag{1.9}$$

where

$$\lambda=\lambda_d=-\frac{1}{\log(1-\gamma)}. \tag{1.10}$$

Following the proof of Erdős and Taylor, without any new idea, one can prove that

$$\lim_{n\to\infty}\frac{\eta(n)}{\log n}=\lambda \qquad \text{a.s.} \tag{1.11}$$

We can present a stronger lower estimate of $\xi(n)$.

Theorem C (Révész [10]) *Let* $d \geq 4$ *and*

$$\psi(n)=\psi(n,B)=\lambda\log n-\lambda B\log\log n. \tag{1.12}$$

Then for any $\varepsilon>0$ *we have*

$$\xi(n)\geq\psi(n,3+\varepsilon) \qquad \text{a.s.}$$

if n *is big enough.*

Erdős and Taylor [3] also investigated the properties of

$$Q(k,n):=\#\{\mathbf{x}:\ \mathbf{x}\in\mathcal{Z}_d,\ \xi(\mathbf{x},n)=k\},$$

i.e. the cardinality of the set of points visited exactly k times in the time interval $[1,n]$. They proved

Theorem D (Erdős and Taylor [3]) *For* $d\geq 3$ *and for any* $k=1,2,\ldots$

$$\lim_{n\to\infty}\frac{Q(k,n)}{n}=\gamma^2(1-\gamma)^{k-1} \qquad \text{a.s.} \tag{1.13}$$

Let

$$\begin{aligned} U(k,n) &:= \#\{j:\ 0<j\le n,\ \xi(\mathbf{S}_j,\infty)=k,\ \mathbf{S}_j\ne\mathbf{S}_\ell\ (\ell=1,2,\dots,j-1)\} \\ &= \#\{\mathbf{x}\in\mathcal{Z}_d:\ 0<\xi(\mathbf{x},n)\le\xi(\mathbf{x},\infty)=k\}. \end{aligned} \tag{1.14}$$

Repeating the proof of Theorem D one can get

$$\lim_{n\to\infty}\frac{U(k,n)}{n}=\gamma^2(1-\gamma)^{k-1}\qquad \text{a.s.} \tag{1.15}$$

for any $k=1,2,\dots$.

Define furthermore

$$R(k,n):=\sum_{j=k}^{\infty}Q(j,n), \tag{1.16}$$

$$V(k,n):=\sum_{j=k}^{\infty}U(j,n). \tag{1.17}$$

It follows that for fixed $k\ge 1$

$$\lim_{n\to\infty}\frac{R(k,n)}{n}=\gamma(1-\gamma)^{k-1}\qquad \text{a.s.} \tag{1.18}$$

$$\lim_{n\to\infty}\frac{V(k,n)}{n}=\gamma(1-\gamma)^{k-1}\qquad \text{a.s.} \tag{1.19}$$

The properties of these quantities were further investigated (for fixed k) by Pitt [8] who proved (1.13), (1.15) and (1.18), (1.19) for general random walk and by Hamana [5], [6] who proved central limit theorems (in general case for $d\ge 3$).

In this paper we study the question whether k can be replaced by a sequence $t(n)=t_n\nearrow\infty$ of positive integers in (1.13), (1.15), (1.18) and (1.19).

Theorem 1: *Let $d\ge 3$, and define*

$$\mu=\mu(t) := \gamma(1-\gamma)^{t-1}, \tag{1.20}$$

$$t_n := [\psi(n,B)],\quad B>2, \tag{1.21}$$

where $\psi(n,B)$ is defined by (1.12). *Then we have*

$$\lim_{n\to\infty}\sup_{t\le t_n}\left|\frac{U(t,n)}{n\gamma\mu(t)}-1\right|=0\qquad \text{a.s.} \tag{1.22}$$

$$\lim_{n\to\infty}\sup_{t\le t_n}\left|\frac{Q(t,n)}{n\gamma\mu(t)}-1\right|=0 \qquad \text{a.s.} \tag{1.23}$$

$$\lim_{n\to\infty}\sup_{t\le t_n}\left|\frac{V(t,n)}{n\mu(t)}-1\right|=0 \qquad \text{a.s.} \tag{1.24}$$

$$\lim_{n\to\infty}\sup_{t\le t_n}\left|\frac{R(t,n)}{n\mu(t)}-1\right|=0 \qquad \text{a.s.} \tag{1.25}$$

Here in $\sup_{t\le t_n}$, t runs through positive integers.

(1.25) of Theorem clearly implies (compare to Theorem C)

Corollary 1.1 *Let $d\ge 3$. Then for any $\varepsilon>0$ we have almost surely*

$$\xi(n)\ge \lambda\log n-(2+\varepsilon)\log\log n$$

if n is big enough.

First we present some more notations. For $\mathbf{x}\in\mathcal{Z}_d$ let $T_{\mathbf{x}}$ be the first hitting time of $\mathbf{x}$, i.e. $T_{\mathbf{x}}=\min\{i\ge 1:\mathbf{S}_i=\mathbf{x}\}$ with the convention that $T_{\mathbf{x}}=\infty$ if there is no i with $\mathbf{S}_i=\mathbf{x}$. Let $T=T_{\mathbf{0}}$. In general, for a subset A of $\mathcal{Z}_d$, let T_A denote the first time the random walk visits A, i.e. $T_A=\min\{i\ge 1:\ \mathbf{S}_i\in A\}=\min_{\mathbf{x}\in A}T_{\mathbf{x}}$. Let $\mathbf{P}_{\mathbf{x}}(\cdot)$ denote the probability of the event in the bracket under the condition that the random walk starts from $\mathbf{x}\in\mathcal{Z}_d$. We denote $\mathbf{P}(\cdot)=\mathbf{P}_{\mathbf{0}}(\cdot)$.

Introduce further

$$q_{\mathbf{x}} := \mathbf{P}(T<T_{\mathbf{x}}), \tag{1.26}$$

$$s_{\mathbf{x}} := \mathbf{P}(T_{\mathbf{x}}<T). \tag{1.27}$$

In words, $q_{\mathbf{x}}$ is the probability that the random walk, starting from $\mathbf{0}$, returns to $\mathbf{0}$, before reaching $\mathbf{x}$ (including $T<T_{\mathbf{x}}=\infty$), and $s_{\mathbf{x}}$ is the probability that the random walk, starting from $\mathbf{0}$, hits $\mathbf{x}$, before returning to $\mathbf{0}$ (including $T_{\mathbf{x}}<T=\infty$).

2. Preliminary facts and results

First we present some lemmas needed to prove Theorem.

Introduce the following notations:

$$\begin{aligned}
X_i(t) &= X_i = \\
&= \begin{cases} 1 & \text{if } \mathbf{S}_j \neq \mathbf{S}_i \ (j = 1,2,\ldots,i-1),\ \xi(\mathbf{S}_i,\infty) \geq t, \\ 0 & \text{otherwise,} \end{cases} \\
Y_i(t,n) &= Y_i = \\
&= \begin{cases} 1 & \text{if } \mathbf{S}_j \neq \mathbf{S}_i \ (j = 1,2,\ldots,i-1),\ \xi(\mathbf{S}_i,n) \geq t, \\ 0 & \text{otherwise,} \end{cases} \\
\rho_i &= \rho_i(t) = I\{X_i = 1\}(\min\{j:\ \xi(\mathbf{S}_i,j) \geq t\} - i), \\
\mu_i &= \mu_i(t) = \gamma(i)(1-\gamma)^{t-1},
\end{aligned}$$

$t = 1,2,\ldots$, $i = 1,2,\ldots$, where $I\{\cdot\}$ denotes the usual indicator function.

Recall the definitions of $\gamma(i)$, γ and $\mu = \mu(t)$ in (1.4) (1.5) and (1.20). Furthermore let

$$\sigma_n^2 = \sigma_n^2(t) := \mathbf{E}\left(\sum_{i=1}^n X_i - n\mu\right)^2. \tag{2.1}$$

Clearly we have

$$R(t,n) = \sum_{i=1}^n Y_i,$$

$$V(t,n) = \sum_{i=1}^n X_i.$$

Lemma 2.1. (Dvoretzky and Erdős [2])

$$\mathbf{P}(\mathbf{S}_i \neq \mathbf{S}_j,\ j = 1,2,\ldots,i-1) = \mathbf{P}(\xi(\mathbf{0},i-1) = 0) = \gamma(i).$$

The following lemma is a trivial consequence of Theorem A.

Lemma 2.2.

$$\mathbf{P}(n < \rho_i(t) < \infty) \leq \frac{O(1)t^{d/2}}{n^{d/2-1}},$$

$$\mu \leq \mu_i \leq \left(1 + \frac{O(1)}{i^{d/2-1}}\right)\mu,$$

$$\mathbf{E}X_i = \mu_i.$$

The next lemma can be obtained by elementary calculations.

Lemma 2.3.

$$n\mu \le \mathbf{E}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mu_i \le n\mu + \mu a_n O(1),$$

where

$$a_n = \sum_{i=1}^{n} \frac{1}{i^{d/2-1}} = \begin{cases} O(1) & \text{if} \quad d > 4, \\ O(1)\log n & \text{if} \quad d = 4, \\ O(1)n^{1/2} & \text{if} \quad d = 3. \end{cases}$$

Lemma 2.4. *Let $n > 3^3$. Then*

$$\sigma_n^2 \le n\mu + \mu a_n O(1) - n^2\mu^2 + 2(I + II + III), \tag{2.2}$$

where

$$\begin{aligned} I &= \sum_{1 \le i < j \le n} \mathbf{P}(X_i = 1,\ X_j = 1,\ \rho_i \ge n^\alpha), \\ II &= \sum_{1 \le i < j \le \min(i+3n^\alpha, n)} \mathbf{P}(X_i = 1,\ X_j = 1,\ \rho_i < n^\alpha), \\ III &= \sum_{1 \le i < i+3n^\alpha < j \le n} \mathbf{P}(X_i = 1,\ X_j = 1,\ \rho_i < n^\alpha), \\ \alpha &= 2/d. \end{aligned}$$

Proof. Clearly we have

$$\begin{aligned} \sigma_n^2 &= \mathbf{E}\left(\sum_{i=1}^{n} X_i\right)^2 + n^2\mu^2 - 2n\mu\mathbf{E}\sum_{i=1}^{n} X_i = \\ &= \mathbf{E}\sum_{i=1}^{n} X_i + 2\sum_{1 \le i < j \le n} \mathbf{E}X_iX_j + n^2\mu^2 - 2n\mu\sum_{i=1}^{n}\mu_i \le \\ &\le n\mu + \mu a_n O(1) + 2\sum_{1 \le i < j \le n} \mathbf{E}X_iX_j - n^2\mu^2. \end{aligned}$$

Further

$$\sum_{1 \le i < j \le n} \mathbf{E}X_iX_j = \sum_{1 \le i < j \le n} \mathbf{P}\{X_i = 1,\ X_j = 1\} = I + II + III.$$

Hence Lemma 2.4 is proved.

Now let $A^{(\mathbf{x})}$ denote the two-point set $\{\mathbf{0}, \mathbf{x}\}$ and let $\Xi(A^{(\mathbf{x})}, \infty) = \xi(\mathbf{0}, \infty) + \xi(\mathbf{x}, \infty)$ denote its total occupation time.

Lemma 2.5. *For* $\mathbf{x} \in \mathcal{Z}_d$, $\mathbf{x} \neq \mathbf{0}$, *define* $\gamma_{\mathbf{x}} := \mathbf{P}(T_{\mathbf{x}} = \infty)$ *and recall the definitions of* $q_{\mathbf{x}}$ *and* $s_{\mathbf{x}}$ *in* (1.26) *and* (1.27)*. Then*

$$\begin{aligned}
\gamma_{\mathbf{e}_i} &= \gamma_{-\mathbf{e}_i} = \gamma, \quad i = 1, 2, \ldots, d, && (2.3)\\
\gamma_{\mathbf{x}} &\geq \gamma, && (2.4)\\
q_{\mathbf{x}} &= 1 - \frac{\gamma}{1 - (1 - \gamma_{\mathbf{x}})^2}, && (2.5)\\
s_{\mathbf{x}} &= (1 - \gamma_{\mathbf{x}})(1 - q_{\mathbf{x}}), && (2.6)\\
q_{\mathbf{x}} + s_{\mathbf{x}} &= 1 - \frac{\gamma}{2 - \gamma_{\mathbf{x}}}, && (2.7)\\
\mathbf{P}(\Xi(A^{(\mathbf{x})}, \infty) = j) &= (1 - q_{\mathbf{x}} - s_{\mathbf{x}})(q_{\mathbf{x}} + s_{\mathbf{x}})^j, \quad j = 0, 1, \ldots. && (2.8)
\end{aligned}$$

Proof. We show (2.3) first. For symmetric reason, $\gamma_{\pm\mathbf{e}_i} = \gamma_{\pm\mathbf{e}_j}$, $i, j = 1, \ldots, d$. Hence

$$1 - \gamma = \sum_{i=1}^{d} \mathbf{P}(\mathbf{S}_1 = \mathbf{e}_i)(1 - \gamma_{\mathbf{e}_i}) + \sum_{i=1}^{d} \mathbf{P}(\mathbf{S}_1 = -\mathbf{e}_i)(1 - \gamma_{-\mathbf{e}_i}) = 2\sum_{i=1}^{d} \frac{1}{2d}(1 - \gamma_{\mathbf{e}_1}) = 1 - \gamma_{\mathbf{e}_1},$$

proving (2.3).

To show (2.4), observe that starting from the origin, before hitting $\mathbf{x}$ with $\|\mathbf{x}\| > 1$, the random walk should hit first the sphere $S(\mathbf{x}, 1) := \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| = 1\}$. Hence

$$1 - \gamma_{\mathbf{x}} = \mathbf{P}(T_{S(\mathbf{x},1)} < \infty)(1 - \gamma) \leq 1 - \gamma. \tag{2.9}$$

Now let $Z(A)$ denote the number of visits in the set A up to the first return to zero, i.e.

$$Z(A) = \sum_{n=1}^{T} I\{\mathbf{S}_n \in A\}. \tag{2.10}$$

Observe that

$$\mathbf{P}(Z(A^{(\mathbf{x})}) = j + 1, T < \infty) = \begin{cases} q_{\mathbf{x}} & \text{if } j = 0, \\ s_{\mathbf{x}}^2 q_{\mathbf{x}}^{j-1} & \text{if } j = 1, 2, \ldots \end{cases} \tag{2.11}$$

Summing up in (2.11) we get

$$\sum_{j=0}^{\infty} \mathbf{P}(Z(A^{(\mathbf{x})}) = j + 1, T < \infty) = q_{\mathbf{x}} + \frac{s_{\mathbf{x}}^2}{1 - q_{\mathbf{x}}} = \mathbf{P}(T < \infty) = 1 - \gamma. \tag{2.12}$$

On the other hand, one can easily see that

$$\begin{aligned} 1-\gamma &= \mathbf{P}(T<\infty)=\mathbf{P}(T<T_{\mathbf{x}})+\mathbf{P}(T>T_{\mathbf{x}},T<\infty) \\ &= \mathbf{P}(T<T_{\mathbf{x}})+\mathbf{P}(T>T_{\mathbf{x}})\mathbf{P}_{\mathbf{x}}(T<\infty) \\ &= \mathbf{P}(T<T_{\mathbf{x}})+\mathbf{P}(T>T_{\mathbf{x}})\mathbf{P}(T_{\mathbf{x}}<\infty)=q_{\mathbf{x}}+s_{\mathbf{x}}(1-\gamma_{\mathbf{x}}), \end{aligned}$$

i.e.

$$1-\gamma=q_{\mathbf{x}}+s_{\mathbf{x}}(1-\gamma_{\mathbf{x}}) \tag{2.13}$$

Now (2.12) and (2.13) easily imply (2.5) and (2.6), hence also (2.7).

Equation (2.8) was proved in [1] for general random walk. For completeness a short proof is presented here. The probability that the random walk, starting from $\mathbf{0}$, returns to $\mathbf{0}$ without hitting $\mathbf{x}$, is $q_{\mathbf{x}}$, while $s_{\mathbf{x}}$ is the probability that the random walk starting from $\mathbf{0}$ hits $\mathbf{x}$ without returning to $\mathbf{0}$. Similarly, for symmetric reason, $q_{\mathbf{x}}$ is also the probability of the random walk starting from $\mathbf{x}$ returns to $\mathbf{x}$ without hitting $\mathbf{0}$, and $s_{\mathbf{x}}$ is also the probability of the random walk starting from $\mathbf{x}$ hits $\mathbf{0}$ in finite time, without returning to $\mathbf{x}$. Hence, the probability that the random walk starting from any point of $A^{(\mathbf{x})}$, returns to $A^{(\mathbf{x})}$ in finite time, is $q_{\mathbf{x}}+s_{\mathbf{x}}$. This gives (2.8).

Similarly to Theorem A, we prove

Lemma 2.6.

$$\begin{aligned} 1-\gamma_{\mathbf{x}}(n):=\mathbf{P}(T_{\mathbf{x}}<n) &= 1-\gamma_{\mathbf{x}}+\frac{O(1)}{n^{d/2-1}}, &(2.14)\\ q_{\mathbf{x}}(n):=\mathbf{P}(T<\min(n,T_{\mathbf{x}})) &= q_{\mathbf{x}}+\frac{O(1)}{n^{d/2-1}}, &(2.15)\\ s_{\mathbf{x}}(n):=\mathbf{P}(T_{\mathbf{x}}<\min(n,T)) &= s_{\mathbf{x}}+\frac{O(1)}{n^{d/2-1}}, &(2.16) \end{aligned}$$

and $O(1)$ is uniform in $\mathbf{x}$.

Proof. For the proof of (2.14) see Jain and Pruitt [7].

To prove (2.15) and (2.16), observe that

$$\begin{aligned} q_{\mathbf{x}}-q_{\mathbf{x}}(n) &= \mathbf{P}(T<T_{\mathbf{x}},\, n\le T<\infty)\le\mathbf{P}(n\le T<\infty)=\gamma(n)-\gamma, \\ s_{\mathbf{x}}-s_{\mathbf{x}}(n) &= \mathbf{P}(T_{\mathbf{x}}<T,\, n\le T_{\mathbf{x}}<\infty)\le\mathbf{P}(n\le T_{\mathbf{x}}<\infty)=\gamma_{\mathbf{x}}(n)-\gamma_{\mathbf{x}}. \end{aligned}$$

Lemma 2.7. *Let $i < j$. Then for $t \geq 1$ integer we have*

$$\mathbf{P}(X_i = 1, X_j = 1) \leq C\mu^2 \left(1 + \frac{t^{d/(d-2)}}{(j-i)^{d/2}} \left(\frac{2}{2-\gamma}\right)^{2t}\right), \tag{2.17}$$

where C is a constant, independent of i, j, t and $\mu = \mu(t) = \gamma(1-\gamma)^{t-1}$.

Proof. Using (2.8) of Lemma 2.5, we get

$$\begin{aligned}
&\mathbf{P}(X_i = 1, X_j = 1) \\
&\leq \sum_{\mathbf{x} \in \mathcal{Z}_d} \mathbf{P}(\mathbf{S}_j - \mathbf{S}_i = \mathbf{x}, \xi(\mathbf{S}_i, \infty) - \xi(\mathbf{S}_i, i) + \xi(\mathbf{S}_j, \infty) - \xi(\mathbf{S}_j, i) \geq 2t - 1) \\
&= \sum_{\mathbf{x} \in \mathcal{Z}_d} \mathbf{P}(\mathbf{S}_{j-i} = \mathbf{x}) \mathbf{P}(\Xi(A^{(\mathbf{x})}, \infty) \geq 2t - 1) \\
&= \sum_{\mathbf{x} \in \mathcal{Z}_d} \mathbf{P}(\mathbf{S}_{j-i} = \mathbf{x})(q_{\mathbf{x}} + s_{\mathbf{x}})^{2t-1} = \sum_{\mathbf{x} \in \mathcal{Z}_d, \|\mathbf{x}\| \leq R} + \sum_{\mathbf{x} \in \mathcal{Z}_d, \|\mathbf{x}\| > R},
\end{aligned}$$

where R will be chosen later. For estimating the first sum, we use $\gamma_{\mathbf{x}} \geq \gamma$ (cf. (2.4) of Lemma 2.5), hence by (2.7)

$$q_{\mathbf{x}} + s_{\mathbf{x}} = 1 - \frac{\gamma}{2 - \gamma_{\mathbf{x}}} \leq \frac{2(1-\gamma)}{2-\gamma}.$$

On the other hand,

$$\mathbf{P}(\mathbf{S}_{j-i} = \mathbf{x}) \leq \frac{C_1}{(j-i)^{d/2}}, \qquad \mathbf{x} \in \mathcal{Z}_d$$

with some constant C_1, not depending on $\mathbf{x}$ (cf. Spitzer [11], page 72).

Since the cardinality of the set $\{\|\mathbf{x}\| \leq R\}$ is a constant multiple of R^d, we have

$$\sum_{\mathbf{x} \in \mathcal{Z}_d, \|\mathbf{x}\| \leq R} \leq \frac{C_2 R^d}{(j-i)^{d/2}} \left(\frac{2(1-\gamma)}{2-\gamma}\right)^{2t} \tag{2.18}$$

with some constant C_2.

For estimating the second sum, we use $1 - \gamma_{\mathbf{x}} \leq C_3 R^{-d+2}$ for $\|\mathbf{x}\| > R$ (cf. Révész [9], page 241), hence

$$q_{\mathbf{x}} + s_{\mathbf{x}} \leq 1 - \gamma + C_4 R^{-d+2} = (1-\gamma)\left(1 + \frac{C_4}{(1-\gamma)R^{d-2}}\right).$$

Now choose $R = t^{1/(d-2)}$. Then

$$(q_{\mathbf{x}} + s_{\mathbf{x}})^{2t-1} \le C_5(1-\gamma)^{2t}.$$

Here the constant C_5 is independent of both $\mathbf{x}$ and t. Since

$$\sum_{\mathbf{x}\in\mathcal{Z}_d} \mathbf{P}(\mathbf{S}_j - \mathbf{S}_i = \mathbf{x}) = 1,$$

we have

$$\sum_{\mathbf{x}\in\mathcal{Z}_d,\ \|\mathbf{x}\|>R} \le C_5(1-\gamma)^{2t} = C_6\mu^2.$$

this together with (2.18) (putting $R = t^{1/(d-2)}$ there) proves Lemma 2.7.

In the subsequent lemmas t_n is defined by (1.21).

Lemma 2.8. *For $t \le t_n$, any $\varepsilon > 0$ and large enough n we have*

$$I \le O(1)n^{2/d+\varepsilon}\left(n + \left(\frac{2}{2-\gamma}\right)^{2t_n}\right)\mu^2(t). \tag{2.19}$$

Proof. Now we need to estimate the probability

$$\mathbf{P}(X_i = 1, X_j = 1, \rho_i \ge n^\alpha).$$

Define the events B_k by

$$B_k = \{\xi(\mathbf{S}_i,\infty) - \xi(\mathbf{S}_i, i) + \xi(\mathbf{S}_j,\infty) - \xi(\mathbf{S}_j, i) = k\}$$

and consider the k time intervals between the consecutive visits of $\{\mathbf{S}_i, \mathbf{S}_j\}$. Then at least one of these intervals is larger than

$$\frac{\rho_i(t)}{k} \ge \frac{n^\alpha}{k} \tag{2.20}$$

(provided that $\{X_i = 1,\ X_j = 1,\ \rho_i \ge n^\alpha\}$). Denote this event by D_k. Similarly to the proof of Lemma 2.7 we have

$$\mathbf{P}(X_i = 1, X_j = 1, \rho_i \ge n^\alpha) \le \sum_{\mathbf{x}\in\mathcal{Z}_d} \mathbf{P}(\mathbf{S}_j - \mathbf{S}_i = \mathbf{x},\ \cup_{k\ge 2t-1} B_k D_k)$$

$$\le \sum_{\mathbf{x}\in\mathcal{Z}_d} \mathbf{P}(\mathbf{S}_{j-i} = \mathbf{x}) \sum_{k\ge 2t-1} \mathbf{P}(B_k D_k \,|\, \mathbf{S}_j - \mathbf{S}_i = \mathbf{x}).$$

The event $B_k D_k$, under the condition $\mathbf{S}_j - \mathbf{S}_i = \mathbf{x}$, means that placing a new origin at the point $\mathbf{S}_i$, and starting the time at i, there are exactly k visits in the set $A^{(\mathbf{x})}$, and at least one time interval between consecutive visits is larger than n^α/k. Hence applying (2.8) of Lemma 2.5 and (2.15), (2.16) of Lemma 2.6, we get

$$\mathbf{P}(B_k D_k \,|\, \mathbf{S}_j - \mathbf{S}_i = \mathbf{x}) \le k(1 - q_\mathbf{x} - s_\mathbf{x})(q_\mathbf{x} + s_\mathbf{x})^{k-1} \left(q_\mathbf{x} + s_\mathbf{x} - q_\mathbf{x}\left(\frac{n^\alpha}{k}\right) - s_\mathbf{x}\left(\frac{n^\alpha}{k}\right)\right)$$

$$\le O(1) k \left(\frac{k}{n^\alpha}\right)^{d/2-1} (1 - q_\mathbf{x} - s_\mathbf{x})(q_\mathbf{x} + s_\mathbf{x})^{k-1} \le O(1) k^{d/2} n^{2/d-1} (q_\mathbf{x} + s_\mathbf{x})^{k-1},$$

where $O(1)$ is uniform in k and $\mathbf{x}$, hence

$$\begin{aligned} \sum_{k \ge 2t-1} \mathbf{P}(B_k D_k \,|\, \mathbf{S}_j - \mathbf{S}_i = \mathbf{x}) &\le O(1) n^{2/d-1} \sum_{k \ge 2t-1} k^{d/2} (q_\mathbf{x} + s_\mathbf{x})^{k-1} \\ &\le O(1) n^{2/d-1} t^{d/2} (q_\mathbf{x} + s_\mathbf{x})^{2t-2}. \end{aligned}$$

Proceeding now as in the proof of Lemma 2.7, we can estimate

$$\mathbf{P}(X_i = 1, X_j = 1, \rho_i \ge n^\alpha) \le O(1) t^{d/2} n^{2/d-1} \mu^2(t) \left(1 + \frac{t^{d/(d-2)}}{(j-i)^{d/2}} \left(\frac{2}{2-\gamma}\right)^{2t}\right)$$

and summing up for $1 \le i < j \le n$, we get

$$I \le O(1) n^{2/d} t_n^{d/2} \left(n + t_n^{d/(d-2)} \left(\frac{2}{2-\gamma}\right)^{2t_n}\right) \mu^2(t),$$

since $t \le t_n$. But $t_n < \lambda \log n$, therefore any power of t_n can be estimated by n^ε, hence (2.19) follows.

Lemma 2.9. *For $t \le t_n$, any $\varepsilon > 0$ and large enough n we have*

$$II \le O(1) n^{2/d+\varepsilon} \left(n + n^{1-2/d} \left(\frac{2}{2-\gamma}\right)^{2t_n}\right) \mu^2(t). \tag{2.21}$$

Proof. Using the estimate in Lemma 2.7 and summing up for i, j with $1 \le i < j \le \min(i + 3n^\alpha, n)$, using again that $t_n < \lambda \log n$, a simple calculation shows (2.21).

Lemma 2.10. *For $t \le t_n$, any $\varepsilon > 0$ and large enough n we have*

$$III \le \frac{\mu^2(t)n^2}{2} + O(1)n^{3/2}\mu^2(t). \tag{2.22}$$

Proof. Let

$$\begin{aligned}
A &= \{\mathbf{S}_i \text{ is a new point i.e. } \mathbf{S}_i \ne \mathbf{S}_j \ j = 1, 2, \dots, i-1\},\\
B &= \{\xi(\mathbf{S}_i, i + n^\alpha) - \xi(\mathbf{S}_i, i) \ge t - 1\},\\
D &= \{\mathbf{S}_j \text{ is a new point}\},\\
E &= \{\xi(\mathbf{S}_j, \infty) - \xi(\mathbf{S}_j, j) \ge t - 1\},\\
D \subset G &= \left\{\xi(\mathbf{S}_j, j) - \xi\left(\mathbf{S}_j, i + \frac{2(j-i)}{3}\right) = 0\right\},\\
B \subset H &= \{\xi(\mathbf{S}_i, \infty) - \xi(\mathbf{S}_i, i) \ge t - 1\}.
\end{aligned}$$

Recall the definition of $\gamma(n)$ in Section 1 and let $j > i + 3n^\alpha$. Then

$$\begin{aligned}
&\mathbf{P}\{X_i = 1,\ X_j = 1,\ \rho_i < n^\alpha\} \le \mathbf{P}\{ABDE\} \le\\
&\le \mathbf{P}(ABGE) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(G)\mathbf{P}(E) \le\\
&\le \mathbf{P}(A)\mathbf{P}(H)\mathbf{P}(G)\mathbf{P}(E) =\\
&= \gamma(i+1)(1-\gamma)^{t-1}\gamma((j-i)/3)(1-\gamma)^{t-1}.
\end{aligned}$$

Clearly we have

$$\begin{aligned}
III &\le \sum \gamma(i+1)(1-\gamma)^{t-1}\gamma((j-i)/3)(1-\gamma)^{t-1} \le\\
&\le \gamma^2(1-\gamma)^{2t-2} \sum \left(1 + \frac{O(1)}{(j-i)^{d/2-1}}\right)\left(1 + \frac{O(1)}{i^{d/2-1}}\right) \le\\
&\le \gamma^2(1-\gamma)^{2t-2}\left[\binom{n}{2} + O(1)(K + L + M)\right]
\end{aligned}$$

where the summation goes for $\{i, j : 1 \le i < i + 3n^\alpha < j \le n\}$ and

$$\begin{aligned}
K &= \sum \frac{1}{i^{d/2-1}} \le na_n,\\
L &= \sum \frac{1}{(j-i)^{d/2-1}} \le na_n,\\
M &= \sum \frac{1}{i^{d/2-1}} \frac{1}{(j-i)^{d/2-1}} \le na_n.
\end{aligned}$$

Using $a_n = O(1)n^{1/2}$ (see Lemma 2.3) we have (2.22).

Lemma 2.11. *For $t \leq t_n$, any $\varepsilon > 0$ and large enough n we have*

$$\sigma_n^2 = O(1)[n\mu(t) + \mu^2(t)n^{1.8}]. \tag{2.23}$$

Proof is based on Lemmas 2.4, 2.8, 2.9 and 2.10. The numerical values of λ can be obtained by a result of Griffin [4]:

$$\begin{aligned} 1 - \gamma_3 &= 0.341, \\ 1 - \gamma_4 &= 0.193, \\ 1 - \gamma_5 &= 0.131, \\ 1 - \gamma_6 &= 0.104. \end{aligned}$$

Consequently

$$\begin{aligned} \lambda_3 &= 0.929, \\ \lambda_4 &= 0.608, \\ \lambda_5 &= 0.492, \\ \lambda_6 &= 0.442. \end{aligned}$$

By using $t_n < \lambda \log n$, one can verify (numerically)

$$\left(\frac{2}{2-\gamma}\right)^{2t_n} < n^{2\lambda \log(2/(2-\gamma))} < n^{0.75}$$

for $d = 3$ and hence also for all $d \geq 3$. By choosing an appropriate ε and putting the estimations (2.19), (2.21), (2.22) into (2.2), we can see, that the term $n^2\mu^2$ cancels out and all the other terms are smaller than the right hand side of (2.23), proving Lemma 2.11.

Lemma 2.11 implies

Lemma 2.12. *For any $0 < C < B$, $t \leq t_n$ and large enough n we have*

$$\sigma_n(\log n)^{C/2} \leq O(1)((n\mu(t))^{1/2}(\log n)^{C/2} + \mu(t)n^{0.9}(\log n)^{C/2}) = o(1)n\mu(t).$$

3. Proof of the Theorem

First we prove (1.24).

By Markov's inequality for any $C > 0$ we have

$$\mathbf{P}(|V(t,n) - n\mu(t)| \geq \sigma_n (\log n)^{C/2}) \leq (\log n)^{-C}.$$

By Lemma 2.12, if $C < B$,

$$\mathbf{P}(|V(t,n) - n\mu(t)| \geq o(1) n\mu(t)) \leq (\log n)^{-C}.$$

Consequently, since $t_n < \lambda \log n$,

$$\mathbf{P}\left(\sup_{t \leq t_n + 1} \frac{|V(t,n) - n\mu(t)|}{n\mu(t)} \geq o(1) \right) \leq O(1)(\log n)^{-C+1}. \tag{3.1}$$

Choose $C > 2,\ n(k) = \exp(k/\log k)$. (3.1) and Borel-Cantelli lemma imply

$$\lim_{k\to\infty} \sup_{t \leq t(n(k))+1} \left| \frac{V(t,n(k))}{n(k)\mu(t)} - 1 \right| = 0 \quad \text{a.s.} \tag{3.2}$$

Let $n(k) \leq n < n(k+1)$. Then for $t \leq t_n$ we have

$$V(t,n(k)) \leq V(t,n) \leq V(t,n(k+1))$$

and

$$\lim_{k\to\infty} \frac{n(k+1)}{n(k)} = 1.$$

Hence for any $\varepsilon > 0$ and large enough n,

$$\frac{V(t,n)}{n\mu(t)} \leq \frac{V(t,n(k+1))}{n(k+1)\mu(t)} \frac{n(k+1)}{n} \leq (1+\varepsilon) \quad \text{a.s.,}$$

since $t \leq t_n \leq t(n(k+1))$. Similarly,

$$\frac{V(t,n)}{n\mu(t)} \geq \frac{V(t,n(k))}{n(k)\mu(t)} \frac{n(k)}{n} \geq (1-\varepsilon) \quad \text{a.s.}$$

Hence we have (1.24).

Now we turn to the proof of (1.25).

Let

$$M(t,n) = V(t,n) - R(t,n) = \sum_{i=1}^{n} (X_i - Y_i).$$

Observe that $X_i \geq Y_i$ and hence $M(t,n)$ is non-negative and non-decreasing in n. Moreover, by Lemma 2.2

$$\mathbf{E}(X_i - Y_i) = \mathbf{P}(X_i - Y_i = 1) \leq \mathbf{P}(X_i = 1, n - i \leq \rho_i(t) < \infty) \leq \frac{O(1)\mu(t)t^{d/2}}{(n-i)^{d/2-1}}.$$

Consequently

$$0 \leq \frac{\mathbf{E}M(t,n)}{n\mu(t)} \leq \frac{O(1)(\log n)^{d/2}}{n^{1/2}}.$$

By Markov's inequality

$$\mathbf{P}\left(\sup_{t \leq t_n} \frac{M(t,n)}{n\mu(t)} > \varepsilon\right) \leq \frac{O(1)(\log n)^{d/2+1}}{n^{1/2}}.$$

On choosing $n_k = k^{2+\delta}$, $\delta > 0$, Borel-Cantelli lemma implies

$$\lim_{k \to \infty} \sup_{t \leq t_{n_k}} \frac{M(t,n_k)}{n_k\mu(t)} = 0 \qquad \text{a.s.}$$

Using the monotonicity of $M(t,n)$ in n, interpolating between n_k and n_{k+1} we get

$$\lim_{n \to \infty} \sup_{t \leq t_n} \frac{M(t,n)}{n\mu(t)} = 0 \qquad \text{a.s.}$$

This combined with (1.24) gives (1.25).

(1.23) and (1.22) are immediate from (1.25) and (1.24), since $Q(t,n) = R(t,n) - R(t+1,n)$ and $U(t,n) = V(t,n) - V(t+1,n)$.

This completes the proof of the Theorem.

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