# ABOUT THE MEASURE OF HEAVILY VISITED POINTS OF THE BROWNIAN MOTION

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**Abstract**

We consider those space points of the Wiener process, the local time of which is at least a constant multiple of the local time of the origin. The upper and lower class behavior of the Lebesque measure of these points are investigated.

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# 1. INTRODUCTION.

Let $\{W(t); t \geq 0\}$ be a real valued Wiener process, and denote its local time by

$$L(x,t) = \frac{d}{dx}\int_0^t \mathbf{1}\{W(s) \leq x\}\, ds \tag{1.1}$$

where $\mathbf{1}\{A\}$ denotes the indicator function of the event $A$. Furthermore let

$$T_r = \inf\{t > 0;\ L(0,t) \geq r\}. \tag{1.2}$$

It is wellknown that $T_r$ is a so-called "nice clock". The hardly visited points of the Wiener process were studied by Földes and Révész. A point in the state space is called hardly visited if its local time is nonzero, but less than a finite constant. They asked; what can we say about the measure of those points which are hardly visited? Their answer was the following result.

**Theorem A.** [FR,92] *For any fixed* $q > 0$

$$\frac{2q}{j_0^2} \leq \limsup_{r\to\infty} \frac{\mu(y;\ 0 < L(y,T_r) < q)}{\log\log r} \leq \frac{4q}{j_0^2} \qquad \text{a.s.} \tag{1.3}$$

*where* $j_0$ *is the smallest positive root of the Bessel function*

$$J_0(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{2^{2k}(k!)^2}, \tag{1.4}$$

and $\mu(.)$ denotes the ordinary Lebesgue measure. Eisenbaum and Shi revisited this problem and they proved that in (**??**) the lower bound is sharp, that is to say

**Theorem B** [ES, 99] *For any fixed* $q > 0$

$$\limsup_{r\to\infty} \frac{\mu(y;\ 0 < L(y,T_r) < q)}{\log\log r} = \frac{2q}{j_0^2} \qquad \text{a.s..} \tag{1.5}$$

They investigated the lower class behavior of the hardly visited points as well;

**Theorem C** [ES,99]*For any fixed* $q > 0$

$$\liminf_{r\to\infty} \mu(y;\ 0 < L(y,T_r) < q) \log\log r = 2\,q \qquad \text{a.s.} \tag{1.6}$$

In this note we want to investigate the measure of those points which are heavily visited in the sense that they are visited constant times as much as the origin. This investigation will also be based on the nice clock setting that is to say the $T_r$ diffusion. Let us define for $\beta > 0$

$$Y_r(\beta) \overset{def}{=} \mu(y; L(y,T_r) > \beta L(0,T_r)) = \mu(y; L(y,T_r) > \beta\, r). \tag{1.7}$$

First we want to investigate the lim sup and lim inf behavior of $Y_r(\beta)$. Let

$$\varphi_r^p(\lambda) = \mathbf{E}\left(\exp(-\lambda\mu(y; L(y,T_r) > p))\right). \tag{1.8}$$

**Theorem 1.1.**

$$\varphi_r^p(\lambda) = \mathbf{E}\left(\exp(-\lambda\mu(y; L(y,T_r) \geq p))\right) = \left(R_r^p(\lambda)\right)^2 \tag{1.9}$$

*where for $p > 0$*

$$R_r^p(\lambda) = \begin{cases} 1 - \frac{r}{p}\frac{K_0(\sqrt{2\lambda p})}{K_2(\sqrt{2\lambda p})} & \text{if} \quad r \leq p \\ \frac{\sqrt{2r}}{p\sqrt{\lambda}}\frac{K_1(\sqrt{2\lambda r})}{K_2(\sqrt{2\lambda p})} & \text{if} \quad r \geq p \end{cases} \tag{1.10}$$

*and for $p = 0$*

$$R_r^0(\lambda) = \sqrt{2\lambda r}K_1(\sqrt{2\lambda r}). \tag{1.11}$$

*here $K_i(.)$ stand for the Bessel functions of the third kind.*

For the upper tail of the distribution of $Y_r(\beta)$ we prove

**Theorem 1.2.** *For any $\beta \geq 0$*

$$\mathbf{P}(Y_r(\beta) > xr) = \mathbf{P}(Y_1(\beta) > x) \sim \frac{1}{x} \quad \text{as} \qquad x \to \infty. \tag{1.12}$$

The following integral test characterizes the upper class behavior of $Y_r(\beta)$.

**Theorem 1.3.** *Let $f(x)$ be nondecreasing , with $\lim_{x\to\infty} f(x) = +\infty$ and let*

$$I(f) = \int_1^\infty \frac{dx}{xf(x)}. \tag{1.13}$$

*For any $\beta \geq 0$*

- *If* $I(f) < +\infty$ *then*

(1.14) $$\lim_{r\to\infty} \frac{Y_r(\beta)}{r\, f(r)} = 0. \qquad \text{a.s.}$$

- *If* $I(f) = +\infty$ *then*

(1.15) $$\limsup_{r\to\infty} \frac{Y_r(\beta)}{rf(r)} = +\infty \qquad \text{a.s.}$$

It is worthwhile to spell out the $\beta = 0$ case of the above theorem. Introduce

(1.16) $$m^*(r) = \max_{0\le s\le T_r} W(s), \qquad m_*(r) = -\min_{0\le s\le T_r} W(s)$$

and

(1.17) $$Q(r) = m^*(r) + m_*(r).$$

Then clearly $Q(r) = Y_r(0)$ is the Lebesgue measure of the points visited by the Wiener process up to $T_r$. ( the range up to $T_r$)

**Consequence 1.1** *Theorem* 1.3 *remains valid if* $Y_r(\beta)$ *is replaced with* $Q(r)$.

**Remark 1.** It was proved in [F,89] that Theorem 1.3 holds if $Y_r(\beta)$ is replaced by $m^*(r)$ or $m_*(r)$. From this result, the above consequence would not have been hard to conclude. However the fact that for $Q(r)$ and $Y_r(\beta)$ the same integral test hold, indicates that roughly speaking $Y_r(\beta)$ independently from the value of $\beta$ can be as big as the range itself.

To treat the liminf behavior we need the asymptotics of the lower tail. We prove that

**Theorem 1.4.** *For any fixed* $0 \le \beta < 1$ *and for any* $\epsilon > 0$ *we have for* $x \to 0$

(1.18) $$\exp\left(-\frac{2K(\beta)(1+\epsilon)}{x}\right) < \mathbf{P}(Y_r(\beta) \le xr) = \mathbf{P}(Y_1(\beta) \le x) < \exp\left(-\frac{2K(\beta)(1-\epsilon)}{x}\right)$$

*where* $K(\beta) = \left(1 - \sqrt{\beta}\right)^2$. *For* $\beta = 1$ *we have for* $x \to 0$

(1.19) $$\mathbf{P}(Y_1(1) \le x) \sim 2x.$$

*For* $\beta > 1$ *we have*

(1.20) $$\mathbf{P}(Y_1(\beta) = 0) = \left(1 - \frac{1}{\beta}\right)^2 \qquad \text{and for } x \to 0 \qquad \mathbf{P}(0 < Y_1(\beta) \le x) \sim c(\beta)\sqrt{x},$$

*where* $c(\beta)$ *is a constant the value of which is unimportant.*

**Theorem 1.5.** *For any* $0 \le \beta < 1$

$$\liminf_{r\to\infty} \; Y_r(\beta)\,\frac{\log\log r}{r} = 2K(\beta) \qquad \text{a.s.} \tag{1.21}$$

It is interesting to compare Theorem 1.5 with the following result;

**Theorem D.** ([F,89]) *For any* $0 \le \beta < 1$

$$\mathbf{P}\left(\liminf_{r\to\infty} \inf_{|y|<\frac{rK(\beta)}{2\log\log r}} \frac{L(y,T_r)}{r} = \beta\right) = 1. \tag{1.22}$$

Theorem D roughly tells us that for big $r$, on the symmetric interval around the origin of length $rK(\beta)/\log\log r$ every point has a local time which is at least $\beta r$. On the other hand our Theorem 1.5 tells us that the total Lebesgue measure of those points whose local time is at least $\beta r$ is at least twice as much .

**Remark 2.** Let us define

$$\rho^+_{(b)} = \inf\{x \ge 0, L(x,T_r) = b\} \qquad \rho^-_{(b)} = \inf\{x \le 0, L(x,T_r) = b\} \tag{1.23}$$

$$\rho_{(b)} = \rho^+_{(b)} + \rho^-_{(b)}. \tag{1.24}$$

For $b = \beta r$ with $0 \le \beta < 1$, $\rho_{(\beta r)}$ is the Lebesgue measure of those $x$-s which are visited before hitting the $\beta r$ level. The Laplace transform of $\rho^+_{(\beta r)}$ and $\rho^-_{(\beta r)}$ are wellknown, see e.g. [IM,65] (we are only interested in the $0 \le \beta \le 1$ case);

$$\mathbf{E}(\exp(-\lambda\rho_{(\beta r)}))) = \left(\sqrt{\frac{1}{\beta}}\frac{K_1(\sqrt{2\lambda r})}{K_1(\sqrt{2\lambda\beta r})}\right)^2 \qquad 0 \le \beta \le 1. \tag{1.25}$$

Observe that $Y_r(\beta) \ge \rho_{(\beta r)}$. (**??**) combined with the asymptotics for the large values of the Bessel functions (see (**??**) ) would confirm that the asymptotic distribution of $\rho_{(\beta r)}$ and $Y_r(\beta)$ are the same, so we could get most of Theorem 1.4 from that observation as well. The fact that $\rho_{(\beta r)}$ and $Y_r(\beta)$ has the same liminf behavior suggests that the factor 2 difference observed above between Theorem D and Theorem 1.5. which seems to be natural to be attributed to the possible return of the $L(x,T_r)$ above the level $\beta r$ after hitting it, should instead be attributed to the lack of symmetry of hitting the level $\beta r$ on the positive and negative side. We again spell out the $\beta = 0$ case as follows;

**Consequence 1.2**

$$\liminf_{r\to\infty} \; Q(r)\,\frac{\log\log r}{r} = 2 \qquad \text{a.s.} \tag{1.26}$$

In case $\beta > 1$ we have the following results.

**Theorem 1.6.** *For any $\beta > 1$ we have*

$$\liminf_{r\to\infty} \; Y_r(\beta) = 0 \qquad \text{a.s.} \tag{1.27}$$

Finally in case of $\beta = 1$ we prove

**Theorem 1.7.**

$$\liminf_{r\to\infty} Y_r(1)\,\frac{\log r}{r} = 0 \qquad \text{a.s.} \tag{1.28}$$

*For any $\epsilon > 0$*

$$\liminf_{r\to\infty} Y_r(1)\,\frac{(\log r)^{2+\epsilon}}{r} = +\infty \qquad \text{a.s.} \tag{1.29}$$

**Remark 3.** Clearly (**??**) is not sharp. From (**??**) one suspects that the correct exponent of $\log r$ in (**??**) should be $1+\epsilon$.
After getting the results in this nice clock setting, the next natural question is what can we say about the general case. We ask what can we say about

$$X_t(\beta) \stackrel{def}{=} \mu(y; L(y,t) > \beta\, L(0,t)). \tag{1.30}$$

Our answer is summarized in the following theorems.

**Theorem 1.8.** *Let $f(x)$ and $I(f)$ be as in Theorem* 1.3. *For any $\beta \geq 0$*

- *If $I(f) < +\infty$ then*

$$\lim_{t\to\infty} \frac{X_t(\beta)}{L(0,t)\, f(L(0,t))} = 0. \qquad \text{a.s.} \tag{1.31}$$

- *If $I(f) = +\infty$ then*

$$\limsup_{t\to\infty} \frac{X_t(\beta)}{L(0,t) f(L(0,t))} = +\infty \qquad \text{a.s.} \tag{1.32}$$

Then again introducing the Lebesgue measure $V(t)$ of the points visited up to t;

$$M^*(t) = \max_{0 \le s \le t} W(s), \qquad M_*(t) = -\min_{0 \le s \le t} W(s) \tag{1.33}$$

and

$$V(t) = M^*(t) + M_*(t), \tag{1.34}$$

we get that

**Consequence 1.3** *Theorem 1.8 holds if $X_t(\beta)$ is replaced by $V(t)$.*

**Theorem 1.9.** *For all $0 \le \beta < 1$*

$$\liminf_{t \to \infty} \frac{X_t(\beta)}{L(0,t)} \log \log t = 2K(\beta) \qquad \text{a.s..} \tag{1.35}$$

Here we get the following special case ($\beta = 0$);

**Consequence 1.4**

$$\liminf_{t \to \infty} \frac{V(t)}{L(0,t)} \log \log t = 2 \qquad \text{a.s..} \tag{1.36}$$

**Remark 3.** The lower half of the above statement is not new as it was proved by Knight [K,73], that

$$\liminf_{t \to \infty} \frac{\sup_{0 \le s \le t} |W(s)|}{L(0,t)} \log \log t = 1 \qquad \text{a.s..} \tag{1.37}$$

which implies that

$$\liminf_{t \to \infty} \frac{V(t)}{L(0,t)} \log \log t \ge 2 \qquad \text{a.s..} \tag{1.38}$$

**Theorem 1.10.** *For any $\beta > 1$ we have*

$$\liminf_{t \to \infty} X_t(\beta) = 0 \qquad \text{a.s..} \tag{1.39}$$

Finally we have in case $\beta = 1$

**Theorem 1.11.**

(1.40)
$$\liminf_{t\to\infty} \quad X_t(1)\frac{\log L(0,t)}{L(0,t)} = 0 \qquad \text{a.s..}$$

*For any* $\epsilon > 0$

(1.41)
$$\liminf_{t\to\infty} \quad X_t(1)\frac{(\log L(0,t))^{2+\epsilon}}{L(0,t)} = +\infty \qquad \text{a.s.}$$

It is easy to observe that as $X_{T_r}(\beta) = Y_r(\beta)$ it is enough to prove the divergent part of Theorem 1.3 (**??**), and the convergent part of Theorem 1.8 (**??**). Furthermore as

$$\lim_{r\to\infty} \log\log T_r/\log\log r = 1$$

we only have to prove that in Theorem 1.5 in (**??**) $2K(\beta)$ is an upper bound, and in Theorem 1.9 it is a lower bound in (**??**). Similarly concerning Theorems 1.6 and 1.10 we only have to prove Theorem 1.6. As to Theorems 1.7 and 1.11 we need to prove only (**??** ) of Theorem 1.7, and (**??**) of Theorem 1.11.

# 2. PRELIMINARIES.

**Fact 1.** **Theorem F.** Borodin [B,89] *Let* $f(r)$ $r \in (0,h)$ *be a piecewise continuous function with* $f(0) = 0$. *Then*

(2.1)
$$\mathbf{E}\left(\exp\left\{-\int_{-\infty}^{\infty} f(L(y,T_r))\,dy\right\}, \sup_{y\in R} L(y,T_r) < h\right) = R^2(r)\mathbf{1}_{[0,h]}(r)$$

*where* $R(.)$ *is the continuous solution of the problem*

(2.2)
$$\begin{gathered} 2rR'' - f(r)R = 0 \\ R(+0) = 1, \qquad R(h-0) = 0. \end{gathered}$$

**Remark 4.** In case $h = +\infty$ the boundary condition $R(h-0)$ is replaced by the condition $R(+\infty)$ is bounded.

**Fact 2.** For the Bessel functions $K_\nu(.)$ we have the following asymptotics (see [GR,80] formulas 8.446-447);

(2.3)
$$K_0(z) = -\log\frac{z}{2} + \psi(1) - \left(\frac{z}{2}\right)^2\log\frac{z}{2} + \left(\frac{z}{2}\right)^2\psi(2) + o(z^3) \quad \text{as} \quad z \to 0,$$

$$K_1(z) = \frac{1}{z} + \frac{z}{2} \log \frac{z}{2} + Dz + o(z^2) \quad \text{as} \quad z \to 0 \tag{2.4}$$

$$K_2(z) = \frac{2}{z^2} - \frac{1}{2} + o(z^{2-\epsilon}) \quad \text{as} \quad z \to 0 \tag{2.5}$$

where

$$\psi(1) = -\mathbf{C}, \qquad \psi(2) = 1 - \mathbf{C}, \qquad D = \frac{2\mathbf{C} - 1}{4}$$

where $\mathbf{C}$ is the Euler constant.

As to the asymptotics at infinity we have:

$$K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x) \left(1 + \frac{4\nu^2 - 1}{8x} + O\left(\frac{1}{x^2}\right)\right) \quad \text{as } x \to \infty. \tag{2.6}$$

Most of the time we will need only the first term of the above formula, which we spell out separately;

$$K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x) \quad \text{as } x \to \infty. \tag{2.7}$$

**Fact 3.** Scale change property: For any $\beta > 0$

$$\frac{Y_r(\beta)}{r} \stackrel{\mathcal{D}}{=} Y_1(\beta) \tag{2.8}$$

Consequently we have for the distribution function;

$$1 - F^\beta(x) \stackrel{def}{=} \mathbf{P}(Y_1(\beta) > x), \tag{2.9}$$

$$\mathbf{P}(Y_r(\beta) > x) = 1 - F^\beta\left(\frac{x}{r}\right). \tag{2.10}$$

We mention here the following Tauberian theorem;

**Fact 4. Theorem G.** (see in [Do,50] page 511, Theorem 3) *Let $g(x) \geq 0$ and suppose that for some $\gamma \geq 0$, and $c \geq 0$*

$$\int_0^\infty e^{-sx} g(x)\, dx \sim \frac{c}{s^\gamma} L\left(\frac{1}{s}\right) \qquad \text{as} \quad s \to 0 \tag{2.11}$$

*holds, where $L(.)$ is slowly varying at infinity. Then we have*

$$\int_0^t g(x)\,dx \sim \frac{c}{\Gamma(\gamma+1)}\,t^\gamma L(t) \qquad \text{as} \quad t \to \infty. \tag{2.12}$$

**Fact 5.** Let $F(x)$ be a distribution function in $[0,\infty)$ and define

$$U(x) = \int_0^x (1-F(y))\,dy \qquad \text{and} \qquad \hat{U}(s) = \int_0^\infty e^{-sx}\,dU(x). \tag{2.13}$$

According to Theorem 3.9.1 ( [BGT,89 ], page 172) if $\ell(.)$ is slowly varying at infinity, $c > 0$, then the following two statements are equivalent.

$$\frac{U(\lambda x) - U(x)}{\ell(x)} \to c\log\lambda \qquad x \to \infty \quad \text{for all } \lambda > 0 \tag{2.14}$$

$$\frac{\hat{U}(\frac{1}{\lambda x}) - \hat{U}(\frac{1}{x})}{\ell(x)} \to c\log\lambda \qquad x \to \infty \quad \text{for all } \lambda > 0. \tag{2.15}$$

If (**??**) hold we say that $U \in \Pi_\ell$ ( $U$ is in the de Haan class) with $\ell$- index $c$.
By Theorem 3.6.8 of ([BGT,89], page 159) if $U(.)$ satisfies (**??**) then

$$1 - F(x) \sim c\frac{\ell(x)}{x} \qquad \text{as} \qquad x \to \infty. \tag{2.16}$$

**Fact 6.** For any nondecreasing function $f(x)$ for which $\lim_{x\to\infty} f(x) = +\infty$ and any $\rho > 1$ the following sum and integral

$$\sum_{k=1}^\infty \frac{1}{f(\rho^k)} \qquad \text{and} \qquad \int_1^\infty \frac{1}{xf(x)}\,dx \tag{2.17}$$

are equiconvergent.

The following is a special case of de Bruijn's exponential type Tauberian theorem. To see it in full generality we refer to Bingham et al [BGT,89] page 254.

**Fact 7.** Let Z be a nonnegative random variable and let $a > 0$ be a **constant**. Then the following two conditions are equivalent.

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} \log \mathbf{E}(e^{-sZ}) = -a \tag{2.18}$$

$$\lim_{x\to 0} x\log \mathbf{P}(Z\le x) = -\frac{a^2}{4}. \tag{2.19}$$

**Fact 8. Theorem H.** (see e. g. [F,89] Theorem 1) Let $m(r)$ be $m^*(r)$ or $m_*(r)$ . (see (**??**)). Define $I(f)$ as in Theorem 1.3. then we have

$$\mathbf{P}\left(m(r) > rf(r) \quad i.o.\right) = \begin{cases} 0 & \text{if} \quad I(f) < \infty, \\ 1 & \text{if} \quad I(f) = \infty. \end{cases}$$

**Fact 9.** (see e.g. Ito -McKean [IM,65], Knight [K,81]) $\{L(x,T_r);\ x\ge 0\}$ and $\{L(x,T_r);\ x\le 0\}$ are diffusions in $x$, both have the same generator, they are in natural scale, started from $L(0,T_r)=r$, furthermore the two diffusions are independent. Hence for any $\beta>1$ we have that the probability that $L(x,T_r)$ hits $\beta r$ before it hits 0 is $1/\beta$. Thus

$$\mathbf{P}(\sup_{x\ge 0} L(x,T_r)\ge \beta r) = \frac{1}{\beta}. \tag{2.20}$$

Furthermore by the above independence we can conclude that

$$\mathbf{P}(\mu(y; L(y,T_r) > \beta r) = 0) = \left(1-\frac{1}{\beta}\right)^2. \tag{2.21}$$

# 3. PROOFS OF THE THEOREMS.

**Proof of Theorem 1.1.**
For $p>0$ apply Theorem F with

$$f(r) = \begin{cases} 0 & \text{if} \quad r<p \\ \lambda & \text{if} \quad r\ge p \end{cases} \tag{3.1}$$

and $h=+\infty$. Then by the theorem

$$\varphi_r^p(\lambda) = \mathbf{E}\left(\exp(-\lambda\mu(y; L(y,T_r) > p))\right) = \left(R_r^p(\lambda)\right)^2 \tag{3.2}$$

where $R_r^p(\lambda)$ is the bounded solution of the following problem;

$$\begin{aligned} 2rR''(r) &= 0 \quad \text{if} \quad r<p \\ 2rR''(r) - \lambda R(r) &= 0 \quad \text{if} \quad r\ge p \\ R(+0) &= 1. \end{aligned} \tag{3.3}$$

The general solution of the above problem is;

$$
\begin{aligned}
R^p(r) &= Ar + 1 \quad \text{if} \quad r < p \\
R^p(r) = B\sqrt{r}I_1(\sqrt{2\lambda r}) + C\sqrt{r}K_1(\sqrt{2\lambda r}) &\quad \text{if} \quad r \geq p \\
R(+0) &= 1.
\end{aligned} \tag{3.4}
$$

Because of the boundedness of the solution we must have $B = 0$. $A$ and $C$ can be calculated from the continuity of $R(.)$ and $R'(.)$. After some tedious calculations and using the following wellknown identities for the Bessel functions of the third kind (see e.g. [GR,80] formulas 8.486),

$$
K_1'(z) = -K_0(z) - \frac{K_1(z)}{z}, \qquad K_1(z) + \frac{z}{2}K_0(z) = \frac{z}{2}K_2(z) \tag{3.5}
$$

we get that

$$
A = -C\sqrt{\frac{\lambda}{2}}K_0(\sqrt{2\lambda p}), \qquad C = \frac{1}{p\sqrt{\lambda/2}\,K_2(\sqrt{2\lambda p})}. \tag{3.6}
$$

( **??**) and (**??**) easily leads to (**??**).
To get the $p = 0$ case we only have to consider the second line of (**??**) and take the limit as $p \to 0$. Using (**??**) we arrive to (**??**). $\square$

**Proof of Theorem 1.2.**
First recall that (see (**??**))

$$
\varphi_1^\beta(s) = \mathbf{E}\left(\exp(-s\mu(y; L(y,T_1) > \beta))\right) = \mathbf{E}\left(\exp(-sY_1(\beta))\right) \tag{3.7}
$$

and

$$
1 - F^\beta(x) = P(Y_1(\beta) > x). \tag{3.8}
$$

It is easy to show that for $s \to 0$

$$
1 - \varphi_1^\beta(s) = (1 + o(1))\, s \log\left(\frac{1}{s}\right). \tag{3.9}
$$

independently from the value of $\beta$. Integration by part reveals that

$$
\frac{1 - \varphi_1^\beta(s)}{s} = \int_0^\infty e^{-sx}(1 - F^\beta(x))\,dx. \tag{3.10}
$$

One can conclude based on a Tauberian theorem (our Theorem G) that

$$
\int_0^x (1 - F^\beta(u))\,du \sim c\log x \qquad \text{as } x \to \infty \tag{3.11}
$$

However to conclude the asymptotic behavior of $1 - F^\beta(x)$ as $x \to \infty$ one needs to follow a more delicate argument based on Fact 5.

We will show that for

$$\hat{U}^\beta(s) = \int_0^\infty e^{-xs}\, dU^\beta(x) = \int_0^\infty e^{-xs}(1 - F^\beta(x))\, dx = \frac{1 - \varphi_1^\beta(s)}{s} \tag{3.12}$$

where

$$U^\beta(x) = \int_0^x (1 - F^\beta(y))\, dy \tag{3.13}$$

(**??**) holds with $\ell(x) \equiv 1$ and $c = 1$. Thus introducing $s = 1/x$ ($s \to 0$ as $x \to \infty$) we show that for any fixed $\lambda > 0$

$$\lim_{s\to 0}\left(\hat{U}^\beta(\frac{s}{\lambda}) - \hat{U}^\beta(s)\right) = \log\lambda. \tag{3.14}$$

independently from the actual value of $\beta$.
This would imply according to Fact 5 that

$$1 - F^\beta(x) \sim \frac{1}{x} \qquad \text{as} \qquad x \to \infty. \tag{3.15}$$

Thus we have to show that (**??**) holds for all $\beta \geq 0$. We have to consider the $\beta = 0$ $0 < \beta \leq 1$, and the $\beta \geq 1$ cases separately. However observing that for any $0 \leq \beta_1 < \beta_2$

$$Y_1(\beta_1) \geq Y_1(\beta_2) \tag{3.16}$$

we conclude that

$$1 - F^{\beta_1}(x) = P(Y_1(\beta_1) > x) \leq P(Y_1(\beta_2) > x) = 1 - F^{\beta_2}(x). \tag{3.17}$$

Consequently it is enough to show that (**??**) holds for $\beta = 0$ and for $\beta \geq 1$.
We start with the $\beta = 0$ case. By (**??**) we have $\varphi_1^0(s) = 2sK_1^2(\sqrt{2s})$. Furthermore

$$\hat{U}^0(s) = \frac{1 - \varphi_1^0(s)}{s} = \frac{1}{s} - 2K_1^2(\sqrt{2s}). \tag{3.18}$$

Using (**??**) of Fact 2. we have by an easy but tedious calculation that as $s \to 0$

$$\begin{aligned}
&\hat{U}^0(\frac{s}{\lambda}) - \hat{U}^0(s) \\
&\qquad = \quad \frac{\lambda}{s} - 2K_1^2\left(\sqrt{\frac{2s}{\lambda}}\right) - \frac{1}{s} + 2K_1^2(\sqrt{2s})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{s} - 2\left(\sqrt{\frac{\lambda}{2s}} + \sqrt{\frac{s}{2\lambda}}\,\log\sqrt{\frac{s}{2\lambda}} + D\sqrt{\frac{2s}{\lambda}} + o(s)\right)^2 \\
&- \frac{1}{s} + 2\left(\sqrt{\frac{1}{2s}} + \sqrt{\frac{s}{2}}\,\log\sqrt{\frac{s}{2}} + D\sqrt{2s} + o(s)\right)^2 \\
&= \frac{\lambda}{s} - 2\left(\frac{\lambda}{2s} + \log\sqrt{\frac{s}{2\lambda}} + 2D + o(\sqrt{s})\right) \\
&- \frac{1}{s} + 2\left(\frac{1}{2s} + \log\sqrt{\frac{s}{2}} + 2D + o(\sqrt{s})\right) \\
&= -2\log\sqrt{\frac{s}{2\lambda}} + 2\log\sqrt{\frac{s}{2}} + o(\sqrt{s}) = \log\lambda + o(\sqrt{s}). \qquad (3.19)
\end{aligned}
$$

In the above calculation the $o(.)$-s might depend on $\lambda$. Thus we arrive to

$$
\lim_{s\to 0} \hat{U}^{\,0}(\frac{s}{\lambda}) - \hat{U}^{\,0}(s) = \log\lambda. \qquad (3.20)
$$

proving the theorem for $\beta = 0$. Now we have to consider the $\beta \geq 1$ case in a similar manner. We need to use the first line of (**??**). We get that

$$
\frac{1-\varphi^{\beta}(s)}{s} = \frac{2}{\beta s}\frac{K_0(\sqrt{2s\beta})}{K_2(\sqrt{2s\beta})} - \frac{1}{\beta^2 s}\left(\frac{K_0(\sqrt{2s\beta})}{K_2(\sqrt{2s\beta})}\right)^2. \qquad (3.21)
$$

Using (**??**) and (**??**) it is easy to conclude that for any small enough $\epsilon > 0$

$$
\frac{K_0(z)}{K_2(z)} = -\frac{z^2}{2}\log\frac{z}{2} + \psi(1)\frac{z^2}{2} + o(z^{4-\epsilon}) \qquad \text{as} \qquad z \to 0. \qquad (3.22)
$$

and consequently

$$
\left(\frac{K_0(z)}{K_2(z)}\right)^2 = o(z^{4-\epsilon}) \qquad \text{as} \qquad z \to 0. \qquad (3.23)
$$

Based on the above asymptotics, we get that as $s \to 0$

$$
\begin{aligned}
&\hat{U}(\frac{s}{\lambda}) - \hat{U}(s) \\
&= \frac{2\lambda}{\beta s}\left(-\frac{s\beta}{\lambda}\log\sqrt{\frac{s\beta}{2\lambda}} + \psi(1)\frac{s\beta}{\lambda} + o\left(\frac{s}{\lambda}\right)^{2-\epsilon'}\right) + \frac{\lambda}{\beta^2 s}o(s^{2-\epsilon'}) \\
&- \left(\frac{2}{\beta s}\left(-s\beta\log\sqrt{\frac{s\beta}{2}} + \psi(1)\,s\beta + o(s)^{2-\epsilon'}\right) + \frac{1}{\beta^2 s}o(s^{2-\epsilon'})\right). \qquad (3.24)
\end{aligned}
$$

Thus we get

$$\lim_{s\to 0} \hat{U}(\frac{s}{\lambda}) - \hat{U}(s) = \log\lambda \tag{3.25}$$

in this case as well. □

**Proof of the divergent part of Theorem 1.3.**

Assume now that $I(f) = +\infty$. Select and fix an arbitrary big $K > 0$. Let $r_k = \rho^k$ with some $\rho > 1$. We show that for any fixed $\beta > 0$

$$\mathbf{P}(\mu(y; L(y, T_{r_k}) \geq \beta r_k) \geq K r_k f(r_k) \quad \text{i.o.}) = 1. \tag{3.26}$$

We define two sequences of events $\{A_k\}$ $\{B_k\}$ $k = 1, 2, ...$

$$A_k \stackrel{def}{=} \{\mu(y; L(y, T_{r_k}) \geq \beta r_k) > K r_k f(r_k)\} \tag{3.27}$$

$$B_k \stackrel{def}{=} \{\mu(y; L(y, T_{r_k}) - L(y, T_{r_{k-1}}) \geq \beta r_k) > K r_k f(r_k)\}. \tag{3.28}$$

Then clearly $B_k$ implies $A_k$ and by the strong Markov property

$$\mu(y; L(y, T_{r_k}) - L(y, T_{r_{k-1}}) > x) \stackrel{\mathcal{D}}{=} \mu(y; L(y, T_{r_k - r_{k-1}}) \geq x). \tag{3.29}$$

Furthermore the events $\{B_k\}$ $k = 1, 2, ...$ are independent. Thus we only have to prove that

$$\sum_{k=1}^{\infty} \mathbf{P}(B_k) = +\infty. \tag{3.30}$$

Observe now that according to (**??**) and Theorem 1.2 and scale change we have

$$\begin{aligned} \mathbf{P}(B_k) = \mathbf{P}(\mu(y; L(y, T_{r_k - r_{k-1}}) > \beta r_k) > K r_k f(r_k)) &= \\ \mathbf{P}(\mu(y; L(y, T_{r_k - r_{k-1}}) > (r_k - r_{k-1})\beta \frac{r_k}{r_k - r_{k-1}}) > (r_k - r_{k-1})\frac{r_k}{r_k - r_{k-1}} K f(r_k)) &= \\ \mathbf{P}(Y_1(\beta \frac{\rho}{\rho - 1}) > \frac{\rho}{\rho - 1} K f(\rho^k)) &\sim \\ \frac{\rho - 1}{\rho K f(\rho^k)}. \end{aligned} \tag{3.31}$$

Using Fact 6 we conlude from (**??**) that (**??**) holds. Hence

$$\mathbf{P}(B_k \quad \text{i.o.}) = 1$$

which implies that

$$\mathbf{P}(A_k \quad \text{i.o.}) = 1.$$

Now sending $K \to +\infty$ we proved the divergent part of the theorem. $\Box$

**Proof of Theorem 1.4.**:
According to (**??**) - (**??**) and (**??**) for $0 < \beta < 1$ we have

$$\begin{aligned}
\varphi_1^\beta(s) = \left(R_1^\beta(s)\right)^2 &= \frac{2}{s\beta^2}\left(\frac{K_1(\sqrt{2s})}{K_2(\sqrt{2s\beta})}\right)^2 \\
&\sim \frac{2}{s\beta^2}\left(\sqrt{\beta}e^{-2\left(\sqrt{2s}-\sqrt{2s\beta}\right)}\right) \\
&= \frac{2}{s\beta^{\frac{3}{2}}}e^{-\sqrt{s}\sqrt{8}\left(1-\sqrt{\beta}\right)} \quad \text{as} \quad s \to \infty. \qquad (3.32)
\end{aligned}$$

Now we can apply Fact 7 with $a = \sqrt{8}\left(1 - \sqrt{\beta}\right)$ thus we have

$$\lim_{x\to 0} x \log P(Y_1(\beta) \le x) = -2(1 - \sqrt{\beta})^2 = -2K(\beta). \qquad (3.33)$$

In case $\beta = 0$, using (**??**), (**??**) and (**??**) we get that

$$\varphi_1^0(s) \sim \pi\sqrt{\frac{s}{2}}\exp(-\sqrt{8s}). \qquad (3.34)$$

Observe that now Fact 7 is applicable again with $a = \sqrt{8}$, thus (**??**) holds for $\beta = 0$ as well. Concerning now the $\beta = 1$ case we get from the first line of (**??**) with the help of (**??**) and (**??**)

$$R_1^1(s) = 1 - \frac{K_0(\sqrt{2s})}{K_2(\sqrt{2s})} = \sqrt{\frac{2}{s}}\frac{K_1(\sqrt{2s})}{K_2(\sqrt{2s})} \sim \sqrt{\frac{2}{s}} \qquad \text{as } s \to \infty. \qquad (3.35)$$

Consequently by (**??**)

$$\varphi_1^1(s) \sim \frac{2}{s} \qquad \text{as } s \to \infty. \qquad (3.36)$$

Applying now the Tauberian theorem (see e.g. [Fe,70] Theorem 2 on page 445) we get that

$$\mathbf{P}(Y_1(1) \le x) \sim 2x \qquad \text{as } x \to 0, \qquad (3.37)$$

Turning to the $\beta > 1$ case we have seen (see (**??**))

$$R_1^\beta(s) = 1 - \frac{1}{\beta}\frac{K_0(\sqrt{2s\beta})}{K_2(\sqrt{2s\beta})} = \frac{\beta K_2(\sqrt{2s\beta}) - K_0(\sqrt{2s\beta})}{\beta K_2(\sqrt{2s\beta})} =$$

$$\frac{\beta(K_2(\sqrt{2s\beta}) - K_0(\sqrt{2s\beta})) + (\beta-1)K_0(\sqrt{2s\beta})}{\beta K_2(\sqrt{2s\beta})} =$$

$$\frac{2}{\sqrt{2s\beta}}\frac{K_1(\sqrt{2s\beta})}{K_2(\sqrt{2s\beta})} + \left(1-\frac{1}{\beta}\right)\frac{K_0(\sqrt{2s\beta})}{K_2(\sqrt{2s\beta})} \tag{3.38}$$

where in the last line we used (**??**). Now as $\varphi_1^\beta(s) = \left(R_1^\beta(s)\right)^2$ we have that

$$\varphi_1^\beta(s) - \left(1-\frac{1}{\beta}\right)^2 = \varphi_1^\beta(s) - \mathbf{P}(Y_1(\beta)=0) = \int_{0^+}^\infty e^{-su}d\mathbf{P}(Y_1(\beta)\le u). \tag{3.39}$$

We get from (**??**) with easy but tedious calculation using (**??**) and (**??**) again that for $s \to +\infty$

$$\varphi_1^\beta(s) - \left(1-\frac{1}{\beta}\right)^2 \sim \frac{c^*(\beta)}{\sqrt{s}} \tag{3.40}$$

where $c^*(\beta) = \frac{2\sqrt{2}}{\sqrt{\beta}}(1-\frac{1}{\beta})(2-\frac{1}{\beta}) \le 4\sqrt{2}$. Now (**??**) follows from (**??**) and the ordinary Tauberian theorem (see.g. [Fe,70] Theorem 2 on page 445) proving our theorem. □

**Proof of the upper bound (divergent part) of Theorem 1.5**
We will show that for any fixed $0<\beta<1$ and arbitrary small fixed $\eta>0$ there exists a sequence $r_k \to \infty$ such that for

$$B_k \stackrel{def}{=} \left\{\mu(y; L(y,T_{r_k}) > r_k\beta) \le \frac{(1+3\eta)2K(\beta)r_k}{\log\log r_k}\right\} \tag{3.41}$$

$$\mathbf{P}(B_k \quad \text{i.o.}) = 1. \tag{3.42}$$

Let $r_k = k^{2k}$. Define

$$A_k \stackrel{def}{=} \left\{\mu(y; L(y,T_{r_k}) - L(y,T_{r_{k-1}}) > r_k\beta) \le \frac{(1+2\eta)2K(\beta)r_k}{\log\log r_k}\right\}. \tag{3.43}$$

First we show that for $k$ big enough $A_k$ imply $B_k$. Recall the definition of $Q(r)$, in (**??**) the range of the Wiener process up to $T_r$. For arbitrary small $\delta>0$ by Theorem H we have for $k>k_0(\omega)$

$$Q(r_{k-1}) \le r_{k-1}(\log(r_{k-1}))^{(1+\delta)} \le \frac{\eta 2K(\beta)r_k}{\log\log r_k}, \tag{3.44}$$

where the second inequality in (**??**) can be seen with easy calculation using the the definition of $r_k$. Observe that $L(y, T_{r_k}) \neq L(y, T_{r_k}) - L(y, T_{r_{k-1}})$ can only occur if $y$ is visited before $T_{r_{k-1}}$, thus

$$\mu(y; L(y, T_{r_k}) > r_k\beta) \leq \mu(y; L(y, T_{r_k}) - L(y, T_{r_{k-1}}) > r_k\beta) + Q(r_{k-1}) \tag{3.45}$$

hence by (**??**) for $k$ big enough $A_k$ implies $B_k$. Thus we only have to prove

$$\mathbf{P}(A_k \quad \text{i.o.}) = 1. \tag{3.46}$$

Clearly the events $A_k$ are independent and by the strong Markov property and scale change and the lower bound in Theorem 1.4 we have that for any $\epsilon > 0$ and $k$ big enough

$$\begin{aligned}
P(A_k) = \mathbf{P}\left(\mu(y; L(y, T_{r_k - r_{k-1}}) > r_k\beta) \leq \frac{(1+2\eta)2K(\beta)r_k}{\log\log r_k}\right) &= \\
\mathbf{P}\left(\mu(y, L(y, T_1) > \beta\frac{r_k}{r_k - r_{k-1}}) \leq \frac{(1+2\eta)2K(\beta)r_k}{(r_k - r_{k-1})\log\log r_k}\right) &\geq \\
\mathbf{P}\left(\mu(y, L(y, T_1) > \beta) \leq \frac{(1+2\eta)2K(\beta)r_k}{(r_k - r_{k-1})\log\log r_k}\right) &\geq \\
\exp\left(-\frac{2\left(1-\sqrt{\beta}\right)^2(1+\epsilon)(r_k - r_{k-1})}{(1+2\eta)2(1-\sqrt{\beta})^2 r_k}\log\log r_k\right) &\geq \\
\exp\left(-\frac{(1+\epsilon)}{(1+2\eta)}\log\log r_k\right)&.
\end{aligned} \tag{3.47}$$

Clearly when we apply Theorem 1.4 we can select $\epsilon = \eta$. With this selection we conclude that

$$\exp\left(-\frac{(1+\eta)}{(1+2\eta)}\log\log r_k\right) \geq \exp\left(-\log\log r_k\right) \geq \frac{1}{2k\log k}. \tag{3.48}$$

By (**??**) $\sum_k P(A_k) = +\infty$ thus we have (**??**) which implies (**??**). Sending now $\eta \to 0$ gives the theorem. □

**Proof of Theorem 1.6**
First recall that according to Fact 9 for any $\beta > 1$

$$\mathbf{P}(\sup_{x\geq 0} L(x, T_r) \geq \beta r) = \frac{1}{\beta} \tag{3.49}$$

and also

$$\mathbf{P}(\sup_{-\infty<x<+\infty} L(x,T_r)<\beta r)=\mathbf{P}(\mu(y;L(y,T_r)>\beta r)=0)=\left(1-\frac{1}{\beta}\right)^2>0. \tag{3.50}$$

(**??**) combined with the tail $\sigma-$field form of Blumenthal's zero-one law [Bl,57] (see this e.g in Durett [Du, 96] Theorem 2.12, page 17) implies that

$$\mathbf{P}(\liminf(\mu(y;L(y,T_r)>\beta r)=0)=1. \tag{3.51}$$

□

**Proof of Theorem 1.7**
Let $r_k=k^{4k}$. Define for an arbitrary small $\gamma>0$

$$A_k=\{\mu(y;L(y,T_{r_k})-L(y,T_{r_{k-1}})>r_k-r_{k-1})<\frac{\gamma\, r_k}{2\log r_k}\}. \tag{3.52}$$

Then using (**??**) the strong Markov property and scale change we get that

$$\begin{aligned}\mathbf{P}(A_k)=\mathbf{P}(\mu(y;L(y,T_{r_k-r_{k-1}})>r_k-r_{k-1})<\frac{\gamma\, r_k}{2\log r_k}) &\sim \\ \frac{\gamma r_k}{(r_k-r_{k-1})\log r_k}>\frac{\gamma}{\log r_k}=\frac{\gamma}{4k\log k}&.\end{aligned} \tag{3.53}$$

We conclude that $\sum_k \mathbf{P}(A_k)=+\infty$. By the independence of the $A_k$-s and the selection of the sequence $r_k$ we have that

$$\mathbf{P}(A_k \quad \text{i.o.})=1. \tag{3.54}$$

Let

$$B_k=\{\mu(y;L(y,T_{r_k})-L(y,T_{r_{k-1}})>r_k)<\frac{\gamma\, r_k}{2\log r_k}\}. \tag{3.55}$$

Then clearly $A_k\subset B_k$, hence

$$\mathbf{P}(B_k \quad \text{i.o.})=1. \tag{3.56}$$

On the other hand by Theorem 1.3 we have that for k big enough and arbitrary $\epsilon>0$

$$Q(T_{r_{k-1}})\le (k-1)^{4(k-1)}(4(k-1)\log(k-1))^{1+\epsilon}<\frac{\gamma k^{4k}}{2(4k\,\log k)} \tag{3.57}$$

thus (**??**) implies that

$$\mathbf{P}(Y_{r_k}(1) < \frac{\gamma r_k}{\log r_k} \quad \text{i.o.}) = 1 \tag{3.58}$$

Sending now $\gamma \to 0$ proves (**??**). □

**Proof of the convergent part of Theorem 1.8.**
Suppose that for an $f(.)$ satisfying the conditions of the theorem we have $I(f) < +\infty$. Fix $\beta \geq 0$.
First assume that we can prove for $r_k = \rho^k$ with some $\rho > 1$ that for an arbitrary fixed $\epsilon > 0$

$$\sum_{k=1}^{\infty} \mathbf{P}(\mu(y; L(y, T_{r_k}) > \beta r_{k-1}) > \epsilon\, r_{k-1} f(r_{k-1})) < \infty. \tag{3.59}$$

We show that (**??**) implies the convergent part. If (**??**) holds then for $k > k_0(\omega)$ we have

$$\mu(y; L(y, T_{r_k}) > \beta r_{k-1}) \leq \epsilon\, r_{k-1} f(r_{k-1})). \tag{3.60}$$

If $T_{r_{k-1}} \leq t < T_{r_k}$ we have that

$$\begin{aligned} \mu(y; L(y,t) > \beta L(0,t)) \leq \mu(y; L(y, T_{r_k}) > \beta r_{k-1}) \leq \epsilon\, r_{k-1} f(r_{k-1}) \quad = \\ \epsilon L(0, T_{r_{k-1}}) f(L(0, T_{r_{k-1}})) \leq \epsilon\, L(0,t) f(L(0,t)). \end{aligned} \tag{3.61}$$

So we only have to prove (**??**). By Theorem 1.2 and scale change we have

$$\begin{aligned} \mathbf{P}(\mu(y; L(y, T_{r_k}) > \beta r_{k-1}) > \epsilon\, r_{k-1} f(r_{k-1})) &= \mathbf{P}(\mu(y; L(y, T_{\rho^k}) > \beta \rho^{k-1}) > \epsilon\, \rho^{k-1} f(\rho^{k-1})) \\ &= \mathbf{P}\left(\mu\left(y; L(y, T_1) > \frac{\beta}{\rho}\right) > \frac{\epsilon}{\rho} f(\rho^{k-1})\right) \\ &\sim \frac{\rho}{\epsilon} \frac{1}{f(\rho^{k-1})}. \end{aligned} \tag{3.62}$$

But the last expression sums in $k$ as $I(f)$ is finite by Fact 6, which proves the convergent part. □

**Proof of the lower bound (convergent part) of 1.9.**
We will need the following wellknown observation. From the exact distribution of $T_r/r^2$ and the Borel -Cantelli lemma one can easily conclude that for a small enough $C > 0$

$$T_r > \frac{Cr^2}{\log \log r} \qquad \text{a.s.} \tag{3.63}$$

if $r$ is large enough. (**??**) easily implies that for $r$ large enough we have

$$\log\log T_r > \log\log r \qquad \text{a.s..} \tag{3.64}$$

Select and fix an arbitrary small $\epsilon > 0$. Let $r_k = \exp(k^{(1-\epsilon)})$, and now select a small enough $0 < \rho < \epsilon$ such that $\beta(1+\rho) < 1$ and

$$\left(\frac{1-\sqrt{\beta(1+\rho)}}{1-\sqrt{\beta}}\right)^2 > 1-\epsilon \tag{3.65}$$

should hold. Define the events

$$A_k = \left\{ \inf_{T_{r_{k-1}} \le t < T_{r_k}} \frac{\mu(y, L(y,t) > \beta L(0,t)) \log\log t}{L(0,t)} \le 2K(\beta)(1-5\epsilon) \right\}. \tag{3.66}$$

We show that $\sum_k \mathbf{P}(A_k) < +\infty$.

Select $k_0$ big enough such that for $k > k_0$ $\frac{r_k}{r_{k-1}} < 1+\rho$ should hold. Using again scale change and the upper bound in Theorem 1.4, (**??** and (**??**) we get that for $k > k_0$

$$\begin{aligned}
\mathbf{P}(A_k) \le \mathbf{P}\left( \inf_{T_{r_{k-1}} \le t < T_{r_k}} \mu(y, L(y,t) > \beta L(0,t)) \le \frac{2K(\beta)(1-5\epsilon)L(0,T_{r_k})}{\log\log T_{r_{k-1}}} \right) &\le \\
\mathbf{P}\left( \mu(y, L(y, T_{r_{k-1}}) > \beta L(0, T_{r_k})) \le \frac{2K(\beta)(1-5\epsilon) r_k}{\log\log r_{k-1}} \right) &= \\
\mathbf{P}\left( \mu(y, L(y, T_1) > \beta \frac{r_k}{r_{k-1}}) \le \frac{2K(\beta)(1-5\epsilon) r_k}{r_{k-1}\log\log r_{k-1}} \right) &\le \\
\mathbf{P}\left( \mu(y, L(y, T_1) > \beta(1+\rho)) \le \frac{2K(\beta)(1-5\epsilon) r_k}{r_{k-1}\log\log r_{k-1}} \right) &\le \\
\exp\left( -\frac{2(1-\sqrt{\beta(1+\rho)})^2(1-\epsilon) r_{k-1}}{(1-5\epsilon)2(1-\sqrt{\beta})^2 r_k} \log\log r_{k-1} \right) &\le \\
\exp\left( -\frac{(1-\epsilon)^2 r_{k-1}}{(1-5\epsilon) r_k} \log\log r_{k-1} \right) &=: p(k,\epsilon).
\end{aligned} \tag{3.67}$$

Observe now that by the choice of $k_0$, and by the selection of $\rho < \epsilon$ we have for $k > k_0$ that $\frac{r_{k-1}}{r_k} > 1-\epsilon$. Then we have

$$p(k,\epsilon) \le \exp\left( -\frac{(1-\epsilon)^4}{(1-5\epsilon)} \log(k-1) \right) \le \exp\left(-(1+\epsilon)\log(k-1)\right) \tag{3.68}$$

which sums in $k$. Hence $\sum_k \mathbf{P}(A_k) < \infty$, thus

$$\left\{ \inf_{T_{r_{k-1}} \le t < T_{r_k}} \frac{\mu(y, L(y,t) > \beta L(0,t)) \log\log t}{L(0,t)} > 2K(\beta)(1-5\epsilon) \right\} \quad \text{a.s.} \tag{3.69}$$

and by sending $\epsilon \to 0$ we have the theorem. $\square$

**Proof of (??) of Theorem 1.11.**
Select and fix an arbitrary large $K > 0$. Let $r_k = \exp(k^\alpha)$ with some $0 < \alpha < 1$. Define

$$A_k = \{\mu(y, L(y, T_{r_{k-1}}) > r_k) \le \frac{K r_k}{(\log r_k)^\beta}\}. \tag{3.70}$$

We want to show that for an appropriate choice of $\alpha$ and $\beta$

$$\sum_k \mathbf{P}(A_k) < \infty. \tag{3.71}$$

Using the exponential Markov inequality and Theorem 1.1 with $p = r_k$ and $r = r_{k-1}$ we get that for any $\lambda > 0$

$$\begin{aligned} \mathbf{P}(A_k) < \exp\left(\frac{\lambda K r_k}{(\log r_k)^\beta}\right) \mathbf{E}\left(\exp(-\lambda \mu(y, L(y, T_{r_{k-1}}) > r_k)\right) &= \\ \exp\left(\frac{\lambda K r_k}{(\log r_k)^\beta}\right) \left(1 - \frac{r_{k-1}}{r_k} \frac{K_0(\sqrt{2\lambda r_k})}{K_2(\sqrt{2\lambda r_k})}\right)^2 &. \end{aligned} \tag{3.72}$$

Observe that by (**??**) for $x \to \infty$

$$\frac{K_0(x)}{K_2(x)} = \frac{1 - \frac{1}{8x} + O\left(\frac{1}{x^2}\right)}{1 + \frac{15}{8x} + O\left(\frac{1}{x^2}\right)} = 1 - \frac{2}{x} + O\left(\frac{1}{x^2}\right). \tag{3.73}$$

Now it is an easy computation to see that for $k$ big enough we have

$$\frac{r_k - r_{k-1}}{r_k} \sim \frac{r_{k-1}}{r_k} \frac{\alpha}{k^{1-\alpha}} \le \frac{\alpha}{k^{1-\alpha}}. \tag{3.74}$$

Now select $\lambda = \frac{(\log r_k)^\beta}{r_k}$ , then apply (**??**) and (**??**) to get

$$\mathbf{P}(A_k) < e^K \left(1 - \frac{r_{k-1}}{r_k} \frac{K_0\left(\sqrt{2(\log r_k)^\beta}\right)}{K_2\left(\sqrt{2(\log r_k)^\beta}\right)}\right)^2 \quad <$$

$$c_1\left(\frac{\alpha}{k^{1-\alpha}}+\frac{c_2}{(\log r_k)^{\beta/2}}\right)^2 =$$

$$c_1\left(\frac{\alpha}{k^{1-\alpha}}+\frac{c_2}{k^{\alpha\beta/2}}\right)^2 \tag{3.75}$$

where $c_1$ and $c_2$ are positive constant, the first of which depends on $K$. For arbitrary small $\epsilon > 0$ we select now $\beta = 2+\epsilon$ and $\alpha = 2/(4+\epsilon)$. With this choice of $\alpha$ and $\beta$ we have $1-\alpha = \alpha\beta/2 = (2+\epsilon)/(4+\epsilon) > 1/2$. Hence $\sum_k \mathbf{P}(A_k)$ is convergent proving (**??**). Consequently for $k$ big enough

$$\mu(y, L(y, T_{r_{k-1}}) > r_k) \geq \frac{K r_k}{(\log r_k)^{2+\epsilon}} \qquad \text{a.s..} \tag{3.76}$$

For $T_{r_{k-1}} < t < T_{r_k}$ we get from (**??**) by monotonicity and using $r_k = L(0, T_{r_k})$ that

$$\mu(y, L(y,t) > L(0, T_{r_k})) \geq \frac{K L(0, T_{r_k})}{(\log L(0, T_{r_k}))^{2+\epsilon}} \qquad \text{a.s.} \tag{3.77}$$

implying that

$$\mu(y, L(y,t) > L(0,t)) \geq \frac{K L(0,t)}{(\log L(0,t))^{2+\epsilon}}. \tag{3.78}$$

Sending $K \to \infty$ proves the theorem.

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