The structure of high distance Heegaard splittings

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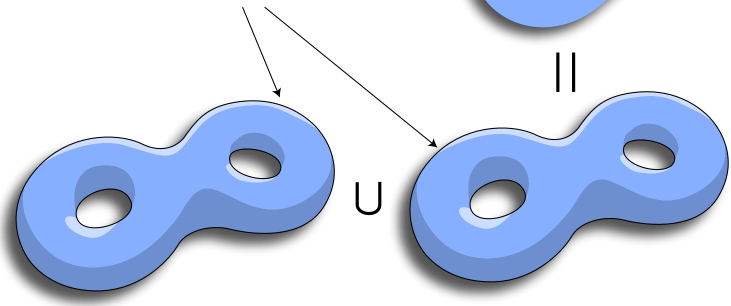
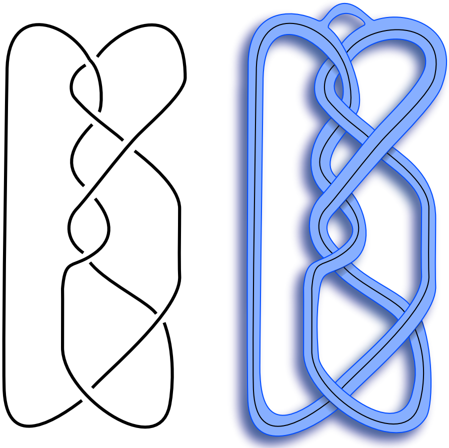
A *Heegaard splitting* is defined by gluing together two *handlebodies*.

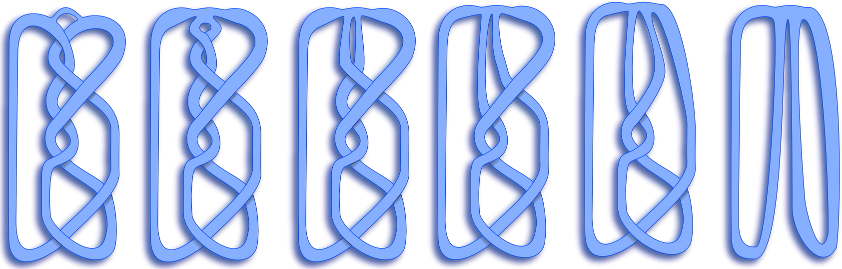


A 2-bridge knot complement

and a genus two surface.

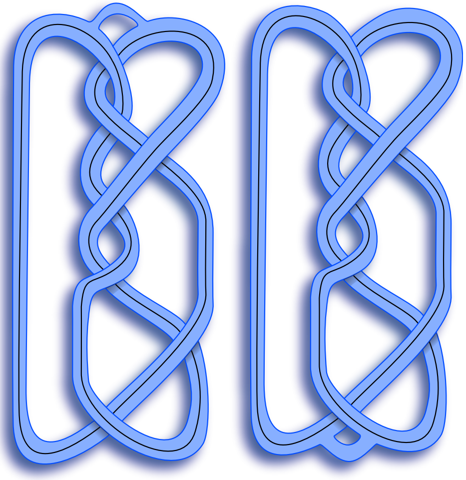


Inside the surface is a compression body

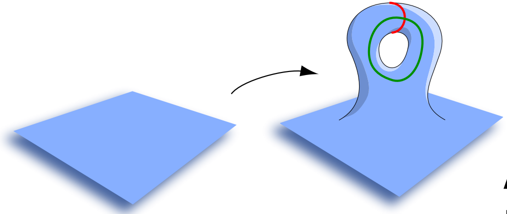


Outside the surface is a handlebody

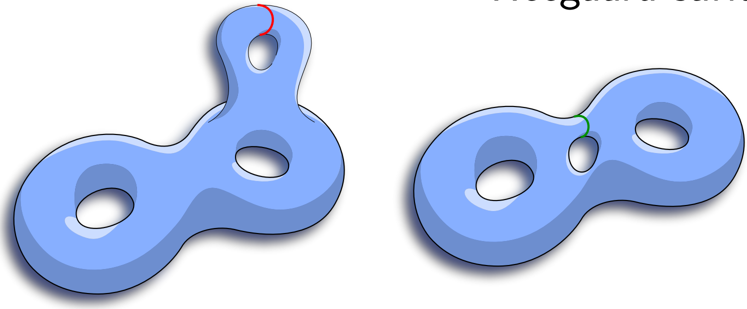
Two different Heegaard surfaces



(Four more not shown.)



A *stabilized* Heegaard surface.

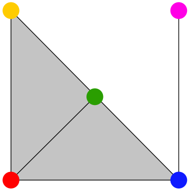


Question: Given a three-manifold, what are all its unstabilized Heegaard splittings?

Answered for:

1. S^3 (Waldhausen)
2. T^3 (Boileau–Otal)
3. Lens spaces (Bonahon–Otal)
4. (most) Seifert fibered spaces (Moriah–Schultens, Bachman–Derby-Talbot, J.)
5. Two-bridge knot complements (Morimoto–Sakuma, Kobayashi)

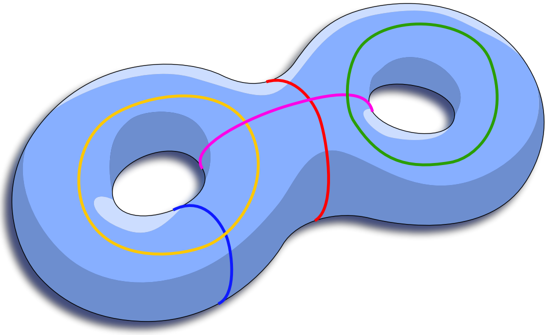
The complex of curves $\mathcal{C}(\Sigma)$:



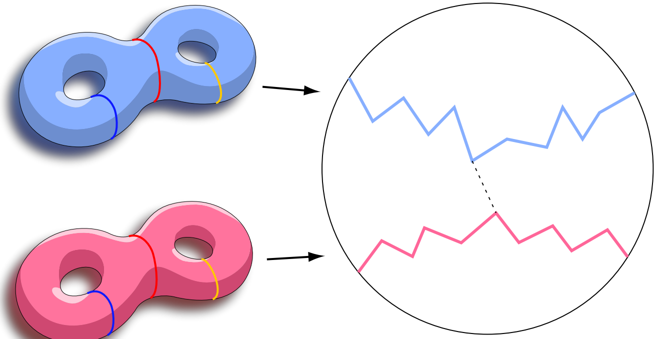
vertices: essential simple closed curves

edges: pairs of disjoint curves

simplices: sets of disjoint curves



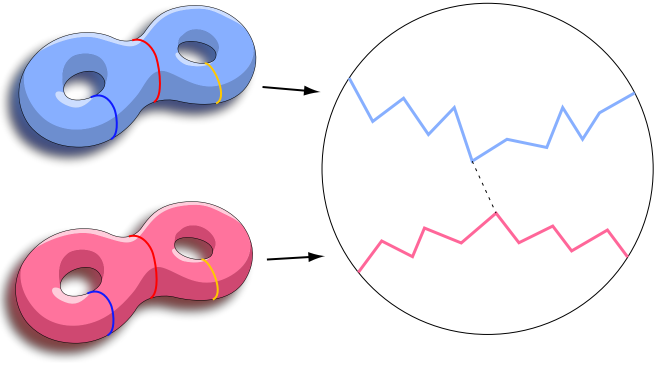
Handlebody sets - loops bounding disks



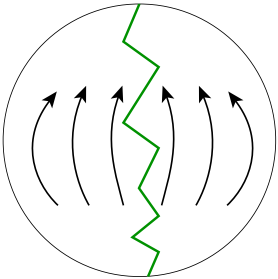
(Hempel) distance $d(\Sigma)$
- between handlebody sets

Theorem (Masur-Minsky): $\mathcal{C}(\Sigma)$ is δ-hyperbolic.

Handlebody sets are quasi-convex.

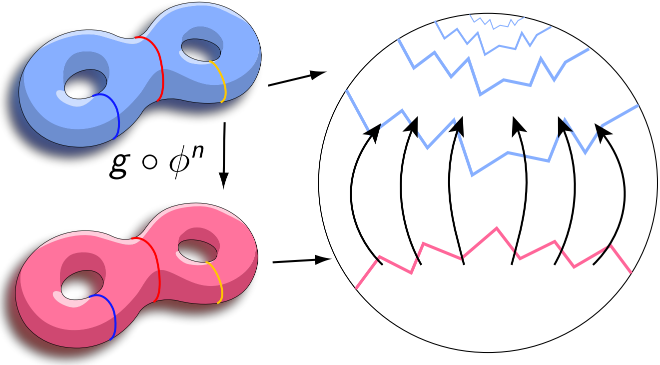


A surface self-homeo ϕ acts on $\mathcal{C}(\Sigma)$.



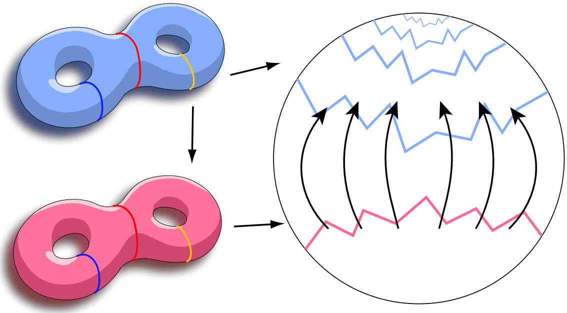
Theorem (Thurston): If ϕ has infinite order and no fixed loops then ϕ is *pseudo-Anosov*.

Theorem (Hempel): Composing the gluing map with (pseudo-Anosov) ϕ^n produces high distance Heegaard splittings.



Theorem (Hartshorn):

Evey incompressible surface in M has genus at least $\frac{1}{2}d(\Sigma)$.

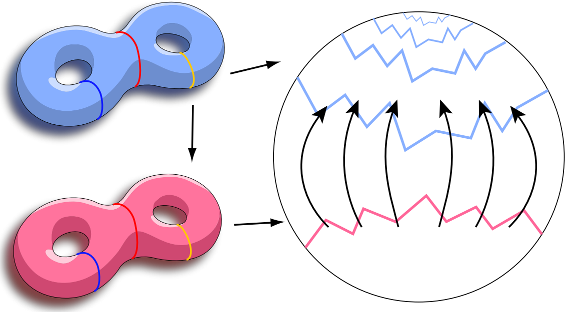


Theorem (Scharlemann-Tomova):

If $\frac{1}{2}d(\Sigma) > genus(\Sigma)$ then

the only unstabilized Heegaard surface in M

of genus less than $\frac{1}{2}d(\Sigma)$ is Σ.



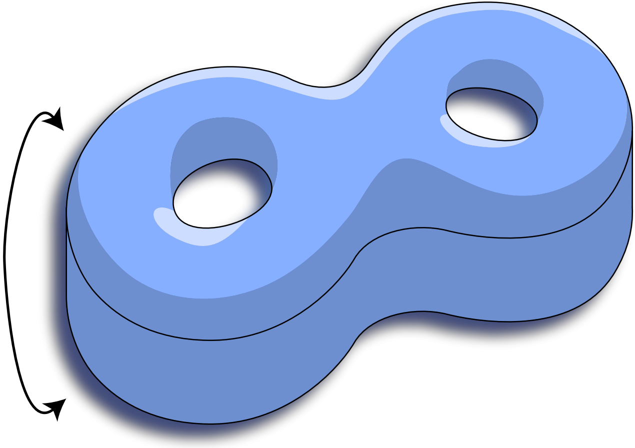
Theorem: Hartshorn's bound is Sharp.

Theorem: For any integers $d \geq 6$ (even), $g \geq 2$,
There is a three-manifold M with a genus g, distance d Heegaard splitting and an unstabilized genus $\frac{1}{2}d + (g-1)$ Heegaard splitting.

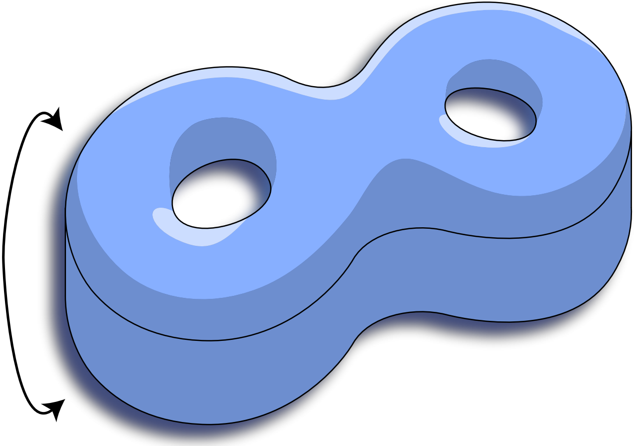
(Off from Scharlemann-Tomova bound by $g-1$.)

Surface bundle $B(\phi)$:

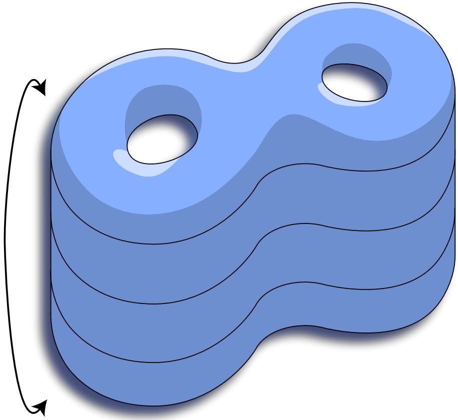
$$\Sigma \times [0,1]/((x,0) \sim (\phi(x),1)).$$



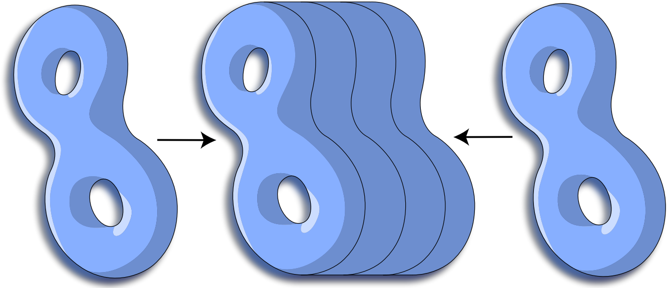
Theorem (Thurston): If ϕ is pseudo-Anosov then $B(\phi)$ is hyperbolic.



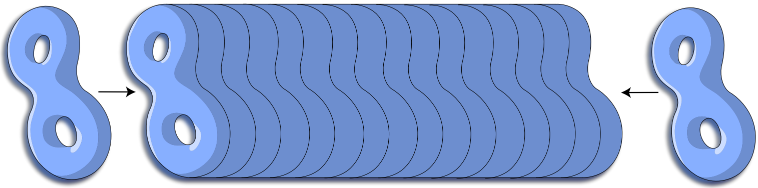
Note: $B(\phi^n)$ is a cyclic cover of $B(\phi)$.



A quasi-geometric Heegaard splitting

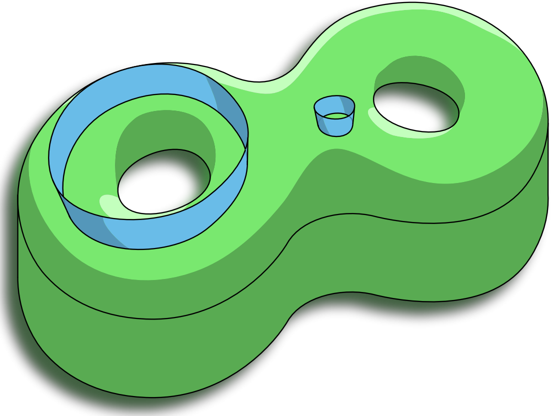


For large n, a better approximation



Namazi-Souto: Can construct a metric with ϵ-pinched curvature.

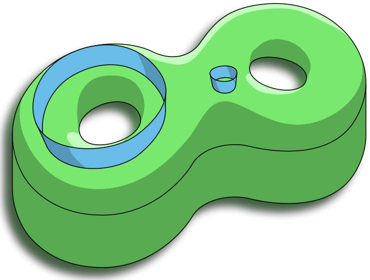
Lemma: Every incompressible surface F intersects every cross section Σ_t essentially.



Choose F to be harmonic so that the induced sectional curvature is less than that of M.

Theorem (Gauss-Bonnet): For bounded curvature, area is proportional to Euler characteristic.

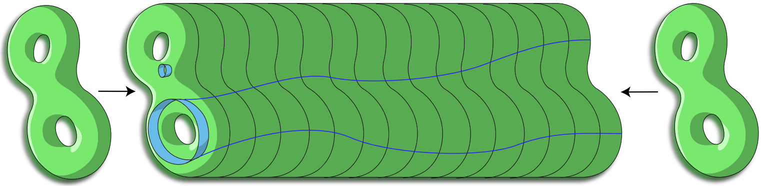
Note: Cross sections have bounded injectivity radius.



So, length of $F \cap \Sigma_t$ is bounded below.

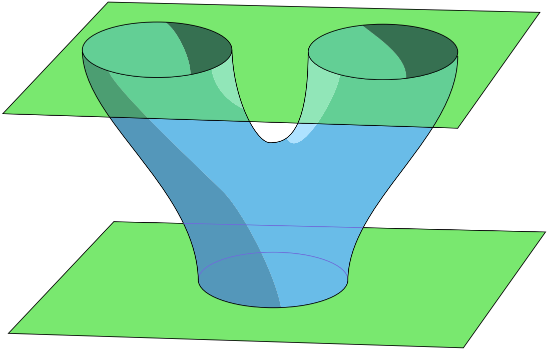
(Hass-Thompson-Thurston):

Integrate over length of product $\Rightarrow$ large area.

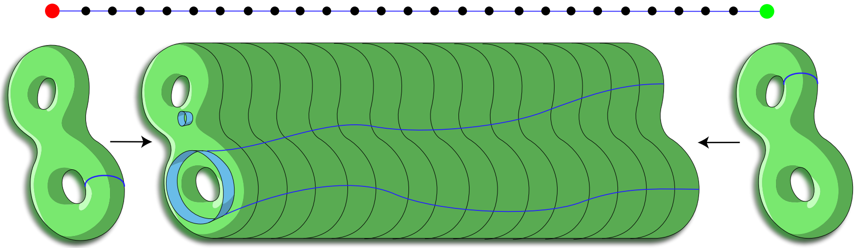


Corollary: Any incomressible surface has high genus.

Theorem (Hartshorn): Every incompressible surface in M has genus at least $\frac{1}{2}d(\Sigma)$.

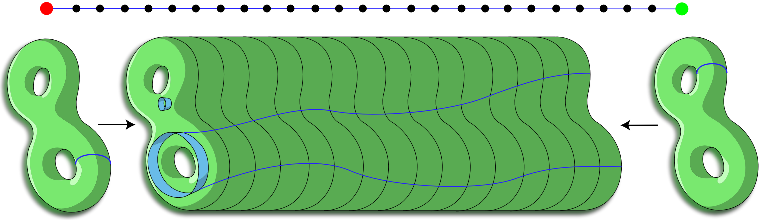


Saddles determine a path in $\mathcal{C}(\Sigma)$.

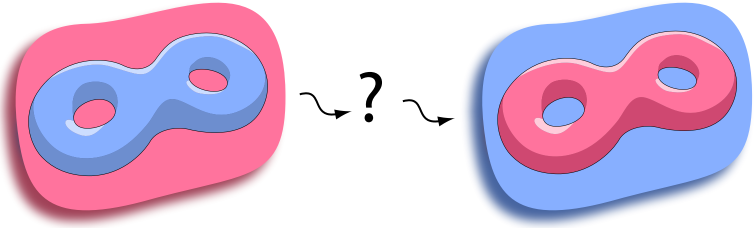


Theorem (Scharlemann-Tomova):

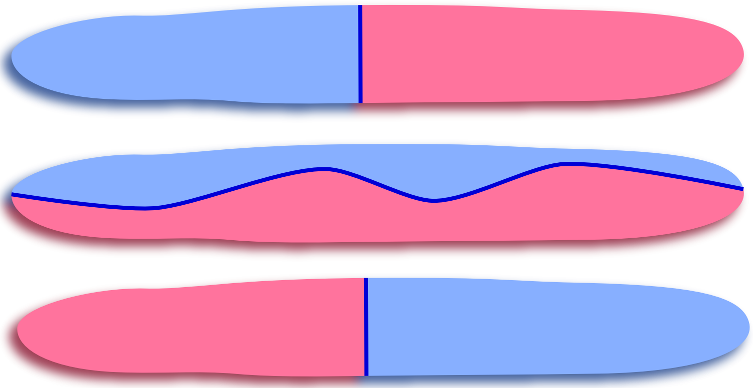
Every unstabilized Heegaard surface in M is Σ or has genus at least $\frac{1}{2}d(\Sigma)$.



Flippable - an isotopy of the surface interchanges the handlebodies

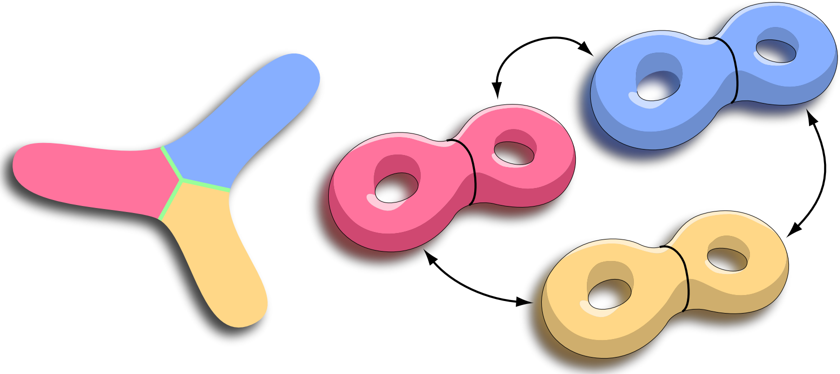


Theorem (Hass-Thompson-Thurston):
High distance Heegaard splittings
are not flippable.

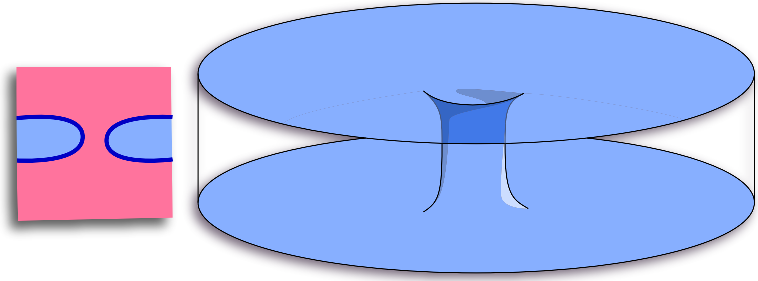


Three handlebody decomposition -

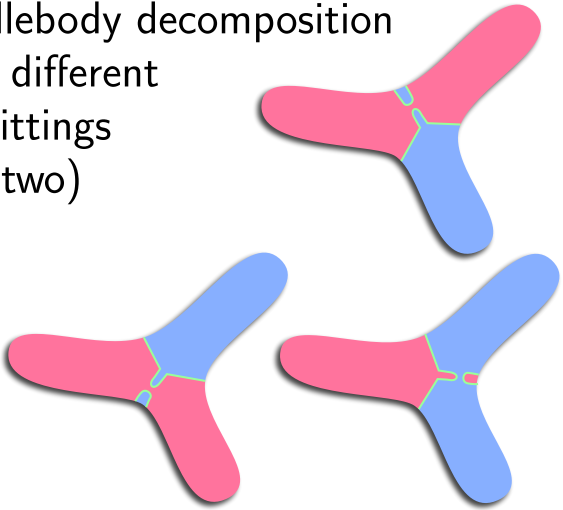
Three handlebodies glued alternately along subsurfaces.



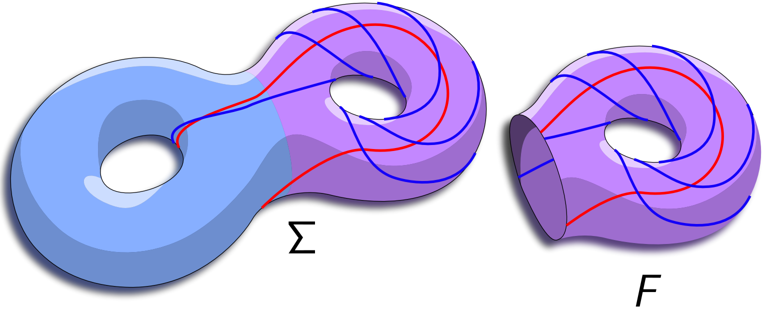
Connect a pair of handlebodies



A three-handlebody decomposition defines three different Heegaard splittings (all distance two)

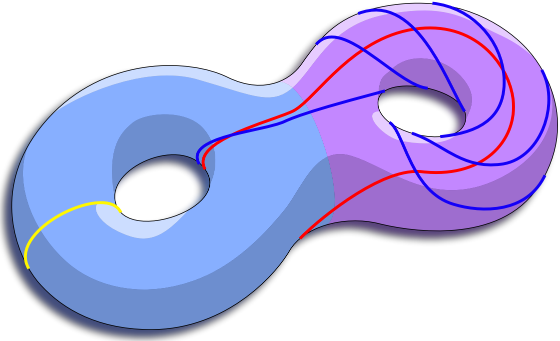


Subsurface projection $d_F(\ell_1, \ell_2)$.

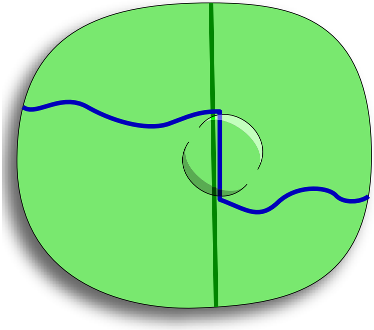


Lemma (Ivanov/Masur-Minsky/Schleimer?):

If $d_F(\ell_1, \ell_2) > n$ then every path from ℓ_1 to ℓ_2 of length n passes through a loop disjoint from F.

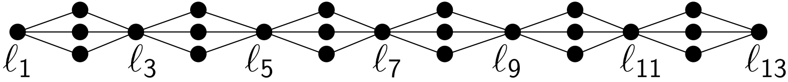


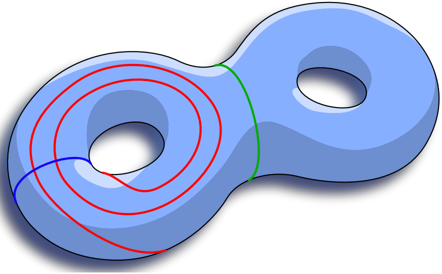
Theorem (J.-Minsky-Moriah): If Σ has a distance d subsurface F then every Heegaard splitting of genus less than $\frac{1}{2}d$ has a subsurface parallel to F.



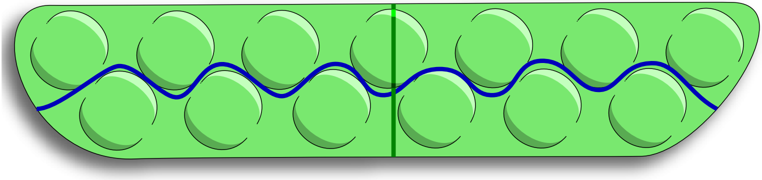
(Ido-Jang-Kobayashi): Flexible geodesics:

$d_{F_j}(\ell_i, \ell_k)$ sufficiently large.

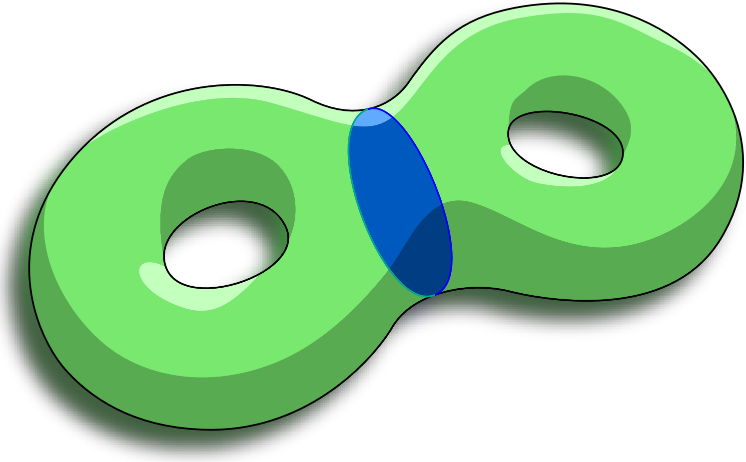




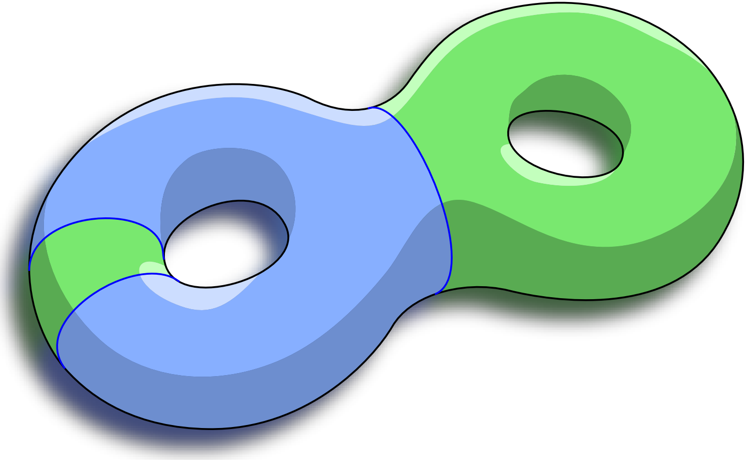
The hyperbolic picture



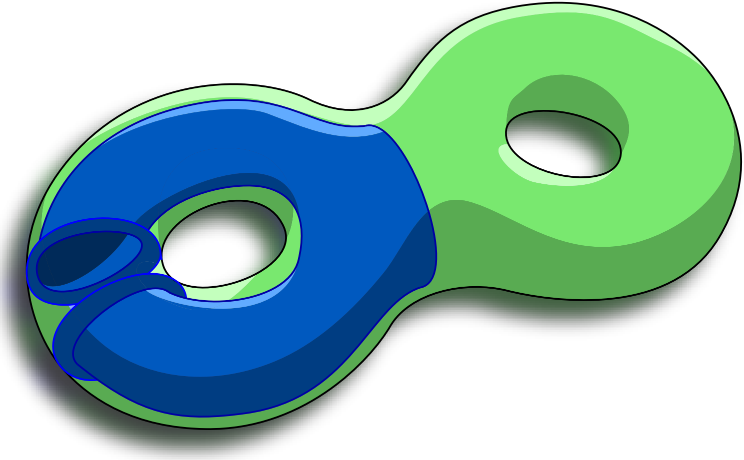
Step 0: $\partial F_0 = \ell_0$



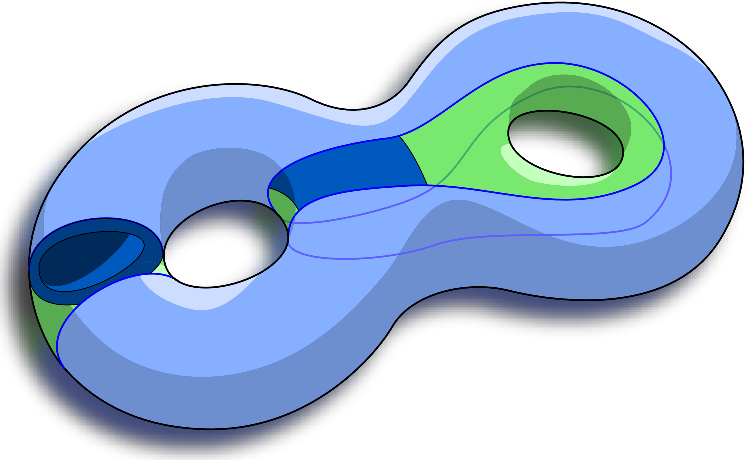
Step 1: $\partial F_1' = \ell_0 \cup \partial N(\ell_1)$



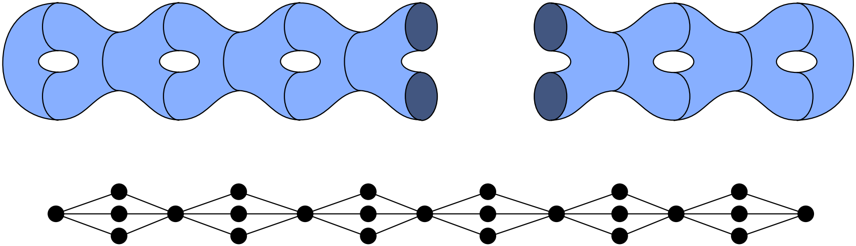
Step 2: $F_1 = F_0 \cup F_1' \cup \{vertical\ annuli\}$



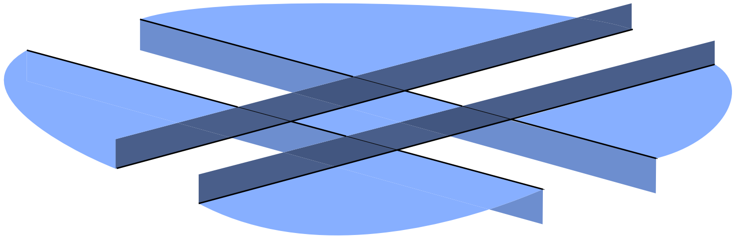
Step 3: $\partial F_2' = \partial N(\ell_1) \cup \ell_2$



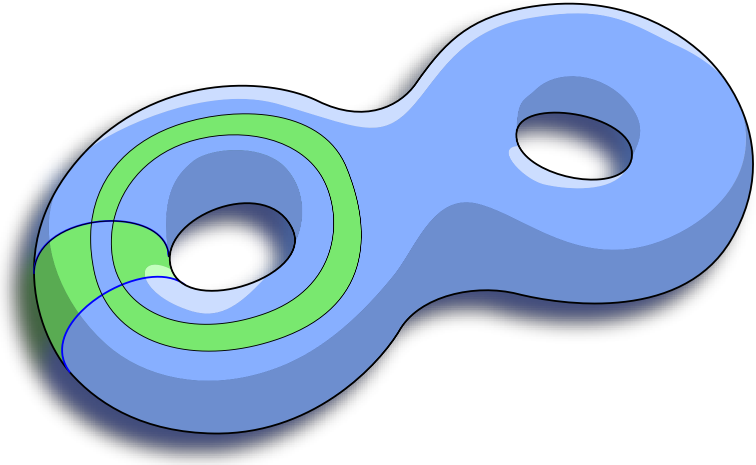
Build from both sides



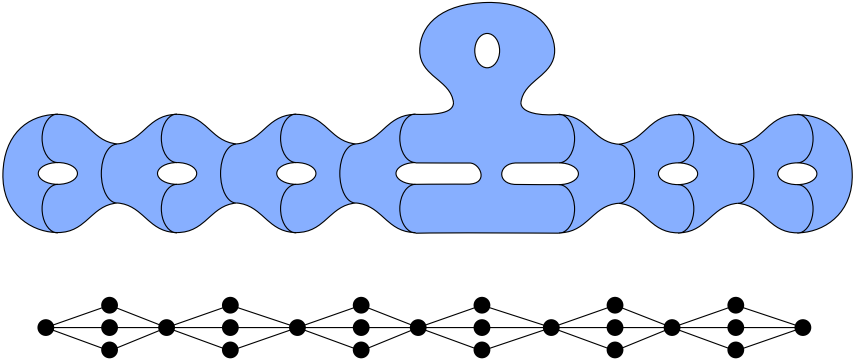
The junction



In the original surface



The full surface:



Theorem: For any integers $d \geq 6$ (even), $g \geq 2$, There is a three-manifold M with a genus g, distance d Heegaard splitting and an unstabilized genus $\frac{1}{2}d + (g - 1)$ Heegaard splitting.