CHAPTER 10

Seifert manifolds

In the previous chapter we have proved various general theorems on three-manifolds, and it is now time to construct examples. A rich and important source is a family of manifolds built by Seifert in the 1930s, which generalises circle bundles over surfaces by admitting some "singular" fibres. The three-manifolds that admit such kind of fibration are now called *Seifert manifolds*.

In this chapter we introduce and completely classify (up to diffeomorphisms) the Seifert manifolds. In Chapter 12 we will then show how to *geometrise* them, by assigning a nice Riemannian metric to each. We will show, for instance, that all the elliptic and flat three-manifolds are in fact particular kinds of Seifert manifolds.

10.1. Lens spaces

We introduce some of the simplest 3-manifolds, the lens spaces. These manifolds (and many more) are easily described using an important three-dimensional construction, called *Dehn filling*.

10.1.1. Dehn filling. If a 3-manifold M has a spherical boundary component, we can cap it off with a ball. If M has a toric boundary component, there is no canonical way to cap it off: the simplest object that we can attach to it is a solid torus $D \times S^1$, but the resulting manifold depends on the gluing map. This operation is called a *Dehn filling* and we now study it in detail.

Let M be a 3-manifold and $T \subset \partial M$ be a boundary torus component.

Definition 10.1.1. A *Dehn filling* of M along T is the operation of gluing a solid torus $D \times S^1$ to M via a diffeomorphism $\varphi\colon \partial D \times S^1 \to T$.

The closed curve $\partial D \times \{x\}$ is glued to some simple closed curve $\gamma \subset T$, see Fig. 10.1. The result of this operation is a new manifold $M^{\rm fill}$, which has one boundary component less than M.

Lemma 10.1.2. *The manifold $M^{\rm fill}$ depends only on the isotopy class of the unoriented curve γ.*

Proof. Decompose S^1 into two closed segments $S^1 = I \cup J$ with coinciding endpoints. The attaching of $D \times S^1$ may be seen as the attaching of a 2-handle $D \times I$ along $\partial D \times I$, followed by the attaching of a 3-handle $D \times J$ along its full boundary.

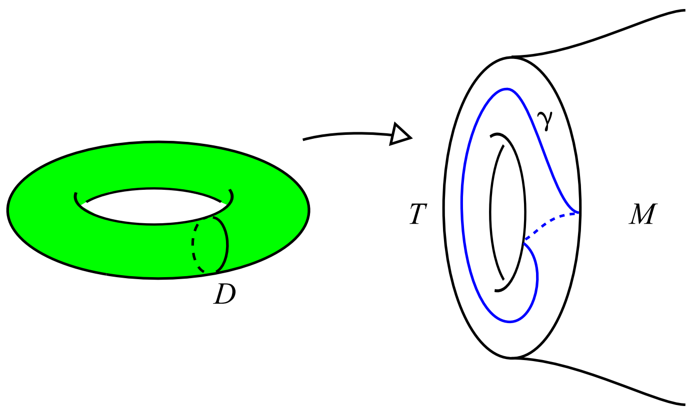


Figure 10.1. The Dehn filling M^{fill} of a 3-manifold M is determined by the unoriented simple closed curve $\gamma \subset T$ to which a meridian ∂D of the solid torus is attached.

If we change γ by an isotopy, the attaching map of the 2-handle changes by an isotopy and hence gives the same manifold. The attaching map of the 3-handle is irrelevant by Proposition 9.2.1. □

We say that the Dehn filling *kills* the curve γ, since this is what really happens on fundamental groups, as we now see.

The *normaliser* of an element $g \in G$ in a group G is the smallest normal subgroup $N(g) \triangleleft G$ containing g. The normaliser depends only on the conjugacy class of $g^{\pm 1}$, hence the subgroup $N(\gamma) \triangleleft \pi_1(M)$ makes sense without fixing a basepoint or an orientation for γ.

Proposition 10.1.3. *We have*

$$\pi_1(M^{\text{fill}}) = \pi_1(M)/_{N(\gamma)}.$$

Proof. The Dehn filling decomposes into the attachment of a 2-handle over γ and of a 3-handle. By Van Kampen, the first operation kills $N(\gamma)$, and the second leaves the fundamental group unaffected. □

Let a *slope* on a torus T be the isotopy class γ of an unoriented homotopically non-trivial simple closed curve. The set of slopes on T was indicated by $\mathscr{S}$ in Chapter 7. If we fix a basis (m, l) for $H_1(T, \mathbb{Z}) = \pi_1(T)$, every slope may be written as $\gamma = \pm(pm + ql)$ for some coprime pair (p, q). Therefore we get a 1-1 correspondence

$$\mathscr{S} \longleftrightarrow \mathbb{Q} \cup \{\infty\}$$

by sending γ to $\frac{p}{q}$. If T is a boundary component of M, every number $\frac{p}{q}$ determines a Dehn filling of M that kills the corresponding slope γ.

Different values of $\frac{p}{q}$ typically produce non-diffeomorphic manifolds M^{fill}: this is not always true - a notable exception is described in the next section - but it holds in "generic" cases.

10.1.2. Lens spaces. The simplest manifold that can be Dehn-filled is the solid torus $M = D \times S^1$ itself. The oriented *meridian* $m = S^1 \times \{y\}$ and *longitude* $l = \{x\} \times S^1$ form a basis for $H_1(\partial M, \mathbb{Z})$.

Definition 10.1.4. The *lens space* $L(p, q)$ is the result of a Dehn filling of $M = D \times S^1$ that kills the slope $qm + pl$.

A lens space is a three-manifold that decomposes into two solid tori. We have already encountered lens spaces in the more geometric setting of Section 3.4.10, and we will soon prove that the two definitions are coherent. Since $L(p, q) = L(-p, -q)$ we usually suppose $p \geqslant 0$.

Exercise 10.1.5. We have $\pi_1\big(L(p, q)\big) = \mathbb{Z}/_{p\mathbb{Z}}$.

Proposition 10.1.6. *We have* $L(0, 1) = S^2 \times S^1$ *and* $L(1, 0) = S^3$.

Proof. The lens space $L(0, 1)$ is obtained by killing m, that is by mirroring $D \times S^1$ along its boundary. The lens space $L(1, 0)$ is S^3 because the complement of a standard solid torus in S^3 is another solid torus, with the roles of m and l exchanged (exercise). □

Exercise 10.1.7. Every Dehn filling of one component of the product $T \times [0, 1]$ is diffeomorphic to $D \times S^1$. Therefore by Dehn-filling both components of $T \times [0, 1]$ we get a lens space.

The solid torus $D \times S^1$ has a non-trivial self-diffeomorphism

$$(x, e^{i\theta}) \longmapsto (xe^{i\theta}, e^{i\theta})$$

called a *twist* along the disc $D \times \{y\}$. The solid torus can also be *mirrored* via the map

$$(x, e^{i\theta}) \longmapsto (x, e^{-i\theta}).$$

Exercise 10.1.8. We have $L(p, q) \cong L(p, q')$ if $q' \equiv \pm q^{\pm 1} \pmod p$.

Hint. Twist, mirror, and exchange the two solid tori giving $L(p, q)$. □

Remark 10.1.9. The meridian m of the solid torus $M = D \times S^1$ may be defined intrinsically as the unique slope in ∂M that is homotopically trivial in M. The longitude l is *not* intrinsically determined: a twist sends l to $m + l$. The solid torus contains infinitely many non-isotopic longitudes, and there is no intrinsic way to choose one of them.

10.1.3. Equivalence of the two definitions. When $p > 0$, we have defined the lens space $L(p, q)$ in two different ways: as the (q, p)-Dehn filling of the solid torus, and as an elliptic manifold in Section 3.4.10. In the latter description we set

$$\omega = e^{\frac{2\pi i}{p}}, \qquad f(z, w) = (\omega z, \omega^q w)$$

and define $L(p, q)$ as $S^3/_\Gamma$ where $\Gamma = \langle f \rangle$ is generated by f. We now show that the two definitions produce the same manifolds.

Proposition 10.1.10. *The manifold $S^3/_{\langle f \rangle}$ is the (q, p)-Dehn filling of the solid torus.*

Proof. The isometry f preserves the central torus

$$T = \left\{(z, w) \;\middle|\; |z| = |w| = \tfrac{\sqrt{2}}{2}\right\}$$

that divides S^3 into two solid tori

$$N^1 = \left\{(z, w) \;\middle|\; |z| \leqslant \tfrac{\sqrt{2}}{2},\ |w| = \sqrt{1 - |z|^2}\right\},$$
$$N^2 = \left\{(z, w) \;\middle|\; |w| \leqslant \tfrac{\sqrt{2}}{2},\ |z| = \sqrt{1 - |w|^2}\right\}.$$

Identify T with $S^1 \times S^1 = \mathbb{R}^2/_{\mathbb{Z}^2}$ in the obvious way, so that $H_1(T) = \mathbb{Z} \times \mathbb{Z}$. The meridians of N^1 and N^2 are $(1, 0)$ and $(0, 1)$. The isometry f act on T as a translation of vector $v = \left(\frac{1}{p}, \frac{q}{p}\right)$. The quotient $T/_{\langle f \rangle}$ is again a torus, with fundamental domain the parallelogram generated by v and $w = (0, 1)$.

The quotients $N^1/_{\langle f \rangle}$, and $N^2/_{\langle f \rangle}$ are again solid tori. Therefore $S^3/_{\langle f \rangle}$ is also a union of two solid tori. Their meridians are the projections of the horizontal and vertical lines in $\mathbb{R}^2$ to $T/_{\langle f \rangle} = \mathbb{R}^2/_{\langle v, w \rangle}$. In the basis (v, w) these meridians are $pv - qw$ and w respectively. Therefore $S^3/_{\langle f \rangle}$ is a $(-q, p)$-Dehn filling on the solid torus, which is diffeomorphic to the (q, p)-Dehn filling by mirroring the solid torus. □

Corollary 10.1.11. *We have $L(1, 0) = S^3$ and $L(2, 1) = \mathbb{RP}^3$.*

Proof. We have $f = \text{id}$ and $f = -\text{id}$, correspondingly. □

10.1.4. Classification of lens spaces. Which lens spaces are diffeomorphic? It is not so easy to answer this question, because many lens spaces like $L(5, 1)$ and $L(5, 2)$ have the same homotopy and homology groups, while there is no evident diffeomorphism between them. A complete answer was given by Reidemeister in 1935, who could distinguish lens spaces using a new invariant, now known as the *Reidemeister torsion*. More topological proofs were discovered in th 1980s by Bonahon and Hodgson. We follow here Hatcher [**26**].

Theorem 10.1.12. *The lens spaces $L(p, q)$ and $L(p', q')$ are diffeomorphic $\iff p = p'$ and $q' \equiv \pm q^{\pm 1}$ (mod p).*

Definition 2. The system of closed curves $u_1, ..., u_g$ and $v_1, ..., v_g$ on the sphere with g handles N is said to constitute a *Heegaard diagram* if the two following conditons are satisfied:

1) the curves $u_1, ..., u_g$ are pairwise nonintersecting and the complement to their union is connected;
2) the curves $v_1, ..., v_g$ are pairwise nonintersecting and the complement to their union is connected.

Let us verify that *Definitions* 1 *and* 2 *are equivalent.* It is clear that the meridians of a handlebody are pairwise nonintersecting and do not disconnect its surface. Hence we need only prove that any Heegaard diagram in the sense of Definition 2 corresponds to the Heegaard splitting of some manifold.

First we claim that if the sphere N with g handles is cut along g nonintersecting circles that do not split N, a sphere S^2 from which $2g$ disks have been deleted is obtained. Suppose that instead we get a surface H with h handles and $2g$ deleted disks. The removal of one disk decreases the Euler characteristic[1] by 1. Hence the Euler characteristic of H is $(2-2h)-2g$. On the other hand, cutting along a circle does not change the Euler characteristic, so that $2-2h-2g = 2-2g$, whence $h = 0$ as claimed.

Now take two copies of the surface N. Cut one copy along the circles u_i (Fig.10.4,c) and the other along v_i (Fig.10.4,d). In both cases we get a sphere with $2g$ holes (Fig.10.4,b and e). These spheres may be homeomorphically deformed so that the boundary circles correspond to the canonical meridians of the handlebodies with g handles (Fig.10.4,a and f). Now it is easy to construct two handlebodies with g handles M_1^3 and M_2^3 together with homeomorphisms of their boundaries onto N taking the meridians of M_1^3 and M_2^3 onto the circles u_i and v_i on N respectively. This gives the required Heegaard splitting. □

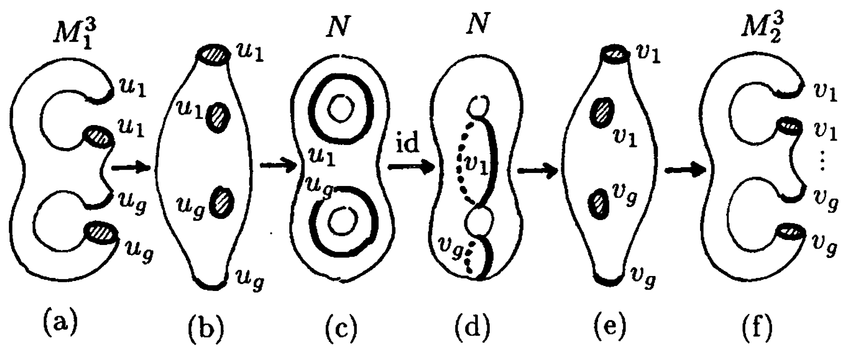


FIGURE 10.4. Heegaard splitting corresponding to Heegaard diagram

§11. Lens spaces

So far, we have demonstrated only a few examples of 3-manifolds presented by Heegaard splittings or diagrams. In this section we consider an infinite series of 3-manifolds that may be conveniently presented in this way: the classical lens spaces.

11.1. The only 3-manifold that can be obtained by gluing two 3-disks by a homeomorphism of their boundaries is the 3-sphere S^3. From two solid tori, as we

[1]The reader not familiar with this notion is referred to 12.6 or 21.2 below.

have seen above, besides the sphere S^3 (see 8.4), one can get projective space $\mathbb{R}P^3$ (Problem 8.4). But S^3 and $\mathbb{R}P^3$ are not the only 3-manifolds obtainable by gluing two solid tori by a homeomorphism of their boundaries. We shall begin by giving a geometrical definition of such manifolds, based on a discrete group action on S^3, and only then discuss their Heegaard presentation.

Suppose p and q are coprime positive integers, and $p \geqslant 3$. On the unit sphere $S^3 \subset \mathbb{C}^2$ consider the action (without fixed points) of the group $\mathbb{Z}/p\mathbb{Z}$ with generator σ by setting

$$\sigma(z, w) = (\exp(2\pi i/p)z, \exp(2\pi iq/p)w).$$

Take the quotient of S^3 by this action of $\mathbb{Z}/p\mathbb{Z}$, i.e., identify each point $x \in S^3$ with the points $\sigma x, ..., \sigma^{p-1}x$. Since the action has no fixed points, it is easy to see that the quotient space is a 3-manifold; it is called a *lens space* and is denoted by $L(p, q)$.

11.2. Let us show that the quotient under the action of $\mathbb{Z}/p\mathbb{Z}$ of each of the two solid tori $|z|^2 \leqslant 1/2$ and $|w|^2 \leqslant 1/2$ is a solid torus, so that the lens space $L(p, q)$ can be glued from two solid tori. Consider the following cellular decomposition of the sphere S^3:

0) zero-dimensional cells $(0, \exp(2\pi ik/p))$;
1) one-dimensional cells $(0, \exp(2\pi i\theta))$, $k/p < \theta < (k+1)/p$;
2) two-dimensional cells $(\rho\exp(2\pi ik/p), w)$, $0 < \rho \leqslant 1$, $|w| = \sqrt{1-\rho^2}$;
3) three-dimensional cells $(\rho\exp(2\pi i\theta), w)$, $0 < \rho \leqslant 1$; $k/p < \theta < (k+1)/p$, $|w| = \sqrt{1-\rho^2}$, $k = 0, 1, ..., p-1$.

There are p cells in each dimension (indexed by the letter k). Under the action of the group $\mathbb{Z}/p\mathbb{Z}$, the cells permute with each other, so that the above cell decomposition of the sphere S^3 induces a cell decomposition of the lens space $L(p, q)$ with one cell in each dimension from 0 to 3. So our lens space can be obtained by taking one of the 3-cells and performing the appropriate identifications on its boundary under the action of $\mathbb{Z}/p\mathbb{Z}$.

Unfortunately, the coordinate presentation of our 3-cells in four-dimensional space $\mathbb{C}^2$ is not very convenient to work with, so we begin by changing to the more natural system of coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$, where our 3-cell will just be the unit 3-disk.

To carry out this change, to the point

$$(\rho\exp(2\pi i\theta), w), \text{ where } 0 < \rho \leqslant 1, \ k/p < \theta < (k+1)/p, \ |w|^2 + \rho^2 = 1,$$

let us assign the point

$$(x_1, x_2, x_3) \in \mathbb{R}^3, \text{ where } x_1 + ix_2 = w, \ x_3 = (2p\theta - 2k - 1)\rho;$$

here $|x_3| \leqslant \rho$ and $x_1^2 + x_2^2 + x_3^2 \leqslant 1$ (Fig.11.1). Points of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ for which $x_3 > 0$ (respectively $x_3 < 0$) are assigned to points for which $\theta = (k+1)/p$ (respectively $\theta = k/p$).

Points with $\theta = k/p$ are taken by the generator $\sigma \in \mathbb{Z}/p\mathbb{Z}$ to points for which we have $\theta = (k+1)/p$. It follows from the definition of our assignment that on the coordinates x_1 and x_2, the element σ acts via rotations by the angle $2\pi q/p$ about the origin; it also identifies the 2-cells of the upper hemisphere with 2-cells of the lower one as shown in Fig.11.2. These identifications produce $L(p, q)$.

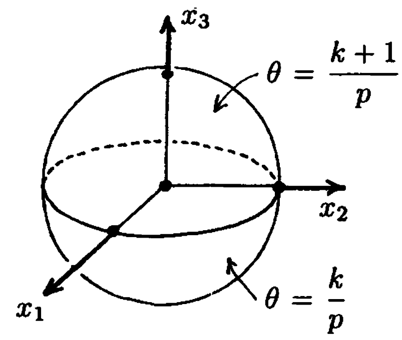


FIGURE 11.1. Points on the boundary of the 3-cell

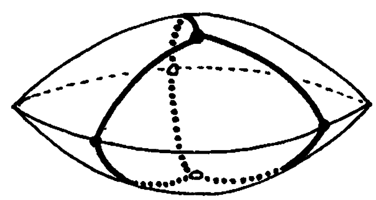


FIGURE 11.2. Identifying the boundary of a "lens"

Traditionally, the 3-cell with identified 2-cells on the boundary is pictured as a somewhat flattened 3-disk that resembles a lens, which is apparently the origin of the term "lens space".

The solid torus $|w|^2 \leqslant 1/2$ intersects the 3-cell from which we constructed the lens space $L(p,q)$ along the solid cylinder (with spherical bases) determined by the inequalities $x_1^2+x_2^2+x_3^2 \leqslant 1$ and $x_1^2+x_2^2 \leqslant 1/2$ (Fig.11.3). Under the identifications due to the action of the element σ, we must glue together the upper base of this cylinder with its lower base, after a rotation by $2\pi q/p$. The result will be the solid torus M_1^3. We leave to the reader the proof of the fact that its complement in the lens space $L(p,q)$ is also a solid torus. (This proof may be obtained by using the constructions described in 11.3.)

Problem 11.1. Find the fundamental group of the lens space $L(p,q)$.

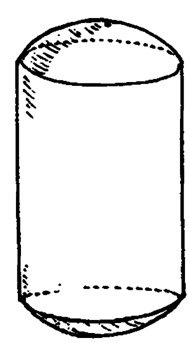


FIGURE 11.3. Cylinder with spherical bases

11.3. It follows from the result of the previous problem that *the lens spaces $L(p,q)$ and $L(p',q')$ are not homeomorphic if $p \neq p'$.* On the other hand, it is obvious from the construction that *the lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic provided $q \equiv q' \mod p$.*

Our next goal is to prove that *the lens spaces* $L(p,q)$ *and* $L(p,q')$ *are homeomorphic provided* $qq' \equiv +1 \mod p$. (Then they will also be homeomorphic when $qq' \equiv -1 \mod p$.)

To do this, cut the 3-cell into tetrahedra by means of p half-planes passing through the x_3-axis and the 0-cells (Fig.11.4). Denote the upper and lower faces of the first tetrahedron by T_1 and S_1, respectively, and the left and right lateral faces by A_1 and B_1, respectively. Denote the faces of the other tetrahedra in a similar way. Initially the faces B_i and A_{i+1} were identical, while the transformation σ identifies the faces S_i and T_{i+q} (here and below, we number the faces modulo p).

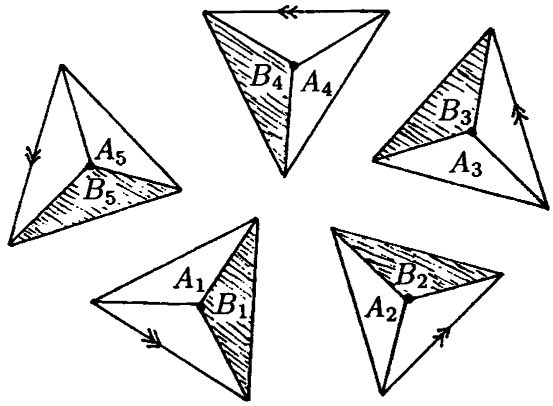


FIGURE 11.4. The lens cut into tetrahedra

Now instead of identifying the lateral faces first, let us begin by identifying the upper and lower faces, and only then identify the lateral ones. Then the adjacent tetrahedra will acquire the numbers $j, j+q, j+2q, ..., j+q'q$. Hence the face B_j will be identified with the face A_{j+1}, which is the same as the face $A_{j+qq'}$, because by assumption the integer q' is the solution of the Diophantine equation $qq' \equiv +1 \mod p$. Since the end result does not depend on the order in which the identifications are performed, the two lens spaces are identical. □

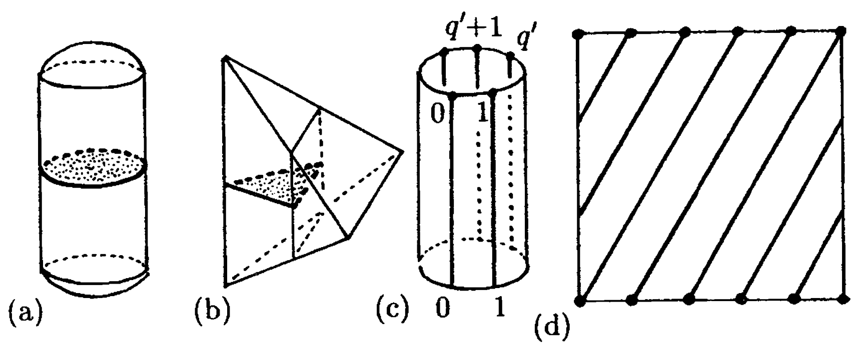


FIGURE 11.5. Heegaard diagram of $L(5,3)$

11.4. Now we are ready to describe the Heegaard diagram of the lens space $L(p,q)$. For the meridional disk of the solid torus $|w|^2 \leqslant 1/2$, we take its section by the plane $x_3 = 0$ (Fig.11.5,a). On Fig.11.5,b we show the part of the meridional disk contained in one of the tetrahedra into which we have cut our "lens". This picture shows that the boundary circle of this disk corresponds to p segments on the other solid torus $|w|^2 \geqslant 1/2$ (Fig.11.5,c). But under our identifications, the lower point with number i is glued to the upper point $i+q'$, where $qq' \equiv 1 \mod p$. Since the integers p and q' are coprime, we obtain one closed curve on the torus (shown in Fig.11.5,d). This is the desired Heegaard diagram.

Comments

The idea of a manifold as a geometric entity not lying in some linear space, but possessing its own intrinsic geometry, is due to B. Riemann. It appeared before the notion of abstract topological space, and in those days was always supplied with a metric. We shall not attempt to describe the progressive "topologization" of the notion of 3-manifold that took place in the first third of the 20th century. The key names, besides Heegaard and Poincaré, are J. W. Alexander, M. Dehn, J. Nielsen, H. Seifert. The classical text that brings together the topological achievements of that period is the *Lehrbuch der Topologie* by H. Seifert and W. Threlfall ([**ST**], 1934).

Most of the material of this chapter is traditional and can be found in several texbooks, e.g. [**Rol**], [**MF**], [**ST**], except for §9, in which we follow the article [**Dow**], and Theorem 8.8, which is standard "folklore", but for which we were not able to find a proof in the literature.

CHAPTER 9 . 3-MANIFOLDS AND SURGERY ON LINKS

A. INTRODUCTION. Up to this point we have been, by and large, concerned with knots and links in their own right. Although knot theory per se is by no means a closed book, many of its more interesting developments in the last decade or two have been in the directions of application of knot-theoretical techniques to other branches of mathematics, or at least other parts of topology. For example, the study of singularities in algebraic geometry has been enriched by the application of knot methods. Milnor's book [1968][2] is a highly recommended introduction to this. Another important application knot theory is in the study of manifolds. In this chapter we will focus attention on three-dimensional manifolds. A technique dating back to Dehn -- now known as surgery -- has become a powerful tool in constructing, and proving theorems about, 3-manifolds. Unless otherwise stated, all closed 3-manifolds M^3 under consideration will be assumed connected and orientable, in other words, $H_1(M) \cong H_3(M) \cong Z$.

B. LENS SPACES. We already know from the Alexander trick that if one attaches two 3-balls together via any homeomorphism of their boundaries, the resulting space is S^3. Consider the analogous construction using two solid tori V_1 and V_2 . If $h : \partial V_2 \to \partial V_1$ is a homeomorphism we may form the space

$$M^3 = V_1 \bigcup_h V_2$$

which is the result of identifying each $x \in \partial V_2$ with $h(x) \in \partial V_1$ in the disjoint union of V_1 and V_2 .

1. EXERCISE : M^3 is a closed connected orientable 3-manifold which depends, up to homeomorphism, only upon the homotopy class of $h(m_2)$ in ∂V_1, where m_2 is a meridian of V_2. [see section 2E].

2. DEFINITION : Choosing fixed longitude and meridian generators ℓ_1 and m_1 for $\pi_1(\partial V_1)$, we may write

$$h_*(m_2) = p\ell_1 + qm_1$$

where p and q are coprime integers. The resulting M^3 is called the <u>lens space</u> of type (p,q) and denoted traditionally by

$$M^3 = L(p,q).$$

In other words a 3-manifold is a lens space if and only if it contains a solid torus, the closure of whose complement is also a solid torus. Some writers don't count S^3 and $S^2 \times S^1$ as lens spaces.

3. EXERCISE : Establish the following homeomorphisms:

$$L(1,q) \cong S^3$$
$$L(0,1) \cong S^2 \times S^1$$
$$L(2,1) \cong RP^3 \text{ (= } S^3 \text{ with antipodal points identified)}$$

4. EXERCISE : Show that $L(p,q) \cong L(p,-q) \cong L(-p,q) \cong L(-p,-q) \cong L(p,q+kp)$ for each integer k .

Thus we adopt the convention that $0 < q < p$. This exhausts the list of lens spaces which are not the 'degenerate' ones: S^3 and $S^2 \times S^1$.

5. EXERCISE : Show that the fundamental group of $L(p,q)$ is the finite cyclic group Z/p.

So two lens spaces $L(p,q)$ and $L(p',q')$ are definitely not homeomorphic -- nor even of the same homotopy type -- unless $p = p'$.

6. EXERCISE : Show that $L(p,q) \cong L(p,q')$ if $\pm qq' \equiv 1 \pmod{p}$.
[Hint: Regard $V_2 \cup_{h^{-1}} V_1$] .

7. REMARK : Among the list $L(p,1)$,..., $L(p,p-1)$ of all lens spaces with group Z_p (recall that q must be prime to p) there are duplications, up to homeomorphism, according to the exercise. Actually lens spaces have been completely classified. According to Brody [1960]* we have that $L(p,q)$ and $L(p,q')$ are

homeomorphic <=> $\pm q' \equiv q^{\pm 1} \pmod{p}$.

They are, according to Whitehead [1941], of the

same homotopy type <=> $\pm qq'$ is a quadratic residue, mod p .

This means that $\pm qq' \equiv m^2 \pmod{p}$ for some m . Thus for example $L(7,1)$ and $L(7,2)$ are 3-manifolds of the same homotopy type which are not homeomorphic.

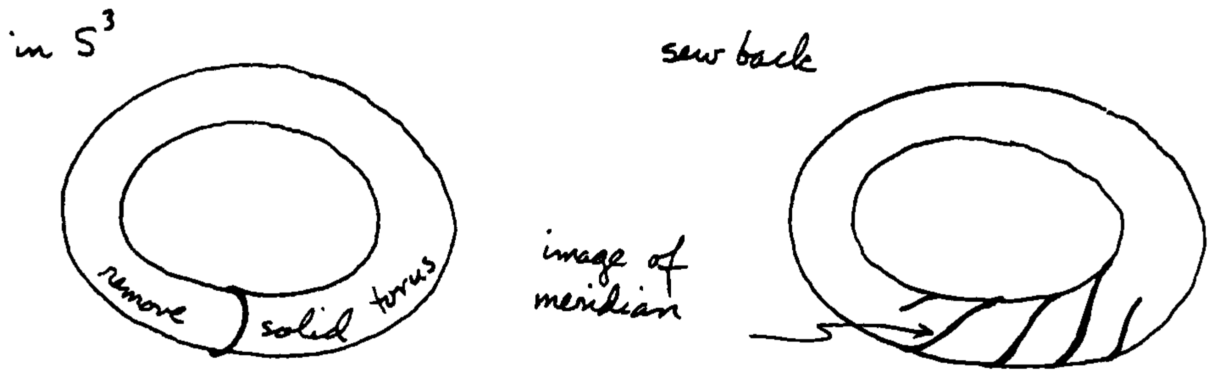
A theorem of Fermat and Euler states that if p is a prime congruent to 3 modulo 4, then for any q , exactly one of $\pm q$ is a quadratic residue mod p . For all other primes p either both or neither of $\pm q$ is a quadratic residue. Thus given $p = 3, 7, 11, \ldots$

* there are earlier proofs, but Brody's actually uses knot theory.

there is only one homotopy type of lens spaces $L(p,q)$. For $p = 5$, 13, ... there are two homotopy types. What if p isn't prime?

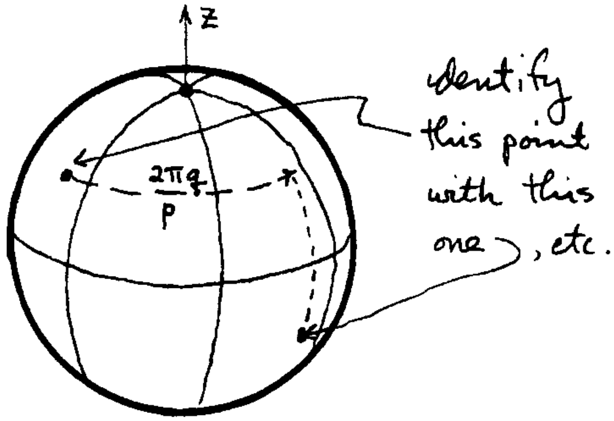
8. OTHER DESCRIPTIONS : There are other ways of describing $L(p,q)$. We list three.

(I) one may regard $L(p,q)$ as the result of removing a tubular neighbourhood of a trivial knot in S^3 and sewing it back so that a meridian lies on a curve which has linking number p with the trivial knot and runs q times along it (i.e. a (q,p) torus knot). Note that the roles of 'meridian' and 'longitude' become reversed if we speak relative to the neighbourhood of the trivial knot, rather than relative to its complementary solid torus.



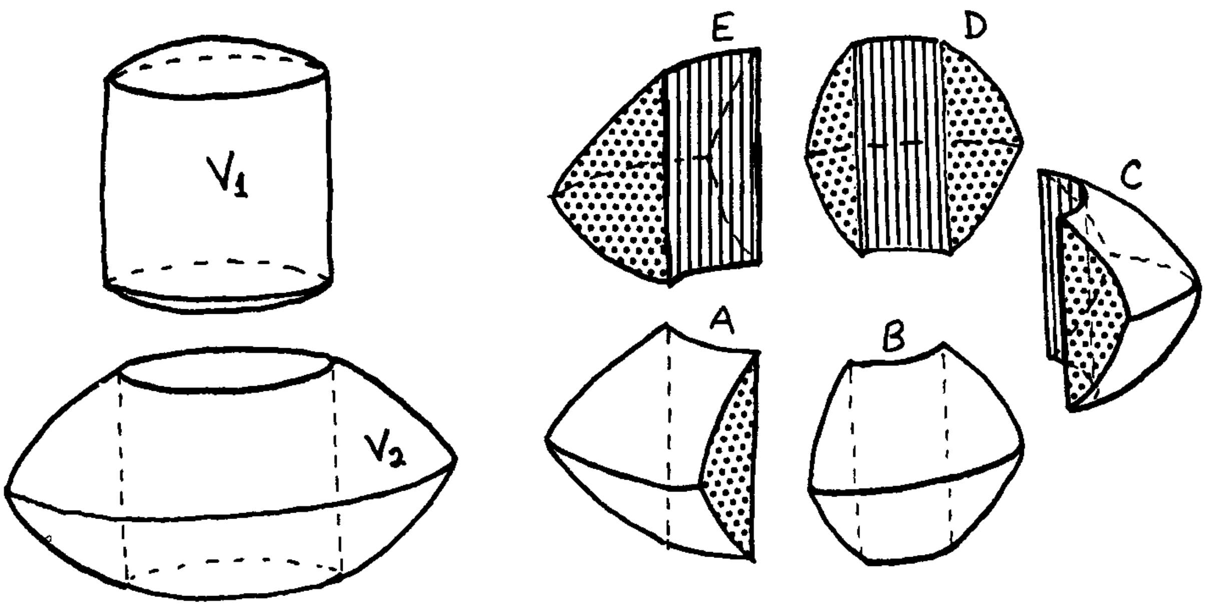
(II) Consider the unit ball B^3 in R^3 . Then identify each point on the upper half $(z \geq 0)$ of ∂B^3 with its image under counter-clockwise rotation by an angle of $2\pi q/p$ about the z-axis, followed by reflection in the $x - y$ plane, as in the following picture.

To construct $L(p,q)$

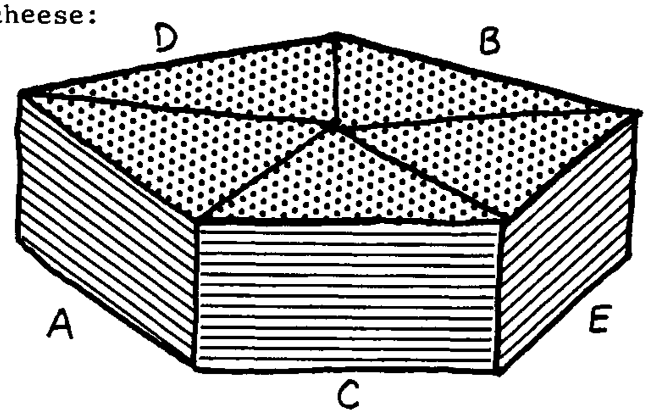


To verify that this identification space is $L(p,q)$ and for reasons which will become apparent later, we depict the B^3 as a lens-shaped solid with edge on the circle $x^2 + y^2 = 1$ and edge-angle $2\pi/p$. Let V_1 be the part of the space inside the cylinder $x^2 + y^2 \leq 1/4$ (with identifications) and let V_2 be the closure of what remains.

It is clear that V_1 is a solid cylinder with ends identified (with a $2\pi q/p$ twist); thus it's a solid torus. That V_2 is also a solid torus can be seen by chopping it into pieces thus (shown for the case $p = 5$, $q = 2$) :



and re-assembling according to the prescribed identifications, like wedges of a big cheese:



We still must identify the parts of V_2 which were separated by the dissection. But this is just sewing the top and bottom of the cheese together (with a twist) to form a solid torus. Now a meridinal curve of V_2 is just one which runs around the perimeter of the big cheese. Thus on V_1 (before identifying top and bottom) it consists of p vertical lines equally spaced on the boundary of the cylinder $x^2 + y^2 = 1/4$. Since top and bottom become twisted by q/p full turns, the meridinal curve of V_2 becomes a p, q curve on V_2 and our identification space is indeed $L(p,q)$.

(III) Here is another description of $L(p,q)$; the most concise of all. Consider S^3 as the unit sphere of C^2 , complex 2-space. Let $\tau : S^3 \to S^3$ be the homeomorphism given by

$$\tau(z_0,z_1) = (z_0 w,\ z_1 w^q) ,$$

where $w = \exp(2\pi i/p)$ is the principal p^{th} root of unity. Then τ is periodic of period p (thus generates a Z/p-action on S^3 .) Consider the orbit space of this action; that is, we identify points

x, y of S^3 if $x = \tau^k(y)$ for some k .

9. EXERCISE : Verify that the orbit space is homeomorphic with $L(p,q)$. [Hint: Show that after stereographic projection, the lens-shaped region of the discussion above may be taken as a fundamental region of the action.]

Another view of this is that S^3 is the universal covering space of $L(p,q)$ and that τ is a generator of the cyclic group of covering translations. This checks with our calculation that the fundamental group of $L(p,q)$ is Z/p (and thus so is the covering translation group of the universal cover).

C. HEEGAARD DIAGRAMS . We now generalize the construction of the previous section. Recall that a handlebody of genus g is the result of attaching g disjoint "1-handles" $D^2 \times [-1,1]$ to a 3-ball B^3 by sewing the parts $D^2 \times \{\pm 1\}$ to $2g$ disjoint disks on ∂B^3 in such a way that the result is an orientable 3-manifold with boundary. Two handlebodies of the same genus are homeomorphic (and vice versa). The boundary of a handlebody of genus g is a closed orientable 2-manifold of genus g , as genus was defined previously. Let H_1 and H_2 be handlebodies of the same genus, g , and let $h : \partial H_2 \to \partial H_1$ be a homeomorphism. Then form the identification space

$$M^3 = H_1 \underset{h}{\cup} H_2$$

as before. Again it is easy to see that M^3 is a closed orientable