The geometry of knot complements

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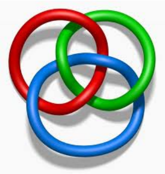
What is a knot ?

A knot is a (smooth) embedding of the circle S^1 in S^3. Similarly, a link of k-components is a (smooth) embedding of a disjoint union of k circles in S^3.

Figure-8 knot

Whitehead link



Borromean rings

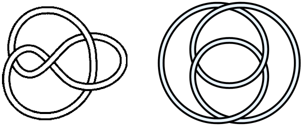
Two knots are equivalent if there is continuous deformation (ambient isotopy) of S^3 taking one to the other.

Goals: Find a practical method to classify knots upto equivalence.

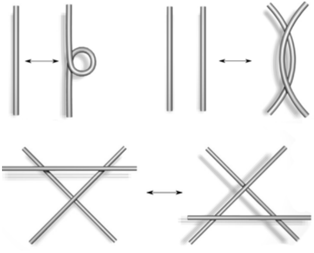
Knot Diagrams

A common way to describe a knot is using a planar projection of the knot which is a 4-valent planar graph indicating the over and under crossings called a knot diagram.

A given knot has many different diagrams.

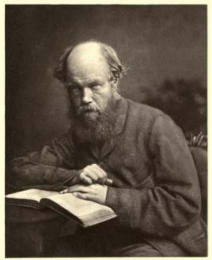


Two knot diagrams represent the same knot if and only if they are related by a sequence of three kinds of moves on the diagram called the Reidemeister moves.



Origins of Knot theory

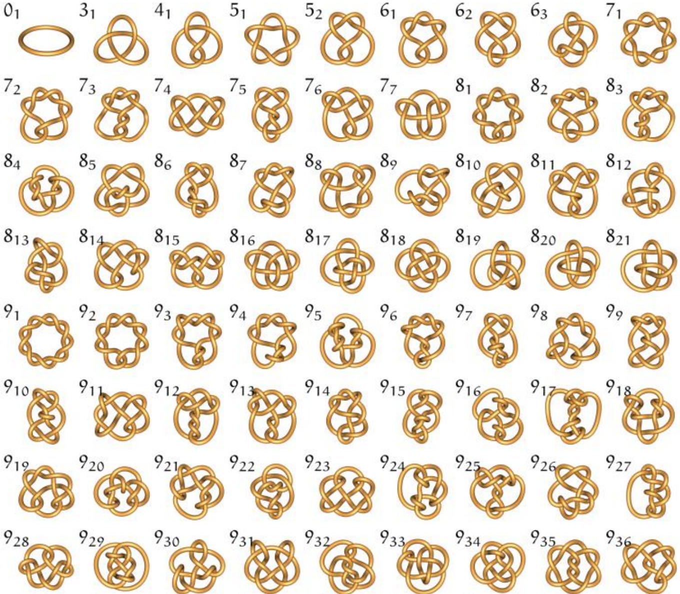
In 1867, Lord Kelvin conjectured that atoms were knotted tubes of ether and the variety of knots were thought to mirror the variety of chemical elements. This theory inspired the celebrated Scottish physicist Peter Tait to undertake an extensive study and tabulation of knots (in collaboration with C. N. Little).



Tait enumerated knots using their diagrammatic complexity called the crossing number of a knot, defined as the minimal number of crossings over all knot diagrams.

However, there is no ether ! So physicists lost interest in knot theory, till about 1980s.

Knot table by crossing number



Knot invariants

A knot invariant is a "quantity" that is equal for equivalent knots and hence can be used to tell knots apart.

Knot invariants appear in two basic flavors:

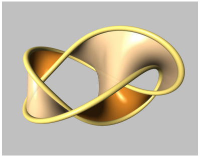
Topological invariants arising from the topology of the knot complement $S^3 - K$ e.g. Fundamental group.

Diagrammatic invariants arising from the combinatorics of the knot diagrams e.g. crossing number.

Topological knot invariants

Let $N(K)$ denote a tubular neighbourhood of the knot K. Then $M = S^3 - \overset{\circ}{N}(K)$ is a 3-manifold with a torus boundary. Examples of basic topological invariants:

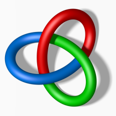
- Alexander polynomial (1927) describes the homology of the infinite cyclic cover as $\mathbb{Z}[t^{\pm}]$-module.
- Invariants from Seifert surfaces (1934) like knot genus, signature, determinant etc.



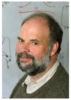
- Representation of $\pi_1(S^3 - K)$ into finite and infinite groups.
- Knot Heegaard Floer homology (Ozsvath-Szabo-Rasmussen, 2003) categorified the Alexander polynomial.

Diagrammatic knot invariants

Any quantity which is invariant under the three Reidemeister moves is a knot invariant. Examples of diagrammatic invariants:



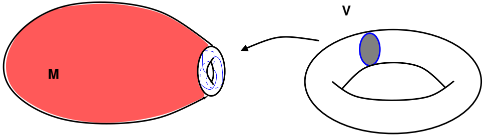
- Tricolorability, Fox n-colorings (Fox, 1956).



- Jones polynomial (1984), discovered via representations of braid groups, led to many new quantum invariants, which can be computed diagrammatically, e.g. Kauffman bracket (1987).
- Using Jones polynomial and relations to graph theory, Tait conjectures from 100 years were resolved (1987).
- Khovanov homology (1999) categorified the Jones polynomial.

Knots and 3-manifolds

Since the boundary of $M = S^3 - N(\overset{\circ}{K})$ is a torus, we can attach a solid torus by choosing a curve (p, q) on ∂M which becomes meridian. The resulting closed 3-manifold is called (p, q)-Dehn filling of M. In general the process of drilling a simple closed curve and filling it with a solid torus is called Dehn surgery.



Theorem (Lickorish-Wallace, 1960)

Any closed, orientable, connected 3-manifold can be obtained by Dehn surgery on a link in S^3.

Now enters geometry ....

In 1980s, William Thurstons seminal work established a strong connection between hyperbolic geometry and knot theory, namely that most knot complements are hyperbolic. Thurston introduced tools from hyperbolic geometry to study knots that led to new geometric invariants, especially hyperbolic volume.

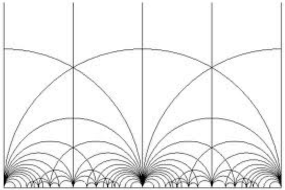
Basic hyperbolic geometry I

- The Poincaré half-space model of hyperbolic 3-space $\mathbb{H}^3 = \{(x, y, t) | t > 0\}$ with metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}t^2}{t^2}$. The boundary of $\mathbb{H}^3$ is $\mathbb{C} \cup \infty$ called the sphere at infinity.
- Geodesic planes ($\mathbb{H}^2$) are vertical planes or upper hemispheres of spheres orthogonal to the xy-plane (with centers on the xy-plane).
- Geodesics are lines or half circles orthogonal to the xy-plane.
- $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$ which acts as Mobius transforms on $\mathbb{C} \cup \infty$ extending this action by isometries.
- The horizontal planes ($t =$constant) are scaled Euclidean planes called horospheres.

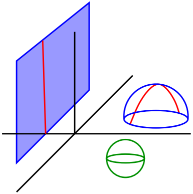
Basic hyperbolic geometry II

Escher's work using Poincare disc

Crochet by Daina Taimina



Hyperbolic upper-half plane



Hyperbolic upper-half space

Hyperbolic 3-manifolds

A 3-manifold M is said to be *hyperbolic* if it has a complete, finite volume hyperbolic metric.

- $\pi_1(M) = \Gamma$ acts by covering translations as isometries and hence has a discrete faithful representation in $\mathrm{PSL}(2, \mathbb{C})$.
- (Margulis 1978) If M is orientable and noncompact then $M = \mathring{M}'$ where $\partial M' = \cup\, T^2$. Each end is of the form $T^2 \times [0, \infty)$ with each section is scaled Euclidean metric, called a cusp.
- (Mostow-Prasad Rigidity, 1968) Hyperbolic structure on a 3-manifold is unique. This implies geometric invariants are topological invariants !

Thurston's Theorems

Theorem (Geometrization of Haken manifolds)

If M is a compact irreducible atoroidal Haken manifold with torus boundary, then the interior of M is hyperbolic.

Theorem (Geometrization of knot complements)

Every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot.

A torus knot is a knot which can be embedded on the torus as a simple closed curve. A satellite knot is a knot that contains an incompressible, non-boundary parallel torus in its complement.

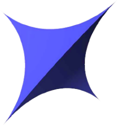
Theorem (Dehn Surgery Theorem)

Let $M = S^3 - K$, where K is a hyperbolic knot. Then (p, q)-Dehn filling on M is hyperbolic for all but finitely many (p, q).

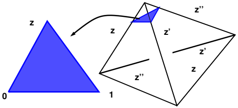
Ideal tetrahedra in $\mathbb{H}^3$

Hyperbolic 3-manifolds are formed by gluing hyperbolic polyhedra.

The basic building block is an ideal tetrahedra which is a geodesic tetrahedra in $\mathbb{H}^3$ with all vertices on $\mathbb{C} \cup \infty$.



An Ideal Tetrahedron



Isometry classes $\leftrightarrow \mathbb{C} - \{0, 1\}$. Every edge gets a complex number z called the edge parameter given by the cross ratio of the vertices. $z' = 1/(1-z)$ and $z'' = (z-1)/z$.

$\mathrm{Vol}(\triangle(z)) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| arg(1-z)$ where $\mathrm{Li}_2(z)$ is the dilogarithm function. $\mathrm{Vol}(\triangle(z)) \leq v_3 \approx 1.01494$, v_3 is the volume of the regular ideal tetrahedron (all dihedral angles $\pi/3$).

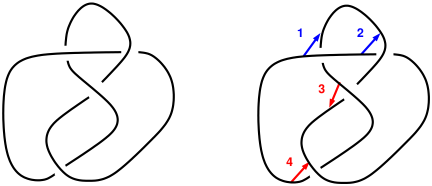
Ideal triangulations

An ideal triangulation of a cusped (non-compact) hyperbolic 3-manifold M is a decompostion of M into ideal tetrahedra glued along the faces with the vertices deleted.

Around every edge, the parameters multiply together to ± 1 ensuring hyperbolicity around the edges. The completeness condition gives a condition on every cusp torus giving a similar equation in the edge parameters. These are called gluing and completeness equations.

Thurston proved that the solution set to these equations is discrete, the paramters are algebraic numbers and give the complete hyperbolic structure on M. $\mathrm{Vol}(M)$ is a sum of volumes of ideal tetrahedra.

Example: Figure-8 knot



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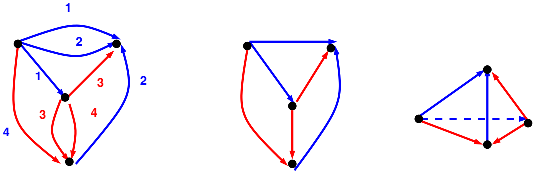

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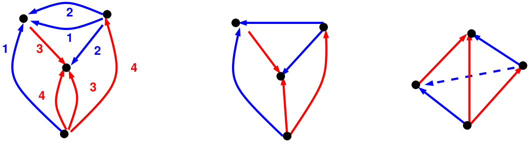


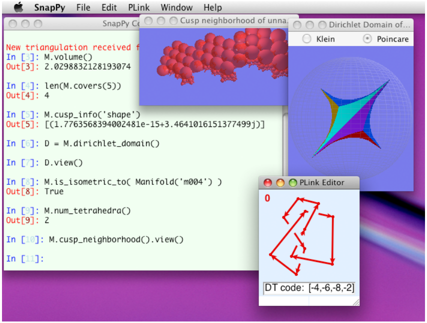
More hyperbolic structures

- Hyperbolic structures on some link complements can be described using circle packings and dual packings via the Koebe-Andreev-Thurston circle packing theorem which relates circle packings to triangulations of S^2.
- Hyperbolic structures on fibered 3-manifolds i.e. surface bundles can be obtained by using a psuedo-Anosov monodromy. The first examples of hyperbolic 3-manifolds were obtained as surface bundles by Jorgensen (1977).
- Hyperbolic 3-manifolds also arise as quotients of arithmetic lattices in $\mathrm{PSL}(2, \mathbb{C})$ e.g. finite index subgroups of $\mathrm{PSL}(2, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers of some number field.

Computing hyperbolic structures

The program SnapPea by Jeff Weeks (1999) computes hyperbolic structures and invariants on 3-manifolds and knots by triangulating and solving gluing equations. It also includes census of hyperbolic manifolds triangulated using at most 7 tetrahedra (4815 manifolds) and census of low volume closed hyperbolic 3-manifolds.

SnapPy by Culler and Dunfield is a modification of SnapPea which uses python interface. Snap by Goodman uses SnapPea to compute arithmetic invariants of hyperbolic 3-manifolds.



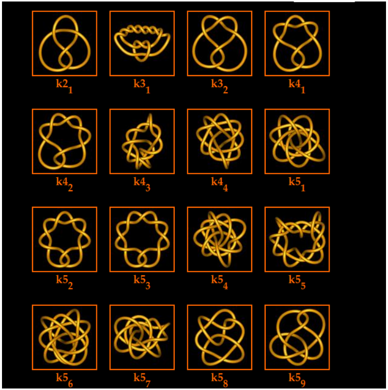
Simplest hyperbolic knots

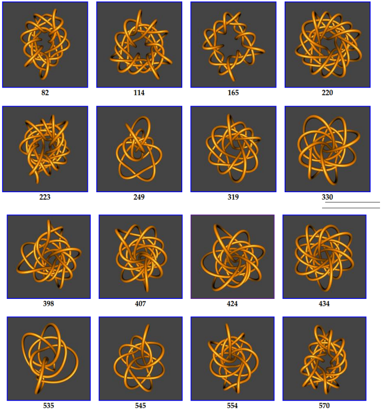
The geometric complexity is the minimum number of ideal tetrahedra used to triangulate a hyperbolic knot complement. The census of hyperbolic knots using this measure of complexity gives a different view of the space of all knots e.g. many of the geometrically simple knots have very high crossing numbers.

Hyperbolic knots with geometric complexity up to 6 tetrahedra were found by Callahan-Dean-Weeks (1999), extended to 7 tetrahedra by Champanerkar-Kofman-Paterson (2004).

Tetrahedra	1	2	3	4	5	6	7	8
Knots	0	1	2	4	22	43	129	299

Simplest hyperbolic knots





Extending the census of simplest hyperbolic knots

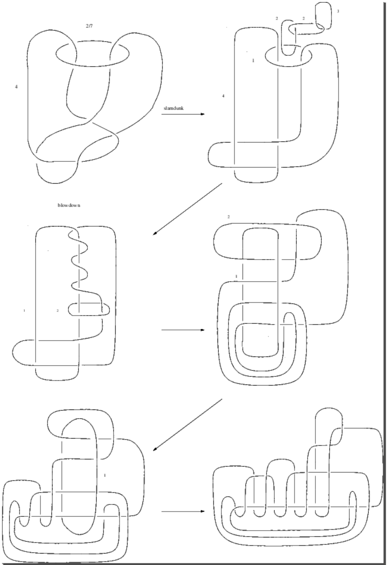
Started as an REU project by Tim Mullen at CSI.

- **Step 1 (Identifying the manifold):** Starting with Thistlethwaite's census of cusped hyperbolic 3-manifolds triangulated with 8 ideal tetrahedra (around 13000 manifolds, included in SnapPy), find one-cusped manifolds which have a Dehn filling homeomorphic to S^3, checked using fundamental group. Many cases involve using program testisom to simplify presentation of fundamental group.
- **Step 2 (Identify the knot):** Find a knot diagram. SnapPy can check isometry from diagram with manifold found in Step 1. Search through existing censuses of knots, generate families and search through them. After the searches we were left with around 20 knots.

Extending the census of simplest hyperbolic knots

Example

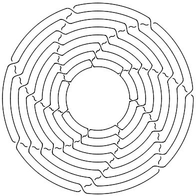
- **Step 3 (Kirby Calculus):** For knots not resolved after Step 2, find a surgery description of knots and use Kirby calculus to find knot diagram.



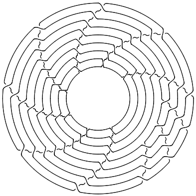
Twisted torus knots

A torus knot is a knot which can be embedded on a torus as a simple closed curve, and is parametrized by the slope p/q of the lift of this curve to $\mathbb{R}^2$. Torus knot $T(p, q)$ has p strands and q overpasses.

A twisted torus knot $T(p, q, r, s)$ is obtained by adding s overpasses to the outermost r strands. Twisted torus knots dominate the census of simplest hyperbolic knots.



$T(9, 7)$



$T(9, 7, 5, 3)$

Volume bounds on twisted torus knots

Theorem (C-Futer-Kofman-Neumann-Purcell, 2010)

Let $T(p,q,r,s)$ be a twisted torus knot. Then

$$\begin{aligned}
\mathrm{Vol}(T(p,q,r,s)) &< 10v_3 && \textit{if } r = 2,\\
\mathrm{Vol}(T(p,q,r,s)) &< v_3(2r+10) && \textit{if } s \bmod r = 0,\\
\mathrm{Vol}(T(p,q,r,s)) &< v_3(r^2+r+10) && \textit{if } s \bmod r \neq 0
\end{aligned}$$

Theorem (C-Futer-Kofman-Neumann-Purcell, 2010)

Choose any sequence $(p_N, q_N) \to (\infty, \infty)$, such that $\gcd(p_N, q_N) = 1$. Then the twisted torus knots $T(p_N, q_N, 2, 2N)$ have volume approaching $10v_3$ as $N \to \infty$.

Open Questions and Conjectures

Its an interesting problem to understand how the topological, diagrammatic and geometric invariants relate to one another.

- What is the topological or geometric interpretation of the Jones polynomial or Khovanov homology ?
- How do the geometric invariants relate to the quantum invariants ? Two big conjecture along these lines are the Volume Conjecture (Kashaev, Murakami, Murakami, 2001) and the AJ Conjecture (Garoufalidis, 2003).
- Understanding geometry of knot complements in terms of the combinatorics of knot diagrams is an active area of research.

Some links

Abhijit's Home page: http://www.math.csi.cuny.edu/abhijit/

KnotAtlas: http://katlas.math.toronto.edu/wiki/

SnapPy: http://www.math.uic.edu/ t3m/SnapPy/

KnotPlot: http://www.knotplot.com/

Thank you