Review for Exam 2

Complex Analysis, MTH 431, Spring 2014

Syllabus

- Exam 2 will be held on Monday April 28th.
- Syllabus for Exam 2: Chapters 5,6,7
- Exam will include definitions, statement of Theorems, true or false questions and some elementary proofs.

Key Concepts

$U \subseteq \mathbb{C}$ open and connected.

Chapter 5

1. Parametrization of a curve, simple curve, closed curve, Piecewise smooth curve, arc length of a smooth curve
2. Parametrization of a line segment joining two complex numbers c and d: $\gamma(t) = (1-t)c + td$, Parametrization of a circle, Parametrization of reversely oriented curve: $\overleftarrow{\gamma}(t) = \gamma(a+b-t)$ for $t \in [a,b]$.
3. Integral of f along γ, Properties of integral of f along γ (a) $\int\limits_{\gamma}(f \pm g) = \int\limits_{\gamma} f \pm \int\limits_{\gamma} g$ (b) $\int\limits_{\gamma} k.f = k\int\limits_{\gamma} f$ (c) $\int\limits_{\overleftarrow{\gamma}} f = -\int\limits_{\gamma} f$
4. Fundamental Theorem of Calculus for $\mathbb{C}$ (Theorem 5.19): *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve. Let $F : U \subseteq \mathbb{C} \to \mathbb{C}$ be such that $F'(z)$ exists and is continuous. Let $\gamma \subset U$. Then,* $\int\limits_{\gamma} F'(z)dz = F(\gamma(b)) - F(\gamma(a))$. If γ is closed, $\int\limits_{\gamma} F'(z)dz = 0$.
5. Absolute value inequality (Theorem 5.23), ML inequality (Theorem 5.24).

Chapter 6

1. A simple closed curve divides the plane into a bounded interior and unbounded exterior regions.
2. Caucy-Goursat theorem (Theorem 6.1).
3. Deformation theorems (Theorems 6.6 and 6.7).

Chapter 7

1. Cauchy's Integral Formula (Theorem 7.1).
2. Cauchy's Integral Formula for $f'(a)$ and $f^{(n)}(a)$ (Theorems 7.4 and 7.5).
3. Morera's theorem, Liouville's theorem (Theorem 7.8 and 7.9).
4. Theorem: If f is entire than $1/f$ cannot be entire.
5. If $p(z)$ is a polynomial of degree $n \geq 1$, then $\exists a \in \mathbb{C}$ such that $p(a) = 0$ and The Fundamental theorem of Algebra (Theorems 7.10, 7.11).
6. Taylor Series of holomorphic f in $N(c, r)$ - Theorem 7.16
7. Uniqueness of Taylor series (Remark 7.17).

All Theorem numbers are from the textbook. Some of the other problems are taken from one of the references. U will always denote an open, connected subset of $\mathbb{C}$, and γ will always denote a contour, unless otherwise stated.

Sample Review Questions

From Chapter 5

1. Show that: $\int_\gamma \frac{1}{z} dz = 2\pi i$ over the circle centered at 0 and radius 3, parameterized as $\gamma(t) = 3e^{it}$, $0 \leq t \leq 2\pi$.
2. Show that

$$\int_\gamma (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

where γ is a circle of radius R centered at a (Theorem 5.13).

3. Integrate $f(z) = \overline{z}$ over the line segment joining 0 to $1 + i$.

4. Integrate $f(z) = \overline{z}^2$ over a circle of radius 2 centered at 0.

5. Integrate $f(z) = z + \overline{z}$ over the circle $|z| = 2$ oriented counterclockwise.

6. Evaluate $\int_\gamma (z^2 - 2z + 3) dz$ over the circle $|z| = 2$ oriented counterclockwise.

7. Evaluate $\int_\gamma e^z dz$ over a simple, curve $\gamma(t)$, $0 \le t \le 1$ which satisfies $\gamma(0) = -4 + i$ and $\gamma(1) = 6 + 2i$. (Hint: Apply Theorem 5.19 on Page 95)

8. Evaluate $\int_\gamma \sin z dz$ over a simple, curve $\gamma(t)$, $0 \le t \le 1$ which satisfies $\gamma(0) = i$ and $\gamma(1) = \pi$. (Hint: Apply Theorem 5.19 on Page 95)

From Chapter 6

1. Evaluate $\int_\gamma (z^2 - 2z + 3) dz$ over the circle $|z| = 2$ oriented counterclockwise using Cauchy's theorem

2. Evaluate $\int_\gamma e^z dz$ over γ consisting of two straight lines, one joining $-4 + i$ to $6 + i$ and other joining $6 + i$ to $6 + 2i$ by using the previous exercise and Theorem 6.6 on Page 115.

3. Evaluate $\int_\gamma \sin z dz$ over γ consisting of two straight lines, one joining i to 0 and other joining 0 to π by using the previous exercise and Theorem 6.6 on Page 115.

4. Compute $\int_\gamma \frac{1}{z^2+1} dz$ where $\gamma(t)$ is a circle centered at $2i$ with radius $\frac{1}{2}$. (Apply Cauchy's theorem!)

5. Compute $\int_\gamma \frac{1}{z^2+1} dz$ where $\gamma(t)$ is a circle centered at $2i$ with radius 2. (Use partial fractions, apply Cauchy's theorem to one term and $\int_\gamma \frac{1}{z-a} dz$ from Chapter 5 to the second term.)

From Chapter 7

1. Suppose f is holomorphic on and inside the circle $z = w + re^{it}$, $0 \le t \le 2\pi$. Show that: $f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it}) dt$.
   (This is a special case of Cauchy's Integral formula.)

2. Compute $\int_0^{2\pi} \sin(e^{it})dt$ using the formula above. ($e^{it} = 0 + 1.e^{it}$, so what is w?)

3. Compute $\int_0^{2\pi} \cos(e^{it})dt$ using the formula above.

4. Compute $\int_0^{2\pi} e^{ie^{it}}dt$ using the exercises above. ($e^{ie^{it}} = \cos(e^{it})+i\sin(e^{it})$)

5. Compute $\int_\gamma \frac{1}{z^2+1}dz$ where $\gamma(t)$ is a circle centered at $2i$ with radius 2. (Use $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{\frac{1}{z+i}}{z-i}$ and apply Cauchy's integral formula!)

6. Evaluate the following integrals:
   (a) $\int_{|z|=2} \frac{e^z}{z}dz$ (b) $\int_{|z|=2} \frac{e^z}{z-3}dz$ (c) $\int_{|z|=2} \frac{e^z}{z(z-3)}dz$
   (d) $\int_{|z|=1} \frac{\sin z}{z}dz$ (e) $\int_{|z|=1} \frac{\sin z}{z^2}dz$ (f) $\int_{|z|=1} \frac{\cos z}{z^3}dz$

7. Evaluate $\int_{|z|=2} \frac{1}{z^2(z-1)^2}dz$ (Use Theorem 7.5)

8. Page 125/ Exercises 7.1b,c

9. State True or False: *Modulus of* $\sin z$ *is bounded between* -1 *and* $1.$, why?

10. Obtain the Taylor series expansion of $\frac{1}{1-z}$, centered at 0 and state its radius of convergence.

11. Obtain the Taylor series expansion of $\frac{1}{(1+z)^2}$, centered at 0 and state its radius of convergence.

12. Obtain the Taylor series expansion of e^z, centered at 1 and state its radius of convergence.

13. Obtain the Taylor series expansion of $\cos(z)$, centered at $\pi/2$ and state its radius of convergence.

Proofs

Theorems 5.13, 5.24 (assuming 5.23), 6.6, 6.7, 7.8 (Morera's Theorem), 7.9 (Liouville's Theorem), Remark 7.17.