Homework 7

Topology, Math 441, Spring 2020
Topic: Quotient Topology, Connectedness
Due: Friday April 17th, 2020

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Reading: Its important to read the following sections to understand quotient topology.

1. pg 56-61 from the text book.
2. Section 3.3 from book *Introduction to Topology* by Adams-Franzosa (skip Example 3.18).
3. Section 3.4 from book *Introduction to Topology* by Adams-Franzosa.

Problems:

1. Give an example to show that the quotient space of a Hausdorff space need not be Hausdorff.
2. (a) Prove that $f : X \to Y$ is a quotient map if subset $K \subset Y$ is closed if and only if $f^{-1}(K)$ is closed.

   (b) A map $f : X \to Y$ is closed if it takes closed sets to closed sets. Using (a) prove that a surjective, continuous, closed map is a quotient map.
3. Consider the equivalence relation on $\mathbb{R}^2$ defined by $(x_1, x_2) \sim (y_1, y_2)$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Describe the quotient space $\mathbb{R}^2/\sim$.
4. Describe or draw a picture of the resulting quotient space. Assume that points are identified only with themselves unless they ar eexplicitly said to be identified with other points.

   (a) The interval $[0, 4] \subset \mathbb{R}$, with integer points identified.

   (b) The sphere with the equator collapsed to a point.

   (c) The real line $\mathbb{R}$ with $[-1, 1]$ collapsed to a point.
5. Prove that $\mathbb{R}$ with the following topology is connected:

   (a) $\mathbb{R}_{PP=0}$, the particular point topology with $p = 0$.

(b) $\mathbb{R}_{EP=0}$, the excluded point topology with $p = 0$.

(c) $\mathbb{R}_{co-fin}$, the co-finite topology on $\mathbb{R}$.

6. Show that if $A \subset X$ is connected then A is contained in a component of X.

7. Prove that $S^2 \subset \mathbb{R}^3$ is connected.

8. Use cutsets to prove the following:

   (a) $[a, b]$ is not homeomorphic to (a, b).

   (b) $[a, b)$ is not homeomorphic to (a, b).

   (c) $(a, b]$ is not homeomorphic to (a, b).

   (d) $(a, b]$ is not homeomorphic to $[a, b]$.

9. Prove that no two-point set separates $\mathbb{R}^2$.

*10. Let X be a topological space and Y be a set. Let $p : X \to Y$ be a surjective map. Prove that the quotient topology on Y induced by p (i.e. using the equivalence relation on X given by $x \sim y \iff p(x) = p(y)$ is the finest topology under which p is continuous.

Handin: 3, 4ab, 5bc, 7, 8ab, 9