Euler's Polyhedral Formula

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Euler

Leonhard Euler (1707-1783)

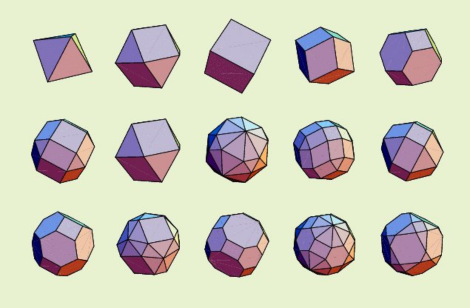
Leonhard Euler was a Swiss mathematician who made enormous contibutions to a wide range of fields in mathematics.

Euler: Some contributions

- Euler introduced and popularized several notational conventions through his numerous textbooks, in particular the concept and notation for a function.
- In analysis, Euler developed the idea of power series, in particular for the exponential function e^x. The notation e made its first appearance in a letter Euler wrote to Goldbach.
- For complex numbers he discovered the formula $e^{i\theta} = \cos\theta + i\sin\theta$ and the famous identity $e^{i\pi} + 1 = 0$.
- In 1736, Euler solved the problem known as the Seven Bridges of Königsberg and proved the first theorem in Graph Theory.
- Euler proved numerous theorems in Number theory, in particular he proved that the sum of the reciprocals of the primes diverges.

Convex Polyhedron

A polyhedron is a solid in $\mathbb{R}^3$ whose faces are polygons.



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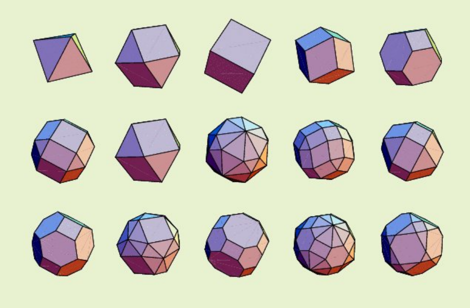
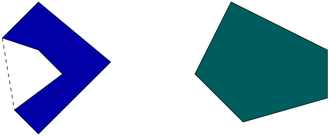


A polyhedron P is convex if the line segment joining any two points in P is entirely contained in P.



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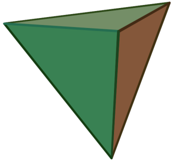
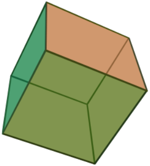
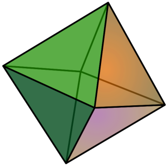


Tetrahedron



Cube



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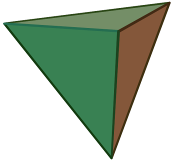

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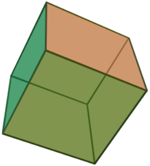
$v = 4, \; e = 6, \; f = 4$



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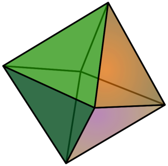

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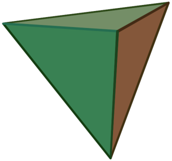

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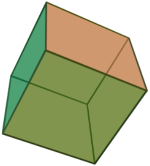

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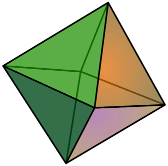

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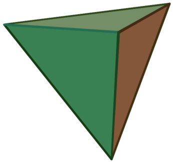

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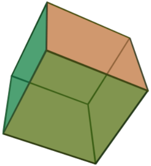

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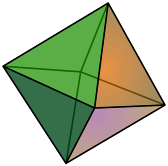

Octahedron

$v = 6, \ e = 12, \ f = 8$

Euler's Polyhedral Formula

Euler mentioned his result in a letter to Goldbach (of Goldbach's Conjecture fame) in 1750. However Euler did not give the first correct proof of his formula.

It appears to have been the French mathematician Adrian Marie Legendre (1752-1833) who gave the first proof using Spherical Geometry.

Adrien-Marie Legendre (1752-1833)

Girard's Theorem: Area of a spherical triangle

Girard's Theorem

The area of a spherical triangle with angles α, β and γ is $\alpha+\beta+\gamma-\pi$.

Corollary

Let R be a spherical polygon with n vertices and n sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\mathrm{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

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Let P be a convex polyhedron in $\mathbb{R}^3$. We can "blow air" to make (boundary of) P spherical.

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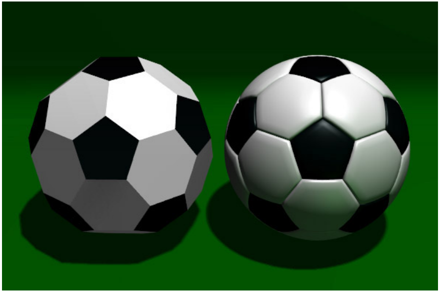

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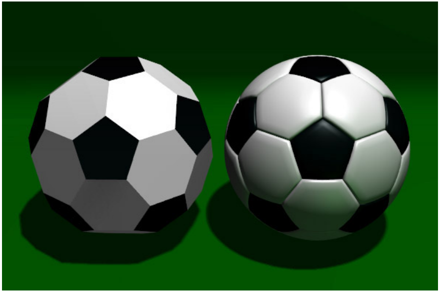

This can be done rigourously by arranging P so that the origin lies in the interior of P and projecting the boundary of P on S^2 using the function $f(x, y, z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$.

It is easy to check that vertices of P go to points on S^2, edges go to parts of great circles and faces go to spherical polygons.

Proof of Euler's Polyhedral Formula

Let v, e and f denote the number of vertices, edges and faces of P respectively. Let $R_1, \ldots, R_f$ be the spherical polygons on S^2.

Since their union is S^2, $\mathrm{Area}(R_1) + \ldots + \mathrm{Area}(R_f) = \mathrm{Area}(S^2)$.

Let n_i be the number of edges of R_i and α_{ij} for $j = 1, \ldots, n_i$ be its interior angles.

$$\begin{aligned}
\sum_{i=1}^{f}\left(\sum_{j=1}^{n_i} \alpha_{ij} - n_i\pi + 2\pi\right) &= 4\pi \\
\sum_{i=1}^{f}\sum_{j=1}^{n_i} \alpha_{ij} - \sum_{i=1}^{f} n_i\pi + \sum_{i=1}^{f} 2\pi &= 4\pi
\end{aligned}$$

Proof of Euler's Polyhedral Formula

Since every edge is shared by two polygons

$$\sum_{i=1}^{f} n_i \pi = 2\pi e.$$

Since the sum of angles at every vertex is 2π

$$\sum_{i=1}^{f} \sum_{j=1}^{n_i} \alpha_{ij} = 2\pi v.$$

Hence $2\pi v - 2\pi e + 2\pi f = 4\pi$ that is $v - e + f = 2$ ∎

Why Five ?

A platonic solid is a polyhedron all of whose vertices have the same degree and all of its faces are congruent to the same regular polygon. We know there are only five platonic solids. Let us see why.

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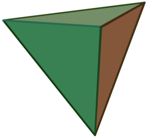	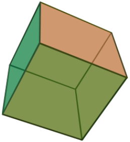	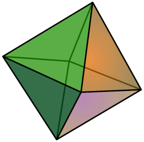	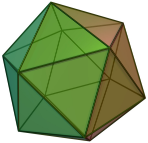	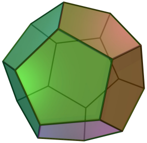
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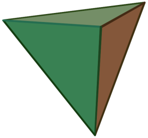	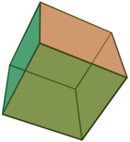	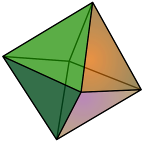	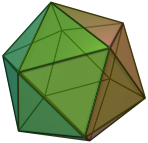	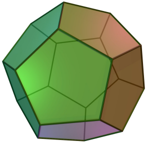
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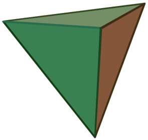	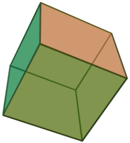	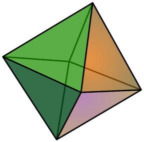	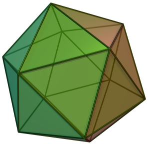	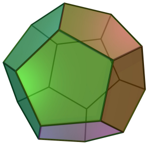
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If $a \geq 6$ or $b \geq 6$ then $\frac{1}{a} + \frac{1}{b} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Hence $a < 6$ and $b < 6$ which gives us finitely many cases to check.

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Plane graphs

Note that we actually proved the Theorem for any (geodesic) graph on the sphere.

Any plane graph can be made into a graph on a sphere by tying up the unbounded face (like a balloon). However one may need to make some modifications (which do not change the count $v - e + f$) to make the graph geodesic on the sphere (keywords: k-connected for $k = 1, 2, 3$).

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Theorem

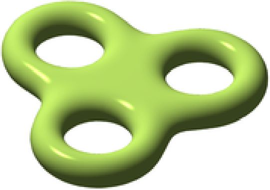
If G is a connected plane graph with v vertices, e edges and f faces (including the unbounded face), then $v - e + f = 2$.

This theorem from graph theory can be proved directly by induction on the number of edges and gives another proof of Euler's Theorem !

Surfaces

What about graphs on other surfaces ?





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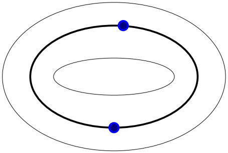

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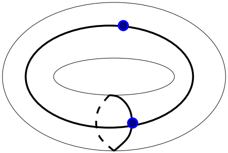

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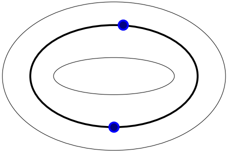
We need the restriction that every face of the graph on the surface is a disk.

Other surfaces



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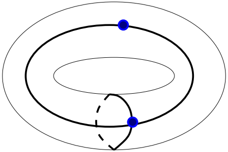

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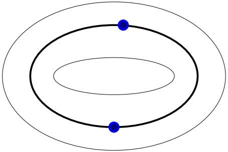

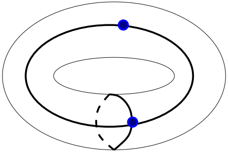

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The number $\chi = v - e + f$ is called the Euler characteristic of the surface. $\chi = 2 - 2g$ where g is the genus of the surface i.e. the number of holes in the surface.