Homework 2

Topology, Math 441, Spring 2020
Topic: Metric Spaces
Due: Monday Feb 24th, 2020

THE CITY UNIVERSITY OF NEW YORK
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Reading:

1. Pages 15-19 from Chapter 2 of the text book.

2. Here is a quick proof of the Cauchy-Schwarz Inequality:

   Theorem 1. *Let* $\vec{x} = (x_1, x_2, \dots, x_n), \vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. *Then*

   $$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \; \|\vec{y}\|.$$

   Proof: Need to prove

   $$(\sum_{i=1}^{n} x_i y_i)^2 \leq (\sum_{i=1}^{n} x_i^2)(\sum_{j=1}^{n} y_j^2) \qquad i.e.$$

   $$\sum_{i=1}^{n} x_i^2 y_i^2 + 2\sum_{i \neq j} x_i y_i x_j y_j \leq \sum_{i,j=1}^{n} x_i^2 y_j^2 \qquad i.e.$$

   $$2\sum_{i \neq j} x_i y_i x_j y_j \leq \sum_{i \neq j}^{n} x_i^2 y_j^2 \qquad i.e.$$

   $$0 \leq \sum_{i \neq j} x_i^2 y_j^2 - 2\sum_{i \neq j} x_i y_i x_j y_j \qquad i.e.$$

   $$0 \leq \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2$$

   which holds. Hence the proof. □

3. Given a vector space V, a *norm* on V is a function $\|\| : V \to \mathbb{R}$ satisfying:

   (a) $\|\vec{x}\| \geq 0$ for all $\vec{x} \in V$ and $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.

   (b) $\|\alpha \, \vec{x}\| = |\alpha| \|\vec{x}\|$ for any scalar α and $\vec{x} \in V$.

   (c) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in V$ (Triangle inequality).

$(V, \|\|)$ is called a *normed linear space*. For example, given a dot product (inner product) on a vector space V, it defines a *norm* by setting $\|\vec{x}\| = (\vec{x} \cdot \vec{x})^{1/2}$.

Problems:

1. Let $(V, \|\|)$ be a normed linear space. Define $d(x, y) = \|x - y\|$. Then d is a metric on V.

2. Prove that $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|\vec{x}\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}$ are norms on $\mathbb{R}^n$.

3. Let (X, d) be a metric space. Define $\delta(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Show that δ is a metric on X.

4. Let (X, d_1) and (Y, d_2) be metric space. Show that

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

   defines a metric on the product $X \times Y$.

5. Let (X, d) be a metric space, $x \in X$ and $0 < r < s$. Show that $B_r(x) \subseteq B_s(x)$. Give an example when they are equal.

6. Show that the following sets are open. Assume $\mathbb{R}^n$ with standard Euclidean metric.

   (a) $\mathbb{R} - \mathbb{Z}$.

   (b) $(a, b) \times (c, d) \subset \mathbb{R}^2$.

   (c) Complement of singleton set in any metric space. This property is known as T_1.

   (d) Complement of a finite set in any metric space.

7. Is $\mathbb{Q} \subset \mathbb{R}$ open ?

8. Let (X, d) be a metric space and let the set X be finite. Show that every subset of X is open. (Hint: First show that singletons are open).

Handin: 2, 4, 6, 8