Final Exam - Part 2 - Take home Exam

Topology, Math 441, Spring 2020
Instructor: Abhijit Champanerkar

Date: May 18th, 2020 **Points:** 100

- Please submit by **emailing me** the solutions in pdf form either typed or written on plain or ruled paper (and scanned), by **11:59 pm on Monday May 18th, 2020**.
- Justify your answers and write clear proofs for full credit. You can assume the continuity of single and multi-variable functions you have seen in Calculus and Advanced Calculus courses.
- **Policy about plagiarsm and cheating:** Please note that you may be asked to explain and justify your solutions to me in an one-on-one online meeting.
- **VERY IMPORTANT** Please include the following honor code statement in your submission, right after your name: **I,** *student name*, **have submitted only those solutions that I fully understand myself.**
- Submit any 5 problems. Please indicate at the begining of your answers which problems you are attempting.

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1. *(20 points)*
   (a) Let A denote the z-axis. Prove that $\mathbb{R}^3 - A$ is connected and path-connected.
   (b) Prove that $\mathbb{R}^3 - S^2$ is disconnected. Please justify all your steps.

2. *(20 points)*
   (a) Show that every set $A \subset \mathbb{R}$ is a compact subset in the finite complement topology of $\mathbb{R}$.
   (b) Give an example to show that arbitrary union of compact sets in a space X is not necessarily compact.

3. *(20 points)* The sum of two subsets A and B of $\mathbb{R}^n$ is defined to be $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.
   (a) Prove that the addition function is continuous i.e. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(a, b) = a + b$ is continuous.
   (b) Prove that the sum of two compact sets in $\mathbb{R}$ is compact

4. *(20 points)* Let X be a topological space. A family $\mathcal{F}$ of subsets of X is called *locally finite* if every point $x \in X$ has an open neighborhood U such that only finitely many members of $\mathcal{F}$ have nonempty intersection with U. Prove that if $\mathcal{F}$ is a locally finite family of closed sets then the union $\cup_{C \in \mathcal{F}}$ is closed.

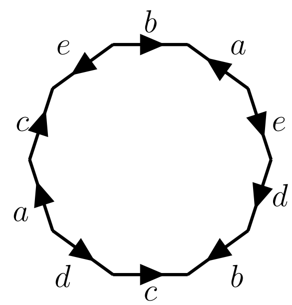
5. *(20 points)* A topological space X has the following property: Suppose that $\mathcal{F} = \{F_j \mid j \in J\}$ is a collection of closed subsets of X such that every finite intersection of subsets from $\mathcal{F}$ is non-empty, then $\cap_{j \in J} F_j \neq \phi$. Prove that X is compact if and only if this propert holds.

6. *(20 points)* Let $S^4 = \{(x, y, z, u, v) \in \mathbb{R}^5 \mid x^2 + y^2 + z^2 + u^2 + v^2 = 1\}$ be the 4-sphere i.e. the set of unit length vectors in $\mathbb{R}^5$. Prove that S^4 is a 4-manifold by finding explicit charts for it.

7. *(20 points)*

   (a) Give an example to show that an identification of sides of polygon which has 3 or more sides identified cannot result in a surface.

   (b) Compute the Euler characteristic of the surface obtained by identifying the sides of the polygon drawn below, and identify which surface it is.



8. *(20 points)*

   (a) Assuming that in the plane every simple closed curve separates the plane in two open sets, one of which is homeomorphic to an open disk, prove that any simple closed curve on S^2 separates it.

   (b) Using cutsets prove that S^2 is not homeomorphic to gT for any $g \geq 1$.