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## Think Globally

By STEVEN STROGATZ

The most familiar ideas of geometry were inspired by an ancient vision - a vision of the world as flat. From parallel lines that never meet, to the Pythagorean theorem discussed in last week's column, these are eternal truths about an imaginary place, the two-dimensional landscape of plane geometry.

Conceived in India, China, Egypt and Babylonia more than 2,500 years ago, and codified and refined by Euclid and the Greeks, this flat-earth geometry is the main one (and often the only one) being taught in high schools today. But things have changed in the past few millennia.

In an era of globalization, Google Earth and transcontinental air travel, all of us should try to learn a little about spherical geometry and its modern generalization, differential geometry. The basic ideas here are only about 200 years old. Pioneered by Carl Friedrich Gauss and Bernhard Riemann, differential geometry underpins such imposing intellectual edifices as Einstein's general theory of relativity. At its heart, however, are beautiful concepts that can be grasped by anyone who's ever ridden a bicycle, looked at a globe or stretched a rubber band. And understanding them will help you make sense of a few curiosities you may have noticed in your travels.

For example, when I was little, my dad used to enjoy quizzing me about geography. Which is farther north, he'd ask, Rome or New York City? Most people would guess New York, but surprisingly they're at almost the same latitude, with Rome being just a bit farther north. On the usual map of the world (the misleading Mercator projection, where Greenland appears gigantic) it looks like you could go straight from New York to Rome by heading due east.

Yet airline pilots never take that route. They always fly northeast out of New York, hugging the coast of Canada. I used to think they were staying close to land for safety's sake, but that's not the reason. It's simply the most direct route, if you take the earth's curvature into account. The shortest path from New York to Rome goes past Nova Scotia and Newfoundland, then heads out over the Atlantic, and finally veers south of Ireland and across France for arrival in sunny Italy.


This kind of path on the globe is called an arc of a "great circle." Like straight lines in ordinary space, great circles on a sphere contain the shortest paths between any two points. They're called "great" because they're the largest circles you can have on a sphere. Conspicuous examples include the equator and the longitudinal circles that pass through the north and south poles.

Another property that lines and great circles share is that they're the straightest paths. That might sound strange - all paths on a globe are curved, so what do we mean by "straightest"? Well, some paths are more curved than others. The great circles don't do any additional curving, above and beyond what they're forced to do by following the surface of the sphere.

Here's a way to visualize this. Imagine you're riding a tiny bicycle on the surface of a globe, and you're trying to stay on a certain path. If it's part of a great circle, you won't ever need to steer. That's the sense in which great circles are "straight." In contrast, if you try to ride along a line of latitude near one of the poles, you'll have to keep turning the handlebars.

Of course, as surfaces go, the plane and the sphere are abnormally simple. The surface of a human body, or a tin can, or a bagel would be more typical - they all have far less symmetry, as well as various kinds of holes and passageways that make them more confusing to navigate. In this more general setting, finding the shortest path between any two points becomes a lot trickier. So rather than delving into technicalities, let's stick to an intuitive approach. This is where rubber bands come in handy.

Specifically, imagine a slippery elastic string that always contracts as far as it can, while remaining confined to the surface. With its help, we can easily determine the shortest path between New York and Rome, or for that matter, between any two points on any surface. Tie the ends of the string to the points of departure and arrival and let the string pull itself tight, while clinging to the surface's contours. When the string is as taut as these constraints allow, voila! It traces the shortest path.

On surfaces just a little more complicated than planes or spheres, something strange and new can happen: many locally shortest paths can exist between the same two points. For example, consider the surface of a soup can, with one point lying directly below the other.


Then the shortest path between them is clearly a line segment, as shown above, and our elastic string would find that solution. So what's new here? The cylindrical shape of the can opens up new possibilities for all kinds of contortions. Suppose we require that the string encircles the cylinder once before connecting to the second point. Now when the string pulls itself taut, it forms a helix, like the curves on old barbershop poles.


This helical path qualifies as another solution to the shortest path problem, in the sense that it's the shortest of the candidate paths nearby. If you nudge the string a little, it would necessarily get longer and then contract back to the helix. You could say it's the "locally" shortest path - the regional champion of all those that wrap once around the cylinder. (By the way, this is why the subject is called "differential" geometry; it studies the effects of small local differences on various kinds of shapes, such as the difference in length between the helical path and its neighbors.)

But that's not all. There's another champ that winds around twice, and another that goes around three times, and so on. There are infinitely many locally shortest paths on a cylinder! Of course,
none of these helices is the "globally" shortest path. The straight-line path is shorter than all of them.

Likewise, surfaces with holes and handles permit many locally shortest paths, distinguished by their pattern of weaving around various parts of the surface. The following video by the mathematician Konrad Polthier of the Free University of Berlin illustrates the non-uniqueness of these locally shortest paths, or "geodesics," on the surface of an imaginary planet shaped like a figure-8, a surface known in the trade as a "two-holed torus":

## (See video online)

The red, yellow and green geodesics all visit very different parts of the planet, thereby executing different loop patterns. But what they all have in common is their superior directness compared to the paths nearby. And just like lines on a plane or great circles on a sphere, these geodesics are the straightest possible curves on the surface. They bend to conform with the surface, but don't bend within it. To make this clear, Polthier has produced another illuminating video.

## (See video online)

Here, a motorcycle rides along a geodesic highway on a two-holed torus, following the lay of the land. The remarkable thing is that the motorcycle's handlebars are locked. It doesn't need to steer to stay on the road. This underscores the earlier intuition that geodesics, like great circles, are the natural generalization of straight lines.

With all these flights of fancy, you may be wondering if geodesics have anything to do with reality. Of course they do. Einstein showed that light beams follow geodesics as they sail through the universe. The famous bending of starlight around the sun, detected in the eclipse observations of 1919, confirmed that light travels on geodesics through curved space-time, with the warping being caused by the sun's gravity.

At a more down-to-earth level, the mathematics of finding shortest paths is critical in everything from the GPS navigation systems in our cars to the routing of traffic on the Internet. In these situations, however, the relevant space is a gargantuan maze of addresses and links, as opposed to the smooth surfaces considered above, and the mathematical issues have to do with the speed of algorithms - what's the most efficient way to find the shortest path through a network? Given the myriad of potential routes, the task would be overwhelming, were it not for the ingenuity of the mathematicians and computer scientists who cracked it.

Sometimes when people say the shortest distance between two points is a straight line, they mean it figuratively, as a way of ridiculing nuance and affirming common sense. In other words, keep it simple. But battling obstacles can give rise to great beauty - so much so that in art, and in math, it's often more fruitful to impose constraints on ourselves. Think of haiku, or sonnets, or telling the story of your life in six words. The same is true of all the math that's been created to find the shortest way from here to there when you can't take the easy way out.

Two points. Many paths. Mathematical bliss.

NOTES:

1. By referring to plane geometry as "flat-earth" geometry, I might seem to be disparaging the subject, but that's not my intent. The tactic of locally approximating a curved shape by a flat one has often turned out to be a useful simplification in many parts of mathematics and physics, from calculus to relativity theory. Plane geometry is the first instance of this great idea.
2. Nor do I mean to suggest that all the ancients thought the world was flat. For an engaging account of Eratosthenes's measurement of the distance around the globe, see:
N. Nicastro, Circumference (St. Martin's Press, 2008).
3. For a more contemporary approach that you might like to try on your own, Robert Vanderbei at Princeton University recently gave a presentation to his daughter's high school geometry class in which he used a photograph of a sunset to show that the earth is not flat, and to estimate its diameter. His slides are posted here.
4. An interactive online demonstration that lets you plot the shortest route between any two points on the surface of the earth is available here. (You'll need to download the free Mathematica Player, which will then allow you to explore hundreds of other interactive demonstrations in all parts of mathematics.)
5. A superb introduction to modern geometry was co-authored by David Hilbert, one of the greatest mathematicians of the $20^{\text {th }}$ century. This classic, originally published in 1952, has been reissued as: D. Hilbert and S. Cohn-Vossen, Geometry and the Imagination (American Mathematical Society, 1999).
6. Several good textbooks and online courses in differential geometry are listed here.
7. Konrad Polthier has produced a number of fascinating educational videos about mathematical topics. Excerpts can be found online here.
Award-winning videos by Polthier and his colleagues appear in the VideoMath Festival collection, available as a DVD from Springer Verlag.
8. The classic algorithm for shortest path problems on networks is due to Edsger Dijkstra. A PDF version of his 1959 paper is available here.
9. Textbook treatments of related routing problems on networks are given online here and here.
10. Steven Skiena has posted an instructive animation of Dijkstra's algorithm.
11. Nature can solve certain shortest path problems by decentralized processes akin to analog computation. For chemical waves that solve mazes, see: O. Steinbock, A. Toth, and K. Showalter, "Navigating complex labyrinths: Optimal paths from chemical waves," Science 267, p. 868 (1995).
Not to be outdone, slime molds can solve them too: T. Nakagaki, H. Yamada, and A. Toth, "Maze-solving by an amoeboid organism," Nature 407, p. 470 (2000).
This slimy creature can even make networks as efficient as the Tokyo rail system: A. Tero et al., "Rules for biologically inspired adaptive network design," Science 327, p. 439 (2010).
12. For an introduction to the mathematics of GPS navigation systems, see: S. Robinson, "Mapping magic," SIAM News (Sep. 26, 2004), available online here.
13. Delightful examples of six-word memoirs are given here and here.

Thanks to Robert Vanderbei, for sending the link to his presentation about estimating the earth's diameter from a photograph of a sunset; Margaret Nelson, for preparing the line drawings; Doug Arnold, Bob Connelly, Paul Ginsparg, Jon Kleinberg, Andy Ruina and Carole Schiffman, for their comments and suggestions; and Konrad Polthier, for generously sharing his videos of geodesics.

