Midterm Exam (take home)

Topology, Math 441, Spring 2020
Instructor: Abhijit Champanerkar

Date: April 15th, 2020 **Points:** 100

- Please submit by emailing me the solutions in pdf form either typed or written on plain or ruled paper, by **11:59 pm on Wednesday April 15th, 2020**.
- Justify your answers and write clear proofs for full credit. You can assume the continuity of single and multi-variable functions you have seen in Calculus and Advanced Calculus courses.
- **Policy about plagiarsm and cheating:** Please note that you may be asked to explain and justify to me in an one-on-one online meeting any solution you write down.
- Please include the following honor code statement in your submission, right after your name: **I,** *student name*, **have submitted only those solutions that I fully understand myself.**

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1. *(20 points)* Identify the set of 2×2 matrices, denoted by $\mathrm{M}(2, \mathbb{R})$ with $\mathbb{R}^4$, thereby giving it the topology inherited from the standard topology from $\mathbb{R}^4$.

   (a) Prove that the set of 2×2 invertible matrices, denoted as $\mathrm{GL}(2, \mathbb{R})$, is an open subset of $\mathrm{M}(2, \mathbb{R})$.

   (b) Prove that the set of 2×2 special orthogonal matrices i.e. matrices A such that $AA^T = \mathrm{Id}$ and $\det(A) = 1$, denoted as $\mathrm{SO}(2, \mathbb{R})$, is a closed subset of $\mathrm{M}(2, \mathbb{R})$.

   (c) Do you think the statements (a) and (b) are true about $n \times n$ matricies ?

2. *(20 points)*

   (a) Let $(p_1, p_2, p_3) \in \mathbb{R}^3$ be any point. Prove that $\mathbb{R}^3 - \{(p_1, p_2, p_3)\} \cong \mathbb{R}^3 - \{(0, 0, 0)\}$.

   (b) Prove that $\mathbb{R}^3 - \{(p_1, p_2, p_3)\}$ is connected.

3. *(20 points)*

   (a) Prove that a space X is connected if and only if there does not exist a continuous, surjective function $f : X \to \{0, 1\}$.

   (b) Prove that the set of 2×2 invertible matrices, denoted as $\mathrm{GL}(2, \mathbb{R})$, is disconnected.

4. *(20 points)* Decide if the following functions are continuous or not. Please give a short justification.

   (a) $f : \mathbb{R}_{dis} \to \mathbb{R}$, $f(x) = 2x$.

   (b) $f : \mathbb{R}_{cofin} \to \mathbb{R}$, $f(x) = 2x$.

   (c) $f : \mathbb{R} \to \mathbb{R}_{\ell}$, $f(x) = 2x$.

   (d) $f : \mathbb{R}_{PP=0} \to \mathbb{R}$, $f(x) = 2x$.

(e) $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$.

5. *(20 points)*

   (a) Prove that any $x \in \mathbb{Z}$ is the limit point of the set $\{3n + 2 \,|n \in \mathbb{Z}_+\}$ in the finite complement topology on $\mathbb{Z}$.

   (b) Prove that we cannot define a metric on $\mathbb{R}$ which will induce the finite complement topology on $\mathbb{R}$.