9

The Sphere

In a sequel to *Flatland*, an octagon by the name of Mr. Puncto surveys some rather large triangles and finds that each has angles that add up to slightly more than 180°! This discovery excites Mr. Puncto, and, after double checking his data, he makes the discovery public. Unfortunately neither the civil authorities nor the scientific establishment shares his excitement. They suspect he is merely inventing excuses to explain some errors in his measurements, and they dismiss him from his job. The true explanation is that the Flatlanders are living not in a plane but on a sphere,

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and on a sphere the angles of a triangle really do add up to more than 180°, as we shall soon see. This incident and much more is described in the book *Sphereland* by Dionys Burger. I heartily recommend *Sphereland* to all readers of the present book (just don't be put off by the rather dull summary of *Flatland* at the beginning).

A triangle drawn on a sphere is called a *spherical triangle*. Each side of a spherical triangle is required to be a geodesic; that is, it is required to be intrinsically straight in the sense that a Flatlander on the sphere would perceive it as bending neither to the left nor to the right. A side of a spherical triangle is thus an arc of a so-called great circle (see Figure 9.1).

From now on we will *measure all angles in radians*, to facilitate easier comparison of angles and areas (in a minute you'll see how and why we want to do this). Recall that π radians = 180°, $\pi/2$ radians = 90°, etc. Except when specified otherwise, we will henceforth assume that all spheres are unit spheres, i.e. they all have radius one.

Exercise 9.1 For each spherical triangle in Figure 9.2 compute (1) the sum of its angles in radians, and (2) its area. To compute the areas, use the fact that the unit sphere has area 4π. For example, the first triangle shown occupies $\frac{1}{8}$ of the sphere, so its area is $(4\pi)/8 = \pi/2$.

Find a formula relating a spherical triangle's angle-sum to its area. This formula appeared in 1629 in

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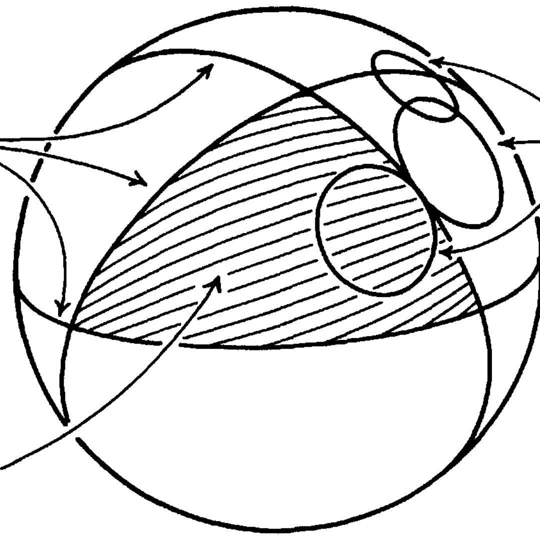

Figure 9.1 On any sphere, the *great circles* are those circles that are as big as possible. A great circle appears straight to a Flatlander on the sphere. By contrast, any lesser circle appears to bend to one side or the other.

the section "De la mesure de la superfice des triangles et polygones sphericques, nouvellement inventee Par Albert Girard" of the book *Invention nouvelle en L'Algebre* by Albert Girard.

You should try to find the formula before reading on, because the following paragraphs give it away. ◫

Exercise 9.2 What is the area of a spherical triangle whose angles in radians are $\pi/2$, $\pi/3$, and $\pi/4$? What is the area of a spherical triangle with angles of 61°, 62°, and 63°? ◫

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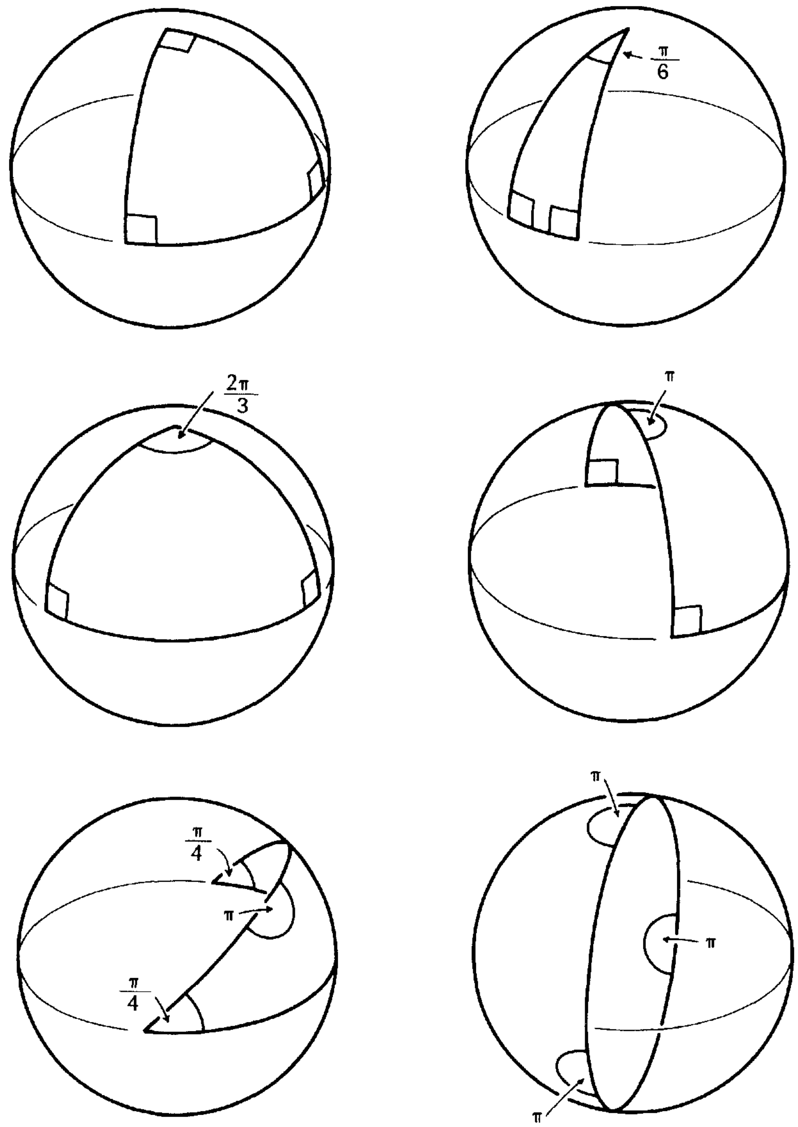

Figure 9.2 Some assorted spherical triangles. Three of the triangles are "degenerate" in the sense that each has one or more angles equal to π. The last triangle occupies an entire hemisphere, and its three sides all lie on the same great circle.

Even though there is no overwhelming need for a proof of the formula you just discovered, I would like to include one anyhow because it is so simple and elegant. (It is not, however, the sort of thing you're likely to stumble onto on your own. I struggled for hours without being able to prove the formula at all.)

First we have to know how to compute the area of a "double lune." A *double lune* is a region on a sphere bounded by two great circles, as shown in Figure 9.3. The largest the angle α can ever be is π, at which point the double lune fills up the entire sphere. So if α is, say, $\pi/3$, then we reason that since $\pi/3$ is $\frac{1}{3}$ the greatest possible angle π, the double lune must

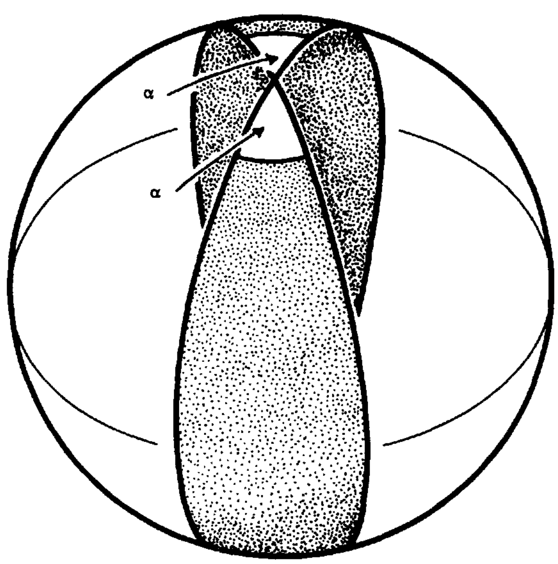

Figure 9.3 A double lune with angle α.

fill up ⅓ the area of the entire sphere, namely $(\frac{1}{3})(4\pi) = 4\pi/3$. Using the same reasoning, we get that the area of a double lune with angle α is $(\alpha/\pi)(4\pi) = 4\alpha$. You can check this formula for some special cases, e.g. $\alpha = \pi/2$ or $\alpha = \pi$.

Now we'll find a formula for the area of a spherical triangle with angles α, β, and γ. First extend the sides of the triangle all the way around the sphere to form three great circles, as shown in Figure 9.4. An "antipodal triangle," identical to the original, is formed on the back side of the sphere. Figure 9.5

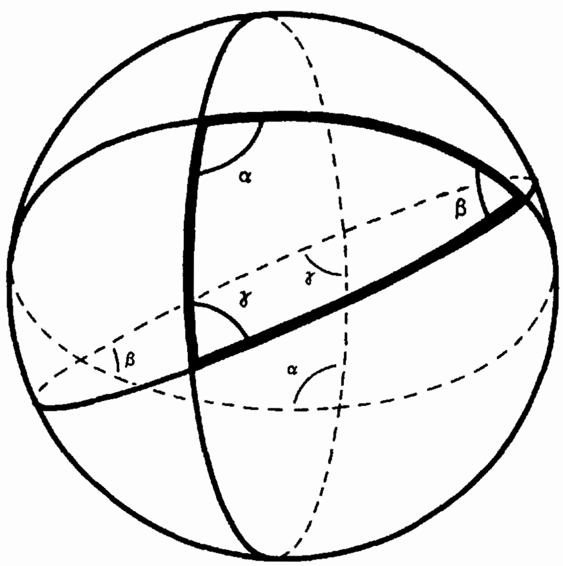

Figure 9.4 Extend the edges of the spherical triangle, and the resulting great circles will form an "antipodal triangle" on the back side of the sphere.

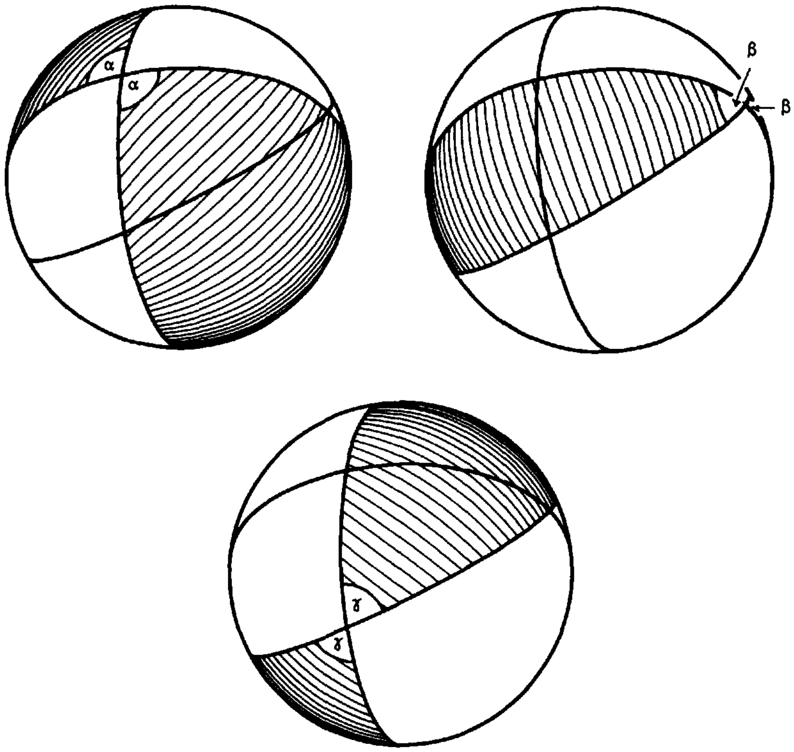

Figure 9.5 Three ways to shade in double lines.

shows three possible ways to shade in double lunes. These double lunes have respective angles α, β, and γ, and therefore their areas are 4α, 4β, and 4γ.

Now look what happens if we shade in all three double lunes simultaneously (Figure 9.6). All parts of the sphere get shaded in at least once, and the original and antipodal triangles each get shaded in three times (once for each double lune). So . . .

[everything was shaded in once] → (area of entire sphere)

[each triangle was shaded in two more times] → (area of original triangle), (area of antipodal triangle)

$$\begin{pmatrix}\text{area of}\\\text{first}\\\text{double}\\\text{lune}\end{pmatrix} + \begin{pmatrix}\text{area of}\\\text{second}\\\text{double}\\\text{lune}\end{pmatrix} + \begin{pmatrix}\text{area of}\\\text{third}\\\text{double}\\\text{lune}\end{pmatrix} = \begin{pmatrix}\text{area of}\\\text{entire}\\\text{sphere}\end{pmatrix} +2\begin{pmatrix}\text{area of}\\\text{original}\\\text{triangle}\end{pmatrix} +2\begin{pmatrix}\text{area of}\\\text{antipodal}\\\text{triangle}\end{pmatrix}$$

$$4\alpha + 4\beta + 4\gamma = 4\pi + 2A + 2A$$

$$4(\alpha + \beta + \gamma) = 4(\pi + A)$$

$$\alpha + \beta + \gamma = \pi + A$$

$$(\alpha + \beta + \gamma) - \pi = A$$

which is just what we wanted to prove! In words, this formula says that the sum of the angles of a spherical triangle exceeds π by an amount equal to the triangle's area.

Exercise 9.3 The formula $(\alpha + \beta + \gamma) - \pi = A$ applies only to triangles on a sphere of radius one. How must you modify the formula to apply to triangles on a sphere of radius two? What about radius three? Write down a general formula for triangles on a sphere of radius r. □

Exercise 9.4 A society of Flatlanders lives on a sphere whose radius is exactly 1000 meters. A farmer has a triangular field with perfectly straight (i.e. geodesic) sides and angles which have been carefully measured as 43.624°, 85.123°, and 51.270°. What is the area of the field? Don't forget to convert the angles to radians. (Bonus Question: How accurately do you

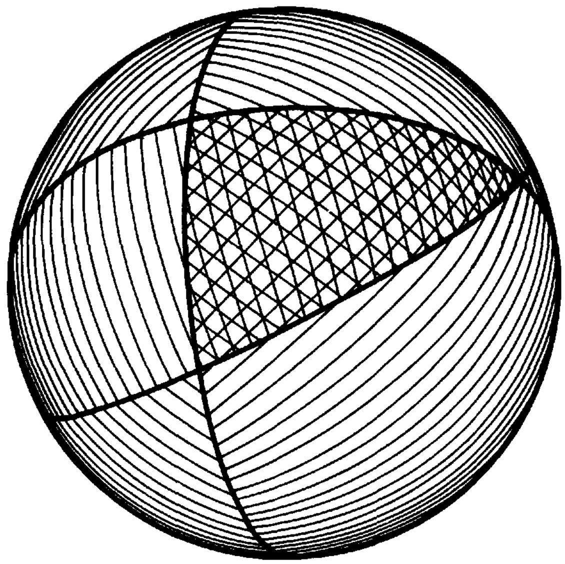

Figure 9.6 Look what happens when we shade in all three double lunes at once.

know the field's area? That is, by plus or minus what percent?) □

Exercise 9.5 A society of Flatlanders lives on a sphere. They carefully survey a triangle and find its angles to be 60.0013°, 60.0007° and 60.0011°, while its area is 5410.52 square meters. What is the area of the entire sphere? □

Exercise 9.6 Estimate the angle-sum of each of the following spherical triangles. Hint: First make a rough estimate of the area enclosed by each triangle, and then apply the formula from Exercise 9.3. Because you have to guess the areas of the triangles, you

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will get only approximate, not exact, answers. The radius of the Earth is roughly 6400 km.

1. The triangle formed by Providence, Newport, and Westerly, Rhode Island. These cities are roughly 50 km apart.
2. The triangle formed by Houston, El Paso, and Amarillo, Texas. These cities are roughly 1000 km apart.
3. The triangle formed by Madras, India; Tokyo, Japan; and St. Petersburg, Russia. These cities are roughly 7000 km apart. □

We've now seen the first major way in which the geometry of a sphere differs from the geometry of a plane. Namely, the sum of the angles of a spherical triangle exceeds π by an amount proportional to the triangle's area, whereas the sum of the angles of a Euclidean (= flat) triangle equals π exactly (study Figure 9.7 for a proof of this last fact).

A piece of a sphere rips open when flattened onto a plane (Figure 9.8). This shows that a circle on a

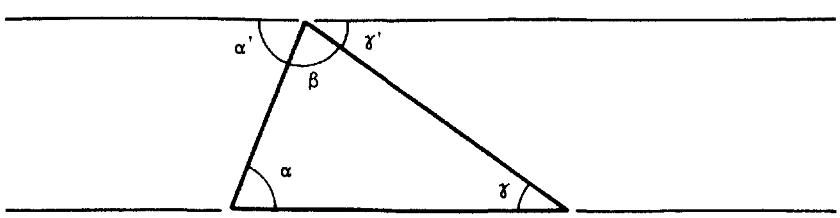

Figure 9.7 A quick proof that the sum of the angles in a Euclidean triangle is π: (1) $\alpha' + \beta + \gamma' = \pi$, (2) $\alpha = \alpha'$ and (3) $\gamma = \gamma'$, therefore (4) $\alpha + \beta + \gamma = \pi$.

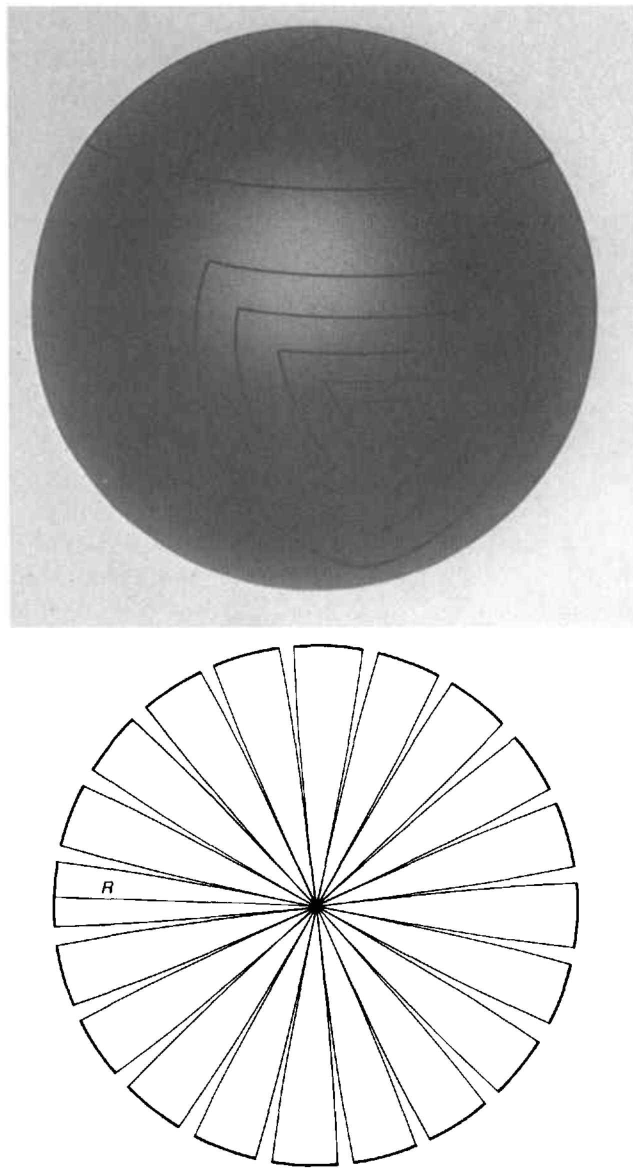

Figure 9.8 A piece of a sphere splits open when flattened.

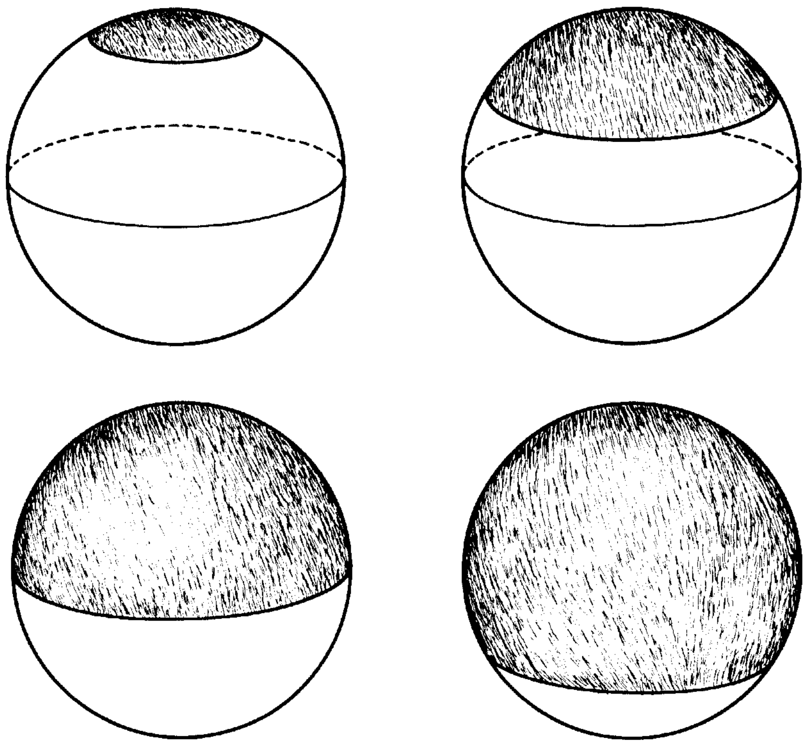

Figure 9.9 The circle's circumference first increases, but then decreases once the circle is past the equator.

sphere has a smaller circumference and encloses less area than a circle of the same radius in a plane. I should stress that the radius of a circle on a sphere is measured along the sphere itself—the way a Flatlander would measure it. Figure 9.9 shows that on a sphere a circle's circumference can actually shrink, even though the circle's (intrinsically measured) radius is still increasing.