Review for Exam 1

Complex Analysis, MTH 431, Spring 2014

Key Concepts

Chapter 2

1. Standard form of a complex number $z = x + iy$
2. Geometric representation of a complex number
3. Conjugate $\overline{z} = x - iy$
4. Properties of conjugates (Page 21)
   (a) $\overline{\overline{z}} = z$
   (b) $\overline{z + w} = \overline{z} + \overline{w}$
   (c) $\overline{zw} = \overline{z}.\overline{w}$
   (d) $z + \overline{z} = 2\text{Re}(z)$
   (e) $z - \overline{z} = 2i\text{Im}(z)$
5. Modulus of a complex number $|z|^2 = z\overline{z}$
6. Properties of modulus: (Theorem 2.1, Page 22)
   (a) $|\text{Re}z| \leq |z|$
   (b) $|\text{Im}z| \leq |z|$
   (c) $|\overline{z}| = |z|$
   (d) $|zw| = |z||w|$
   (e) $|z + w| \leq |z| + |w|$
   (f) $||z| - |w|| \leq |z - w|$
7. Polar form of the complex number $z = r\cos(\theta) + ir\sin(\theta)$
8. Euler's form $z = re^{i\theta}$
9. Converting from one form to another using appropriate relations

   (a) $r = \sqrt{x^2 + y^2}$
   (b) $\cos(\theta) = \frac{x}{r}$, $\sin(\theta) = \frac{y}{r}$ and $\theta \in (-\pi, \pi]$
   (c) $x = r\cos(\theta)$, $y = r\sin(\theta)$

10. Roots of unity

11. Geometric representation of complex set (sketches)

    (a) $\{z : |z - c| < r\}$
    (b) $\{z : |z - c| = k|z - d|\}$
    (c) $\{z : a \leq |z - c| \leq b\}$
    (d) $\{z = x + iy : -a < x < a, \text{and} - b < y < b\}$

Chapter 3

1. r-neighborhood of c, $N(c, r) = \{z : |z - c| < r\}$

2. open set

3. neighborhood $N(a, r)$, closed neighborhood $\overline{N}(a, r)$, circle $k(a, r)$, punctured disc $D'(a, r)$

4. complex function, domain, range, continuity

5. writing $f(z)$ as $u(x, y) + iv(x, y)$, representing functions in z- and w-planes

Chapter 4

U = open connected subset of $\mathbb{C}$

1. $f'(c)$ derivative of a complex function f at a point $c \in \mathbb{C}$

2. Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$

   (a) **Example:** Consider $f(z) = \text{Re}(z)$ or $f(z) = x$. This function is not differentiable at any point in $\mathbb{C}$. (Verify.)

   Whereas, $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$ is differentiable everywhere.

(b) f is differentiable at a point $c \Rightarrow$ Cauchy-Riemann equations are satisfied at c (Page 52/Theorem 4.1)

(c) Cauchy-Riemann equations are satisfied at $c \not\Rightarrow$ differentiability at c (Page 53/Counter example 4.2)

3. holomorphic = differentiable in U

   (a) u_x, u_y, v_x and v_y exist, are continuous in U, and Cauchy-Riemann equations are satisfied in U
   $\Rightarrow f = u + iv$ is differentiable in U (Page 55/Theorem 4.4)

4. entire = differentiable in $\mathbb{C}$

5. f holomorphic on U and $f'(z) \equiv 0 \Rightarrow f$ is constant on U(Page 57/ Theorem 4.9)

6. Goursat's lemma (Page 59/ Theorem 4.11)

7. Converse of Goursat's lemma

8. f holomorphic and $|f|$ is a constant in $N(c, r) \Rightarrow f$ is constant (Page 60/ Theorem 4.13)
   Example: Consider $f(z) = \frac{z}{|z|}$ on $\mathbb{C}\backslash\{0\}$. This function maps all non-zero complex numbers to a circle of radius one. Verify that $|f| = 1$. The theorem implies that this function is not holomorphic on any open neighborhood in its domain.

   Whereas, $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2$ given by $f(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ is differentiable in its domain.

9. Infinite series

10. Geometric series: $\sum\limits_{n=0}^{\infty} z^n$ converges $\Leftrightarrow |z| < 1$
    $\sum\limits_{n=0}^{\infty} z^n = \frac{1}{1-z} \Leftrightarrow |z| < 1$

11. Power series centered at a

12. Convergence of a power series at a point

13. Convergence of a power series in an open neighborhood

14. (Page 61/ Theorem 4.14) (Proof important)
$\sum_{n=0}^{\infty} c_n(z-a)^n$ converges at a point $a+d \Rightarrow \sum_{n=0}^{\infty} c_n(z-a)^n$ converges on $N(a, |d|)$

15. (Page 62/ Theorem 4.15) (Proof important)
Either $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges on $\mathbb{C}$
OR $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges on $N(a, R)$ and diverges on $\mathbb{C} \backslash N(a, R)$
OR $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges <u>only at a</u>

16. Radius of convergence, circle of convergence

17. Ratio test, Root test (includes $R = 0$ and $R = \infty$)

18. (Page 63/ Theorem 4.17)
A power series converges $\Leftrightarrow$ The power series obtained by term-wise differentiation converges.
$\sum_{n=0}^{\infty} c_n(z-a)^n$ and $\sum_{n=0}^{\infty} n c_n(z-a)^{n-1}$ have the same radius of convergence.
A power series converges $\Rightarrow$ it is differentiable and the new series obtained by term-wise differentiation is also convergent
$\Rightarrow$ if a function is defined using a power series, it can be differentiated infinitely many times and each time the radius of convergence stays the same.
Play with $f(z) = \frac{1}{1-z}$.

19. Definition of the exponential function using power series $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

    (a) Observe $\exp(0) = 1$

    (b) Find radius of convergence of $\exp(z)$ using ratio test.

20. Use quotient rule to show $\left(\frac{\exp(z+w)}{\exp(z)}\right)' = 0$.
By Theorem 4.9, $\frac{\exp(z+w)}{\exp(z)} =$ constant.
In particular at $z = 0$, $\frac{\exp(z+w)}{\exp(z)} = \exp(w)$ and so we have $\exp(z+w) = \exp(z)\exp(w)$.

21. Define e to be the sum of the series $\exp(1)$. That is, $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$

    and $e^z := \exp(z)$.

    The function $z \mapsto e^z$ is entire, its derivative is the function itself and $e^{z+w} = e^z e^w$.

    Similarly functions $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ are entire.

22. Using $e^{i(z+w)} = e^{iz}e^{iw}$ and Euler's formula $e^{i*} = \cos(*) + i\sin(*)$, derive angle addition formulas for $\cos(z+w)$ and $\sin(z+w)$.

23. Using $e^{z+w} = e^z e^w$ and $e^0 = e^{z-z}$ show that $e^{-z} = \frac{1}{e^z}$.

24. Show that e^z is always non-zero. (See Page 69/ equation 4.25)

25. Show that $z \mapsto e^{-z}$ is entire.(See Page 69/ line after equation 4.25)

26. Principal logarithm $\log z = \log|z| + i\text{arg}z$ where $\text{arg}z$ is the principal argument of z, $\text{arg}z \in (-\pi, \pi]$.

    $\log(z+w) \neq \log z + \log w$ (why? counterexample)

27. Multifunction is a function that maps each point in the domain, to a set of values in the range.

$$\begin{aligned}
\text{Arg}z &= \{\text{arg}z + 2n\pi : n \in \mathbb{Z}\} \\
\text{Log}z &= \{\text{log}z + 2n\pi i : n \in \mathbb{Z}\} \\
&= \{\log|z| + i(2n\pi + \text{arg}z) : n \in \mathbb{Z}\} \\
&= \log|z| + i\text{Arg}z
\end{aligned}$$

28. $\text{Arg}(z+w) = \text{Arg}z + \text{Arg}w$
    $\text{Log}(z+w) = \text{Log}z + \text{Log}w$

29. Isolated singularity

30. Classification of isolated singularities - removable, poles (simple or order n), essential

31. meromorphic = holomorphic in U except for possibly poles

Sample Review Questions: Chapters 2 and 3

All page numbers and problem numbers are from the textbook used in class (See [3]). Some of the other problems are taken from one of the references.

1. Page 32/ Exercises 2.3-2.5, 2.7, 2.8, 2.16-2.18
2. Page 40/ Ex 3.2
3. Let $z = 1 + 2i$ and $w = 2 - i$. Compute:
   (a) $z + 3w$
   (b) $\overline{w} - z$
   (c) z^3
   (d) $\text{Re}(w^2 + w)$
   (e) $z^2 + \overline{z} + i$
4. Find the modulus and the conjugate of $\frac{3-i}{\sqrt{2}+3i}$.
5. Solve the equation $z^4 + 1 = 0$.
6. Solve the equation $z^4 + 16 = 0$.
7. Sketch the following sets. Determine whether they are open, closed, neither or both and determine their interior, closure & boundary.
   (a) $|z + 3| < 2$
   (b) $|\text{Im}z| < 1$
   (c) $0 < |z - 1| < 2$
   (d) $|z - 1| + |z + 1| = 2$
   (e) $|z - 1| + |z + 1| < 3$
   (f) $2 < |z| \leq 3$
   (g) $E = \{z : z \in \mathbb{R} \quad \text{and} \quad -2 < z < -1\} \cup \{z : |z| < 1\} \cup \{z : z = 1 \quad \text{or} \quad z = 2\}$
8. Write the following functions as $u(x, y) + iv(x, y)$. Discuss the domain and range for each example.

(a) $f(z) = 5i$ constant function

(b) $f(z) = 3z$ linear function

(c) $f(z) = z^2$ quadratic function

(d) $f(z) = \overline{z}$ conjugate

(e) $f(z) = |z|$ modulus

(f) $f(z) = \frac{1}{z}$ inverse

(g) $f(z) = iz$ (90^0) counter-clockwise rotation

9. Sketch the region $|\text{Im}z| < 1$ in the z-plane and the region $w = f(z)$ in the w-plane where $f(z) = 5i$.

10. Sketch the region $|\text{Im}z| < 1$ in the z-plane and the region $w = f(z)$ in the w-plane where $f(z) = \overline{z}$.

11. Sketch the region $|\text{Im}z| < 1$ in the z-plane and the region $w = f(z)$ in the w-plane where $f(z) = iz$.

Sample Review Questions: Chapter 4

U = open, connected subset of $\mathbb{C}$

1. Page 55/ Example 4.6

2. Page 55/ Examples 4.7, 4.8

3. Page 61/ Exercise 4.1

4. Page 66/ Example 4.20

5. Page 69/ derivation of equations 4.22-4.25

6. Page 69/ Exercises 4.5,4.7

7. Page 76/ Example 4.24

8. Using the definition of differentiability at a point determine if the following functions are differentiable at c, for any $c \in \mathbb{C}$.

   (a) $f(z) = z^3$ (Answer: entire)

(b) $f(z) = \overline{z}$ (Answer: diff only at 0)

(c) $f(z) = \overline{z}^2$ (Answer: nowhere diff)

9. Which of the following functions are differentiable (where?) / holomorphic (where?)

   (a) $f(z) = e^{-x}e^{-y}$

   (b) $f(z) = 2x + ixy$

   (c) $f(z) = x^2 + iy^2$

   (d) $f(z) = e^x e^{-iy}$

   (e) $f(z) = \text{Im} z$

10. Prove: If f is holomorphic on U and always real valued, then f is a constant. (Hint: use Cauchy-Riemann equations, show $f' = 0$).

11. Prove: If f and $\overline{f}$ are both holomorphic on U, then f is a constant on U.

12. Suppose that $f = u + iv$ is holomorphic and $u = x^2 + y^2$. Find v.

13. Find a power series (& determine its radius of convergence) of $\frac{1}{1+4z}$.

14. Find a power series (& determine its radius of convergence) of $\frac{1}{3-\frac{z}{2}}$.

15. Find the radius of convergence of $\sum_{k=0}^{\infty} 4^k (z-2)^k$. (Ratio test)

16. Find the radius of convergence of $\sum_{k=0}^{\infty} k^n z^k$, for $n \in \mathbb{Z}$. (Root test)

17. Find the poles (& their orders) of $(z^2+1)^{-3}(z-1)^{-4}$.

18. Give examples (if they exist) of:

   (a) a non-constant holomorphic function defined on an open set, but has $f' = 0$

   (b) f such that $|f|$ is a constant

   (c) f holomorphic such that $|f|$ is constant

   (d) an entire function

(e) a function with one simple pole

(f) a function with exactly two simple poles

(g) a function with exactly one pole of multiplicity 2

(h) a function with a removable singularity

(i) a function with an essential singularity

References

[1] Matthias Beck, Gerald Marchesi, Dennis Pixton and Lucas Sabalka, *A First Course in Complex Analysis*, version 1.3, `http://math.sfsu.edu/beck/complex.html`.

[2] George Cain, *Complex Analysis*,`http://people.math.gatech.edu/~cain/winter99/complex.html`.

[3] (Required Text) John Howie, 2004, *Complex Analysis, Springer Undergraduate Mathematics Series*