Solutions to take home Midterm Exam

1. *(20 points)* Identify the set of 2×2 matrices, denoted by $\mathrm{M}(2,\mathbb{R})$ with $\mathbb{R}^4$, thereby giving it the topology inherited from the standard topology from $\mathbb{R}^4$.

   (a) Prove that the set of 2×2 invertible matrices, denoted as $\mathrm{GL}(2,\mathbb{R})$, is an open subset of $\mathrm{M}(2,\mathbb{R})$.

   (b) Prove that the set of 2×2 special orthogonal matrices i.e. matrices A such that $AA^T = \mathrm{Id}$ and $\det(A) = 1$, denoted as $\mathrm{SO}(2,\mathbb{R})$, is a closed subset of $\mathrm{M}(2,\mathbb{R})$.

   (c) Do you think the statements (a) and (b) are true about $n\times n$ matricies ?

Solution: a) $M(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{R} \right\} \cong \mathbb{R}^4$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,b,c,d)$$

$\det : M(2,\mathbb{R}) \to \mathbb{R}$, $\det(A) = ad - bc$ cts as it is a polynomial

$A \in GL(2,\mathbb{R})$ iff A invertible iff $\det(A) \neq 0$

$\Rightarrow GL(2,\mathbb{R}) = \det^{-1}(\mathbb{R} - 0)$

$\mathbb{R} - 0 \subset \mathbb{R}$ open, det cts $\Rightarrow GL(2,\mathbb{R})$ open.

b) $f : M(2,\mathbb{R}) \times M(2,\mathbb{R}) \longrightarrow M(2,\mathbb{R})$

$f(A,B) = AB$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & \cdots \\ \cdots & \cdots \end{pmatrix}$ not needed

f cts $\quad$ polynomial in every co-ord

$g : M(2,\mathbb{R}) \longrightarrow M(2,\mathbb{R})$

$g(A) = AA^T$ $\quad$ cts as matrix mul cts & $A \mapsto A^T$ cts

$g^{-1}(Id)$ closed as

$\{Id\} \subset M(2,\mathbb{R})$ closed $\quad Id \to (1,0,0,1) \in \mathbb{R}^4$

$\det^{-1}(\{1\})$ closed as det cts & $\{1\} \subset \mathbb{R}$ closed

$SO(2,\mathbb{R}) = g^{-1}(\{Id\}) \cap \det^{-1}(\{1\}) \Rightarrow$ closed.

c) Yes true for $n\times n$ matricies.

2. *(20 points)*

(a) Let $(p_1, p_2, p_3) \in \mathbb{R}^3$ be any point. Prove that $\mathbb{R}^3 - \{(p_1,p_2,p_3)\} \cong \mathbb{R}^3 - \{(0,0,0)\}$.

(b) Prove that $\mathbb{R}^3 - \{(p_1,p_2,p_3)\}$ is connected.

Solution: a) $f: \mathbb{R}^3 \to \mathbb{R}^3$ $f(x,y,z) = (x+p_1, y+p_2, z+p_3)$

linear in every co-ord

$\Rightarrow f$ cts

$f(0,0,0) = (p_1,p_2,p_3)$

$g: \mathbb{R}^3 \to \mathbb{R}^3$ $g(x,y,z) = (x-p_1, y-p_2, z-p_3)$.

g cts

$g(p_1,p_2,p_3) = (0,0,0)$

$g = f^{-1}$

f homeo & $f: \mathbb{R}^3 - (0,0,0) \to \mathbb{R}^3 - (p_1,p_2,p_3)$

$\Rightarrow$ they are homeo.

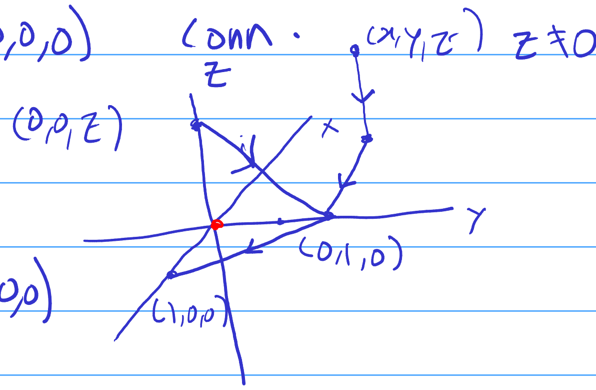
b) Enough to show $\mathbb{R}^3 - (0,0,0)$ conn.

Proof 1:

Enough to show $\mathbb{R}^3 - (0,0,0)$ is path conn.

We will find a path from (x,y,z) to $(1,0,0)$

See picture.



cases: ① $z=0$, ② $z \neq 0$ ③ $z \neq 0, y = 0$ ④ $z \neq 0, y \neq 0$

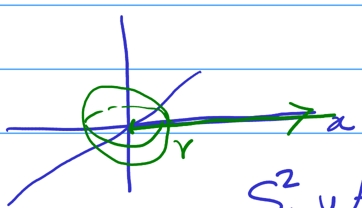
draw piecewise linear path in all these cases.

$\mathbb{R}^3 - (0,0,0)$ path conn $\Rightarrow$ conn.

Proof 2: Assume $S^2 \subset \mathbb{R}^3$ connected

$\Rightarrow S^2_r = \{(x,y,z) \mid x^2+y^2+z^2 = r^2\}$ is conn

$A = \{(r,0,0) \mid r > 0\}$ conn



$\mathbb{R}^3 - (0,0,0) = \bigcup_{r>0} (S^2_r \cup A)$

$S^2_r \cup A$ conn $\forall r$, $\bigcap_{r>0} (S^2_r \cup A) = A \neq \emptyset$ $\Rightarrow$ conn.

3. *(20 points)*

(a) Prove that a space X is connected if and only if there does not exist a continuous, surjective function $f : X \to \{0, 1\}$.

(b) Prove that the set of 2×2 invertible matrices, denoted as $\mathrm{GL}(2, \mathbb{R})$, is disconnected.

Solution: a) Assume $\exists\ f: X \to \{0,1\}$ cts surj.

let $U = f^{-1}(\{0\})$ & $V = f^{-1}(\{1\})$

$\{0,1\}$ has discrete top $\Rightarrow U, V$ open

surj $\Rightarrow U, V \neq \emptyset$ & $U \cup V = X$

$\{0\} \cap \{1\} = \emptyset \Rightarrow U \cap V \neq \emptyset$

$\Rightarrow \{U, V\}$ is a separation of $X \Rightarrow X$ is discon.

Conversely Assume X is discon $\Rightarrow \exists$ separation $\{U, V\}$

define $f: X \to \{0,1\}$ as

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$

$U, V \neq \emptyset \Rightarrow f$ surj

$\{0,1\}$ disc. top $\Rightarrow f^{-1}(\{0\}) = U$, $f^{-1}(\{1\}) = V$

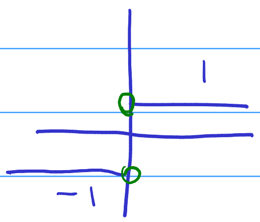
separation $\Rightarrow U, V$ open

$\Rightarrow f$ cts.

b) Define $\mathrm{GL}(2,\mathbb{R}) \xrightarrow{f} \{-1, 1\} \xrightarrow{g} \{0,1\}$ $\quad g(1) = 1$, $g(-1) = 0$

$f(A) = \dfrac{\det(A)}{|\det(A)|}$, det cts, $\det(A) \neq 0$ $A \in \mathrm{GL}(2,\mathbb{R})$

$\Rightarrow f$ cts



$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$, $f\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = -1$

$\Rightarrow f$ surj

$f(x) = \dfrac{x}{|x|}$ $x \neq 0$

$g \circ f$ cts surj $\Rightarrow \mathrm{GL}(2,\mathbb{R})$ is disconn.

$f: \mathbb{R} - 0 \to \{-1, 1\}$ cts

not conn.

4. *(20 points)* Decide if the following functions are continuous or not. Please give a short justification.

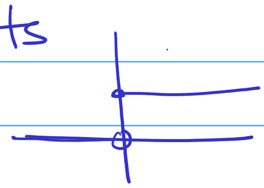
(a) $f : \mathbb{R}_{dis} \to \mathbb{R}$, $f(x) = 2x$. cts

(b) $f : \mathbb{R}_{cofin} \to \mathbb{R}$, $f(x) = 2x$. not cts

(c) $f : \mathbb{R} \to \mathbb{R}_{\ell}$, $f(x) = 2x$. $f^{-1}([0,2)) = [0,1)$ not open $\Rightarrow$ not cts

(d) $f : \mathbb{R}_{PP=0} \to \mathbb{R}$, $f(x) = 2x$. $f^{-1}((2,4)) = (1,2)$ not open $\Rightarrow$ not cts

(e) $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. not cts



5. *(20 points)*

(a) Prove that any $x \in \mathbb{Z}$ is the limit point of the set $\{3n + 2 \mid n \in \mathbb{Z}_+\}$ in the finite complement topology on $\mathbb{Z}$.

(b) Prove that we cannot define a metric on $\mathbb{R}$ which will induce the finite complement topology on $\mathbb{R}$.

Solution: a) U be open nbd of x $\Rightarrow$ $\mathbb{Z} - U$ finite

$\Rightarrow$ $N = \max\{|\alpha| \mid \alpha \in \mathbb{Z} - U\}$

$\Rightarrow$ $3n+2 > N$ $\forall$ $n > N$

$\Rightarrow$ $3n+2 \notin \mathbb{Z} - U$ $\Rightarrow$ $3n+2 \in U$ $\forall n > N$

$\Rightarrow$ $3n+2 \to x$

b) Metric sp. are Hausdorff.

Let U, V be open sets in $\mathbb{R}_{fc}$.

$\Rightarrow$ $\mathbb{R} - U$, $\mathbb{R} - V$ finite.

let $M = \max\{|\alpha| \mid \alpha \notin U, \alpha \notin V\}$ then

$M \in U \cap V$ $\Rightarrow$ $\nexists$ disjoint open sets in $\mathbb{R}_{fc}$

$\Rightarrow$ $\mathbb{R}_{fc}$ is not Hausdorff

$\Rightarrow$ cannot be metric sp.