

1 Summary¹

1. Book 1 contains 5 postulates (including the famous parallel postulate) and 5 common notions, and covers important topics of plane geometry such as the Pythagorean theorem, equality of angles and areas, parallelism, the sum of the angles in a triangle, and the construction of various geometric figures.
2. Book 2 contains a number of lemmas concerning the equality of rectangles and squares, sometimes referred to as “geometric algebra”, and concludes with a construction of the golden ratio and a way of constructing a square equal in area to any rectilinear plane figure.
3. Book 3 deals with circles and their properties: finding the center, inscribed angles, tangents, the power of a point, Thales’ theorem.
4. Book 4 constructs the incircle and circumcircle of a triangle, as well as **regular polygons** with 4, 5, 6, and 15 sides.
5. Book 5, on proportions of magnitudes, gives the highly sophisticated theory of proportion probably developed by Eudoxus, and proves properties such as “alternation” (if $a : b :: c : d$, then $a : c :: b : d$).
6. Book 6 applies proportions to plane geometry, especially the construction and recognition of similar figures.
7. Book 7 deals with elementary number theory: divisibility, prime numbers and their relation to composite numbers, **Euclid’s algorithm** for finding the greatest common divisor, finding the least common multiple.
8. Book 8 deals with the construction and existence of geometric sequences of integers.
9. Book 9 applies the results of the preceding two books and gives the **infinitude of prime numbers** and the construction of all even perfect numbers.
10. Book 10 proves the **irrationality** of the square roots of non-square integers (e.g. $\sqrt{2}$) and classifies the square roots of incommensurable lines into thirteen disjoint categories. Euclid here introduces the term “irrational”, which has a different meaning than the modern concept of irrational numbers. He also gives a formula to produce Pythagorean triples.
11. Book 11 generalizes the results of book 6 to solid figures: perpendicularity, parallelism, volumes and similarity of parallelepipeds.
12. Book 12 studies the volumes of cones, pyramids, and cylinders in detail by using the method of exhaustion, a precursor to integration, and shows, for example, that the volume of a cone is a third of the volume of the corresponding cylinder. It concludes by showing that the volume of a sphere is proportional to the cube of its radius (in modern language) by approximating its volume by a union of many pyramids.

¹https://en.wikipedia.org/wiki/Euclid%27s_Elements

13. Book 13 constructs the **five regular Platonic solids** inscribed in a sphere and compares the ratios of their edges to the radius of the sphere.

Summary Contents of Euclid's Elements

Book	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	Totals
Definitions	23	2	11	7	18	4	22	-	-	16	28	-	-	131
Postulates	5	-	-	-	-	-	-	-	-	-	-	-	-	5
Common Notions	5	-	-	-	-	-	-	-	-	-	-	-	-	5
Propositions	48	14	37	16	25	33	39	27	36	115	39	18	18	465

2 Definitions in Book 1

Definition 1. A point is that which has no part.

Definition 2. A line is breadthless length.

Definition 3. The ends of a line are points.

Definition 4. A straight line is a line which lies evenly with the points on itself.

Definition 5. A surface is that which has length and breadth only.

Definition 6. The edges of a surface are lines.

Definition 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Definition 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definition 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Definition 11. An obtuse angle is an angle greater than a right angle.

Definition 12. An acute angle is an angle less than a right angle.

Definition 13. A boundary is that which is an extremity of anything.

Definition 14. A figure is that which is contained by any boundary or boundaries.

Definition 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16. And the point is called the center of the circle.

Definition 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Definition 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Definition 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Definition 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definition 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Definition 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

3 Postulates and Common Notions in Book 1

Postulate 1. *To draw a straight line from any point to any point.*

Postulate 2. *To produce a finite straight line continuously in a straight line.*

Postulate 3. *To describe a circle with any center and radius.*

Postulate 4. *That all right angles equal one another.*

Postulate 5. *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Common Notion 1. *Things which equal the same thing also equal one another.*

Common Notion 2. *If equals are added to equals, then the wholes are equal.*

Common Notion 3. *If equals are subtracted from equals, then the remainders are equal.*

Common Notion 4. *Things which coincide with one another equal one another.*

Common Notion 5. *The whole is greater than the part.*

4 Propostions in Book 1

Proposition 1. *To construct an equilateral triangle on a given finite straight line.*

Proposition 2. *To place a straight line equal to a given straight line with one end at a given point.*

Proposition 3. *To cut off from the greater of two given unequal straight lines a straight line equal to the less.*

Proposition 4. *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

Proposition 5. *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

Proposition 6. *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

Proposition 7. *Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.*

Proposition 8. *If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.*

Proposition 9. *To bisect a given rectilinear angle.*

Proposition 10. *To bisect a given finite straight line.*

Proposition 11. *To draw a straight line at right angles to a given straight line from a given point on it.*

Proposition 12. *To draw a straight line perpendicular to a given infinite straight line from a given point not on it.*

Proposition 13. *If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.*

Proposition 14. *If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.*

Proposition 15. *If two straight lines cut one another, then they make the vertical angles equal to one another. Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.*

Proposition 16. *In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.*

Proposition 17. *In any triangle the sum of any two angles is less than two right angles.*

Proposition 18. *In any triangle the angle opposite the greater side is greater.*

Proposition 19. *In any triangle the side opposite the greater angle is greater.*

Proposition 20. *In any triangle the sum of any two sides is greater than the remaining one.*

Proposition 21. *If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.*

Proposition 22. *To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.*

Proposition 23. *To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.*

Proposition 24. *If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.*

Proposition 25. *If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.*

Proposition 26. *If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.*

Proposition 27. *If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.*

Proposition 28. *If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.*

Proposition 29. *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.*

Proposition 30. *Straight lines parallel to the same straight line are also parallel to one another.*

Proposition 31. *To draw a straight line through a given point parallel to a given straight line.*

Proposition 32. *In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.*

Proposition 33. *Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.*

Proposition 34. *In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.*

Proposition 35. *Parallelograms which are on the same base and in the same parallels equal one another.*

Proposition 36. *Parallelograms which are on equal bases and in the same parallels equal one another.*

Proposition 37. *Triangles which are on the same base and in the same parallels equal one another.*

Proposition 38. *Triangles which are on equal bases and in the same parallels equal one another.*

Proposition 39. *Equal triangles which are on the same base and on the same side are also in the same parallels.*

Proposition 40. *Equal triangles which are on equal bases and on the same side are also in the same parallels.*

Proposition 41. *If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.*

Proposition 42. *To construct a parallelogram equal to a given triangle in a given rectilinear angle.*

Proposition 43. *In any parallelogram the complements of the parallelograms about the diameter equal one another.*

Proposition 44. *To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.*

Proposition 45. *To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.*

Proposition 46. *To describe a square on a given straight line.*

Proposition 47. *In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.*

Proposition 48. *If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.*