Closure properties of parametric subcompleteness

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Abstract

For an ordinal ε, I introduce a variant of the notion of subcompleteness of a forcing poset, which I call ε-subcompleteness, and show that this class of forcings enjoys some closure properties that the original class of subcomplete forcings does not seem to have: factors of ε-subcomplete forcings are ε-subcomplete, and if $\mathbb{P}$ and $\mathbb{Q}$ are forcing-equivalent notions, then $\mathbb{P}$ is ε-subcomplete iff $\mathbb{Q}$ is. I formulate a Two Step Theorem for ε-subcompleteness and prove an RCS iteration theorem for ε-subcompleteness which is slightly less restrictive than the original one, in that its formulation is more careful about the amount of collapsing necessary. Finally, I show that an adequate degree of ε-subcompleteness follows from the κ-distributivity of a forcing, for $\kappa > \omega_1$.

1 Introduction

Subcomplete forcing was introduced by Jensen in [Jen09], see also [Jen14]. It is a class of forcings that don't add reals but may change cofinalities to be countable (assuming CH, Namba forcing is subcomplete, and Příkrý forcing is subcomplete), and they can be iterated with revised countable support. In order to define subcompleteness, as well as the variants I want to investigate here, I need the following concepts.

Definition 1.1. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal γ such that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$, meaning that if $x \in N$, $f \in L_\gamma(N)$ and $f : x \longrightarrow N$, then $\mathrm{ran}(f) \in N$.

Definition 1.2. For a poset $\mathbb{P}$, $\delta(\mathbb{P})$ is the minimal cardinality of a dense subset of $\mathbb{P}$.

Definition 1.3. Let $N = L^A_\tau = \langle L_\tau[A], \in, A \cap L_\tau[A]\rangle$ be a ZFC^- model, ε an ordinal and $X \cup \{\varepsilon\} \subseteq N$. Then $C^N_\varepsilon(X)$ is the smallest $Y \prec N$ such that $X \cup \varepsilon \subseteq Y$.

Let me define the concept of ε-subcompleteness now, the focus of the present work. The motivation for this form of subcompleteness arose during joint work with Kaethe Minden on [Min17].

Definition 1.4. Let ε be an ordinal. A forcing $\mathbb{P}$ is *ε-subcomplete* if there is a cardinal $\theta > \varepsilon$ which verifies the ε-subcompleteness of $\mathbb{P}$, which means that $\mathbb{P} \in H_\theta$, and for any ZFC^- model $N = L^A_\tau$ with $\theta < \tau$ and $H_\theta \subseteq N$, any $\sigma : \bar{N} \longrightarrow_{\Sigma_\omega} N$ such that $\bar{N}$ is countable, transitive and full and such that $\mathbb{P}, \theta, \varepsilon \in \mathrm{ran}(\sigma)$, any $\bar{G} \subseteq \bar{\mathbb{P}}$ which is $\bar{\mathbb{P}}$-generic over $\bar{N}$, and any $s \in \mathrm{ran}(\sigma)$, the following holds. Letting $\sigma(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V with $p \in G$, there is in $\mathrm{V}[G]$ a σ' such that

1. $\sigma' : \bar{N} \prec N$,

2. $\sigma'(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}, \bar{\varepsilon}) = s, \theta, \mathbb{P}, \varepsilon$,

3. $(\sigma')``\bar{G} \subseteq G$,

4. $C^N_\varepsilon(\mathrm{ran}(\sigma')) = C^N_\varepsilon(\mathrm{ran}(\sigma))$.

In this parlance, $\mathbb{P}$ is subcomplete iff it is $\delta(\mathbb{P})$-subcomplete. It is easy to see that increasing ε weakens the condition of being ε-subcomplete. However, as we shall see, if a forcing $\mathbb{P}$ is ε-subcomplete, for an $\varepsilon > \delta(\mathbb{P})$, then $\mathbb{P}$ is forcing equivalent to a subcomplete forcing. The proof given in [Jen14] that under CH, Namba forcing is subcomplete actually shows that it is ω_2-subcomplete, while the smallest size of a dense subset is 2^{ω_2}. The following facts elucidate the effect of clause 4. in the definition of ε-subcompleteness.

Fact 1.5. *Suppose $\bar{N}$, N are transitive models of ZFC^-, $N = L^A_\tau$, let $\sigma : \bar{N} \prec N$, $\bar{\varepsilon}$ an ordinal in $\bar{N}$ and $\varepsilon = \sigma(\bar{\varepsilon})$. Then $C^N_\varepsilon(\mathrm{ran}(\sigma)) = \bigcup\{\sigma(u) \mid u \in \bar{N} \ \wedge \ \mathrm{card}(u)^{\bar{N}} \le \bar{\varepsilon}\}$.*

Proof. From left to right, if $x \in C^N_\varepsilon(\mathrm{ran}(\sigma))$, then there is a formula φ, a $v \in \bar{N}$ and there is a finite tuple of ordinals $\vec{\xi} < \varepsilon$ such that in N, x is the unique y such that $\varphi(y, \sigma(v), \vec{\xi})$ holds. In $\bar{N}$, let f be the partial function defined by $f(\vec{\zeta}) =$ the unique y such that $\varphi(y, v, \vec{\zeta})$ holds, for $\vec{\zeta} < \bar{\varepsilon}$, if there is a unique such y. Then $u = \mathrm{ran}(f)$ has cardinality at most $\bar{\varepsilon}$ in $\bar{N}$, and $x \in \sigma(u)$. Vice versa, if $x \in \sigma(u)$, for some $u \in \bar{N}$ of $\bar{N}$-cardinality at most $\bar{\varepsilon}$, then let $f : \bar{\varepsilon} \longrightarrow u$ be a surjection in $\bar{N}$. Then $x = \sigma(f)(\xi)$, for some $\xi < \varepsilon$, so $x \in C^N_\varepsilon(\mathrm{ran}(\sigma))$. $\square$

Fact 1.6. *Suppose $\bar{N}$, N are transitive models of ZFC^-, $N = L^A_\tau$, and let $\sigma, \sigma' : \bar{N} \prec N$ be elementary embeddings. Let $\bar{\varepsilon}, \bar{\lambda} \in \bar{N}$ be limit ordinals, with $\mathrm{cf}^{\bar{N}}(\bar{\lambda}) > \bar{\varepsilon}$. Suppose that $\lambda := \sigma(\bar{\lambda}) = \sigma'(\bar{\lambda})$ and $\varepsilon := \sigma(\bar{\varepsilon}) = \sigma'(\bar{\varepsilon})$, and that $C^N_\varepsilon(\mathrm{ran}(\sigma)) = C^N_\varepsilon(\mathrm{ran}(\sigma'))$. Then $\sup \sigma``\bar{\lambda} = \sup \sigma'``\bar{\lambda}$.*

Proof. By symmetry, it suffices to show that for $\xi < \bar{\lambda}$, $\sigma(\xi) < \sup \sigma'``\bar{\lambda}$. By assumption, $\sigma(\xi) \in C^N_\varepsilon(\mathrm{ran}(\sigma'))$, so by Fact 1.5, there is a $u \in \bar{N}$ of $\bar{N}$-cardinality at most $\bar{\varepsilon}$, such that $\sigma(\xi) \in \sigma'(u)$. It follows that $\sigma(\xi) \in \sigma'(u \cap \bar{\lambda})$, since $\sigma(\xi) < \lambda = \sigma'(\bar{\lambda})$. But $u \cap \bar{\lambda}$ is bounded in $\bar{\lambda}$, so $\sigma(\xi) < \sigma'(\zeta)$, for some $\zeta < \bar{\lambda}$ with $u \cap \bar{\lambda} \subseteq \zeta$. $\square$

The theme of these notes is the investigation of the notions of subcompleteness that result from replacing $\delta(\mathbb{P})$ in the original definition with an ordinal ε, or from replacing the *function* $\mathbb{P} \mapsto \delta(\mathbb{P})$ with another function (I consider two such functions, $\mathcal{E}(\mathbb{P})$ and $\mathcal{F}(\mathbb{P})$). Throughout, I'm interested in the interaction between the resulting classes of forcings and forcing equivalence. In Section 2, I show that the class of ε-subcomplete forcings is closed under forcing equivalence and that factors of ε-subcomplete forcings are ε-subcomplete. Subcompleteness itself does not seem to have these closure properties. In Section 4, I introduce the functions $\mathcal{E}$ and $\mathcal{F}$ and I prove a Two Step Theorem for ε-subcomplete forcings. Section 4 explores some closure properties of the class of the $\mathcal{E}$- and $\mathcal{F}$-subcomplete forcings, most importantly, the $\mathcal{F}$-subcomplete forcings are closed under forcing equivalence, and both the $\mathcal{E}$- and the $\mathcal{F}$-subcomplete forcings satisfy a two step theorem. Section 5 is devoted to a proof of an iteration theorem for ε-subcomplete forcings which is less restrictive than the one for subcomplete forcings in that one is required to collapse less than in the original iteration theorems for subcomplete forcings. Finally, in section 6, I show that one can get ε-subcompleteness from κ-distributivity, if $\kappa > \omega_1$ and ε is sufficiently large.

2 ε-subcompleteness and essential subcompleteness

As defined in the introduction, a forcing $\mathbb{P}$ is ε-subcomplete, for an ordinal ε, if it satisfies the definition of subcompleteness, with $\delta(\mathbb{P})$ replaced by ε.

Obviously, if $\varepsilon \le \varepsilon'$ and $\mathbb{P}$ is ε-subcomplete, then it is also ε'-subcomplete.

Definition 2.1. Let $\varepsilon \cdot \mathbb{P}$ be the disjoint union of ε copies of $\mathbb{P}$, with a weakest condition $\mathbb{1}$. Let's say that two forcings $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if every forcing extension of V by $\mathbb{P}$ is equal to some forcing extension of V by $\mathbb{Q}$ and vice versa.

Clearly, forcing equivalence is first order expressible. Note that $\varepsilon \cdot \mathbb{P}$ is forcing equivalent to $\mathbb{P}$. This shows that it's not the case that two forcings $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent in this sense iff their Boolean completions are isomorphic, which is a common misconception, see [Cum10, p. 791], where this is claimed. In the literature, forcings $\mathbb{P}$ and $\mathbb{Q}$ are often defined to be forcing equivalent if there are a dense subset $D \subseteq \mathbb{P}$ and a dense subset $D' \subseteq \mathbb{Q}$ such that $\mathbb{P}\restriction D$ and $\mathbb{Q}\restriction D'$ are isomorphic. This is a different notion of forcing equivalence than the one I am considering here. Note that $\delta(\varepsilon \cdot \mathbb{P}) = \varepsilon\delta(\mathbb{P})$. So there are forcings $\mathbb{Q}$ forcing equivalent to a given $\mathbb{P}$ with $\delta(\mathbb{Q})$ as large as wished.

It follows from all of this that if $\mathbb{P}$ is ε-subcomplete, then $\varepsilon \cdot \mathbb{P}$, which is forcing equivalent to $\mathbb{P}$, is subcomplete.

Definition 2.2. Let's call a poset *essentially subcomplete* if it is ε-subcomplete, for some ε.

In the remainder of this section, a few closure properties of the classes of the ε-subcomplete or essentially subcomplete forcings will be proven, showing that these classes are very natural. The first type of closure property I want to consider is the closure under forcing equivalence.

Lemma 2.3. *If $\mathbb{P}$ is ε-subcomplete and $\mathbb{Q}$ is forcing equivalent to $\mathbb{P}$, then $\mathbb{Q}$ is ε-subcomplete. (In particular, if $\mathbb{P}$ is essentially subcomplete and $\mathbb{Q}$ is forcing equivalent to $\mathbb{P}$, then $\mathbb{Q}$ is essentially subcomplete.)*

Proof. Let θ be a cardinal that verifies the ε-subcompleteness of $\mathbb{P}$, and let's choose θ large enough so that $\mathcal{P}(\mathbb{P} \cup \mathbb{Q}) \in H_\theta$.

Let $\sigma : \bar{N} \prec N = L^A_\tau$ be countable and full, $H_\theta \subseteq N$, $N \models \mathsf{ZFC}^-$, and $s \in N$. I want to assume that $\mathbb{P}, \mathbb{Q}, \theta \in \operatorname{ran}(\sigma)$ as well - this is harmless, by [Jen14, p. 116]. Let $\bar{s}, \bar{\mathbb{P}}, \bar{\mathbb{Q}}, \bar{\theta} = \sigma^{-1}(s, \mathbb{P}, \mathbb{Q}, \theta)$. Let $\bar{H}$ be $\bar{\mathbb{Q}}$-generic over $\bar{N}$.

Since $\bar{\mathbb{P}}$ and $\bar{\mathbb{Q}}$ are forcing equivalent in $\bar{N}$, there is a $\bar{\mathbb{P}}$-generic filter $\bar{G}$ such that $\bar{N}[\bar{G}] = \bar{N}[\bar{H}]$. By ε-subcompleteness of $\mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever G is $\mathbb{P}$-generic, there is in $\mathrm{V}[G]$ an elementary embedding $\sigma' : \bar{N} \prec N$ such that $\sigma'\restriction\{\bar{s}, \bar{\mathbb{P}}, \bar{\mathbb{Q}}, \bar{\theta}\} = \sigma\restriction\{\bar{s}, \bar{\mathbb{P}}, \bar{\mathbb{Q}}, \bar{\theta}\}$, $C^N_\varepsilon(\operatorname{ran}(\sigma')) = C^N_\varepsilon(\operatorname{ran}(\sigma))$ and $\sigma'``\bar{G} \subseteq G$.

So let G be such a $\mathbb{P}$-generic filter, and let σ' be as described. Then σ' lifts to an embedding $\tilde{\sigma}' : \bar{N}[\bar{G}] \prec N[G]$ with $\tilde{\sigma}'(\bar{G}) = G$. Since $\bar{H} \in \bar{N}[\bar{G}]$, it follows by elementarity of $\tilde{\sigma}'$ that $H = \tilde{\sigma}'(\bar{H})$ is $\mathbb{Q}$-generic over N. But since N contains all subsets of $\mathbb{Q}$, this implies that H is $\mathbb{Q}$-generic over V. Moreover, it's true in $\bar{N}[\bar{G}]$ that $\bar{G} \in \bar{N}[\bar{H}]$, so by elementarity, and since $\tilde{\sigma}'(\bar{G}) = G$, this implies that $G \in N[H]$, and so, $\sigma' \in \mathrm{V}[G] \subseteq \mathrm{V}[H]$.

So, σ' has all the desired properties, and since it exists in $\mathrm{V}[H]$, there is some condition $q \in H$ which forces the existence of an embedding with the required properties. $\square$

Observation 2.4. *The class of the essentially subcomplete forcings is the closure of the class of the subcomplete forcings under forcing equivalence.*

Proof. The class of essentially subcomplete forcings is closed under forcing equivalence, by Lemma 2.3, and every subcomplete forcing is essentially subcomplete, so the closure of the class of subcomplete forcings under forcing equivalence is contained in the class of essentially subcomplete forcings. Vice versa, every essentially subcomplete forcing $\mathbb{P}$ is forcing equivalent to a subcomplete forcing, namely, if $\mathbb{P}$ is ε-subcomplete, then $\mathbb{P}$ is forcing equivalent to the subcomplete forcing $\varepsilon \cdot \mathbb{P}$. $\square$

Observation 2.5. *If the forcing axiom holds for a notion of forcing $\mathbb{P}$ (that is, for every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a filter F in $\mathbb{P}$ that intersects each dense set in $\mathcal{D}$), and if forcing with $\mathbb{P}$ necessarily adds a $\mathbb{Q}$-generic filter (in particular this is the case if $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent), then the forcing axiom holds for $\mathbb{Q}$.*

Proof. Let $\mathcal{D}$ be a collection of dense subsets of $\mathbb{Q}$. For each $D \in \mathcal{D}$, let $A_D \subseteq D$ be a maximal antichain. Let $\mathcal{A}$ be an ω_1-sized collection of maximal antichains in $\mathbb{Q}$ containing each A_D, and such that for every $A, A' \in \mathcal{A}$, there is a $B \in \mathcal{A}$ which refines A and A', meaning that for each $b \in B$, there are $a \in A$ and $a' \in A'$ such that $b \leq a$ and $b \leq a'$. For every $A \in \mathcal{A}$, let $A_{\leq}$ be the downward closure of A, i.e., $A_{\leq} = \{p \mid \exists a \in A \quad p \leq a\}$. Then $\mathcal{D}' = \{A_{\leq} \mid A \in \mathcal{A}\}$ is a collection of dense subsets of $\mathbb{Q}$.

Let Γ' be a $\mathbb{P}$-name for a $\mathbb{Q}$-generic filter. So, for every $D \in \mathcal{D}'$, it is forced by $\mathbb{P}$ that Γ' intersects $\check{D}$. This means that the set $\tilde{D} \subseteq \mathbb{P}$ consisting of all $p \in \mathbb{P}$ such that for some $q \in \mathbb{Q}$, $p \Vdash_{\mathbb{P}} \check{q} \in \Gamma' \cap \check{D}$, is dense in $\mathbb{P}$. By the forcing axiom for $\mathbb{P}$, let $\tilde{F}$ be a filter in $\mathbb{P}$ such that $\tilde{F} \cap \tilde{D} \neq \emptyset$, for all $D \in \mathcal{D}'$. Let

$$F = \{q \in \mathbb{Q} \mid \exists p \in \tilde{F} \exists A \in \mathcal{A} \exists r \in A \quad r \leq_{\mathbb{Q}} q \ \wedge \ p \Vdash_{\mathbb{P}} \check{r} \in \Gamma'\}$$

F is then a filter in $\mathbb{Q}$: if $q \in F$ and $q \leq_{\mathbb{Q}} q'$, then $q' \in F$, trivially. And if $q, q' \in F$, then there are $p, p' \in \tilde{F}$, $A, A' \in \mathcal{A}$, $r \in A$ and $r' \in A'$ such that $r \leq q$, $r' \leq q'$ and $p \Vdash_{\mathbb{P}} \quad \check{r} \in \Gamma'$ and $p' \Vdash_{\mathbb{P}} \quad \check{r}' \in \Gamma'$. Let $B \in \mathcal{A}$ be a refinement of A and A'. Then $D := B_{\leq} \in \mathcal{D}'$, and so, $\tilde{F}$ intersects $\tilde{D}$. Let $\tilde{p} \in \tilde{F} \cap \tilde{D}$. Let $\tilde{q} \in D$ be such that $\tilde{p}$ forces with respect to $\mathbb{P}$ that $\check{\tilde{q}} \in \Gamma'$. Then $\tilde{q} \leq a$, for some (unique) $a \in B$, since $D = B_{\leq}$. Hence, $a \in F$. And since $a \in B$ and B refines A, A', it follows that $a \leq q, q'$. This argument also shows that $F \cap D \neq \emptyset$, for all $D \in \mathcal{D}$. $\square$

Lemma 2.6. *The forcing axiom for the class of subcomplete forcings implies the forcing axiom for the class of essentially subcomplete forcings.*

Proof. By Observation 2.5, the forcing axiom for the class of subcomplete forcing implies the forcing axiom for the closure of this class, which is the class of all essentially subcomplete forcings, by Observation 2.4. $\square$

The next closure property says that factors of ε-subcomplete forcings are ε-subcomplete.

Theorem 2.7. *Let $\mathbb{P}$ be a poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset, such that $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete. Then $\mathbb{P}$ is ε-subcomplete.*

Proof. Let θ verify that $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete. I claim that θ also verifies that $\mathbb{P}$ is ε-subcomplete. Let $N = L_\tau[A]$, where $\tau > \theta$ and $H_\theta \subseteq N$. Let s be any given set, $\sigma : \bar{N} \prec N$ be countable and full, $\mathbb{P}, \dot{\mathbb{Q}}, \theta, s \in \operatorname{ran}(\sigma)$, $\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s} = \sigma^{-1}(\mathbb{P}, \dot{\mathbb{Q}}, \theta, s)$ (as before, it is harmless to assume that some additional parameter, in this case $\dot{\mathbb{Q}}$, is in the range of σ). Let $\bar{G} \subseteq \bar{\mathbb{P}}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. We have to show that there is a condition $p \in \mathbb{P}$ such that whenever G is $\mathbb{P}$-generic over V with $p \in G$, then in V$[G]$, there is an elementary embedding $\sigma' : \bar{N} \prec N$ with $\sigma' \restriction \{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\} = \sigma \restriction \{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\}$, $\sigma' “\bar{G} \subseteq G$ and $C_\varepsilon^N(\operatorname{ran}(\sigma')) = C_\varepsilon^N(\operatorname{ran}(\sigma))$.

Now let $\bar{H} \subseteq \dot{\bar{\mathbb{Q}}}^{\bar{G}}$ be $\dot{\bar{\mathbb{Q}}}^{\bar{G}}$-generic over $\bar{N}[\bar{G}]$. By the ε-subcompleteness of $\mathbb{P}*\dot{\mathbb{Q}}$, let $\langle p, \dot{q}\rangle \in \mathbb{P}*\dot{\mathbb{Q}}$ be such that whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V with $\langle p, \dot{q}\rangle \in G * H$, then in $\mathrm{V}[G * H]$, there is an elementary embedding $\rho : \bar{N} \prec N$ with $\rho{\restriction}\{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\} = \sigma{\restriction}\{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\}$, $\rho``(\bar{G} * \bar{H}) \subseteq G * H$ and $C^N_\varepsilon(\mathrm{ran}(\rho)) = C^N_\varepsilon(\mathrm{ran}(\sigma))$. Let $G * H$ be $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V with $\langle p, \dot{q}\rangle \in G * H$, and let ρ be an embedding as described.

Let $\mu > \theta$ be regular in $\mathrm{V}[G * H]$, and consider the structure

$$M = \langle H^{\mathrm{V}[G]}_\mu, \in, N, \sigma, \mathbb{P}, \dot{\mathbb{Q}}, \varepsilon, \theta, s, G\rangle$$

in $\mathrm{V}[G]$, and the infinitary theory $\mathcal{L}$ on M with constants $\underset{\sim}{x}$ for each $x \in M$, a constant symbol $\dot{\sigma}'$ and the basic axioms,[1] the ZFC^--axioms, and the axioms expressing that $\dot{\sigma}' : \underset{\sim}{\bar{N}} \prec N$, $\dot{\sigma}'``\underset{\sim}{\bar{G}} \subseteq \underset{\sim}{G}$, $\dot{\sigma}'{\restriction}\{\underset{\sim}{\bar{\mathbb{P}}}, \underset{\sim}{\dot{\bar{\mathbb{Q}}}}, \underset{\sim}{\bar{\theta}}, \underset{\sim}{\bar{s}}\} = \underset{\sim}{\sigma}{\restriction}\{\underset{\sim}{\bar{\mathbb{P}}}, \underset{\sim}{\dot{\bar{\mathbb{Q}}}}, \underset{\sim}{\bar{\theta}}, \underset{\sim}{\bar{s}}\}$ and $C^{\underset{\sim}{N}}_{\underset{\sim}{\varepsilon}}(\mathrm{ran}(\underset{\sim}{\sigma})) = C^{\underset{\sim}{N}}_{\underset{\sim}{\varepsilon}}(\dot{\sigma}')$.

Then $\mathcal{L}$ is consistent, as witnessed by ρ in $\mathrm{V}[G * H]$. In $\mathrm{V}[G]$, let $\pi : \bar{M} \prec M$, where $\bar{M}$ is countable and transitive, and let $\bar{\mathcal{L}}$ be the infinitary theory on $\bar{M}$ defined as $\mathcal{L}$ is defined over M, with the constants moved by π^{-1}. Then $\bar{\mathcal{L}}$ is consistent, and since $\bar{M}$ is countable in $\mathrm{V}[G]$, it has a solid model (that is, a model whose well-founded part is transitive) $\mathfrak{A}$ in $\mathrm{V}[G]$. Let $k = (\dot{\sigma}')^{\mathfrak{A}}$. Note that since M sees that $\bar{N}$ is countable, $\pi^{-1}(\bar{N}) = \bar{N}$ and $\pi{\restriction}\bar{N} = \mathrm{id}$. We have that $k : \bar{N} \prec \pi^{-1}(N)$, $k``\bar{G} \subseteq \pi^{-1}(G)$ and $k{\restriction}\{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\} = \pi^{-1}(\sigma){\restriction}\{\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}\}$, i.e., $k(\bar{\mathbb{P}}) = \pi^{-1}(\sigma(\bar{\mathbb{P}}))$ etc.

Set $\sigma' = \pi \circ k$. It then follows that $\sigma' : \bar{N} \prec N$, because $(\dot{\sigma}')^{\bar{\mathfrak{A}}} : \bar{N} \longrightarrow \pi^{-1}(N)$, so that if $\vec{a} \in \bar{N}$ and $\varphi(\vec{x})$ is a formula, then $\bar{N} \models \varphi(\vec{a})$ iff $\pi^{-1}(N) \models \varphi(k(\vec{a}))$ iff $N \models \varphi(\pi(k(\vec{a})))$. Similarly, $\sigma'(\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\varepsilon}, \bar{s}) = \mathbb{P}, \dot{\mathbb{Q}}, \varepsilon, s$, because $k(\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \bar{\theta}, \bar{s}) = \pi^{-1}(\mathbb{P}, \dot{\mathbb{Q}}, \theta, s)$, and also, $\sigma'``\bar{G} \subseteq G$, because $k``\bar{G} \subseteq \pi^{-1}(G)$. Finally, a standard argument shows that $C^N_\varepsilon(\mathrm{ran}(\sigma')) = C^N_\varepsilon(\mathrm{ran}(\sigma))$. Here are the details, for the reader's convenience.

The inclusion from left to right is clear because $\mathrm{ran}(k) \subseteq C^{\pi^{-1}(N)}_{\pi^{-1}(\varepsilon)}(\mathrm{ran}(\pi^{-1}(\sigma)))$, which implies that $\mathrm{ran}(\pi \circ k) \subseteq C^N_\varepsilon(\mathrm{ran}(\sigma))$, which implies the desired inclusion.

For the opposite direction, let $c \in C^N_\varepsilon(\mathrm{ran}(\sigma))$. Writing f^N for the canonical Skolem function of N, there is an $n < \omega$, an $\alpha < \varepsilon$ and an $a \in \bar{N}$ such that

$$c = f^N(n, \langle \alpha, \sigma(a)\rangle)$$

We have that $C^{\pi^{-1}(N)}_{\pi^{-1}(\varepsilon)}(\mathrm{ran}(k)) = C^{\pi^{-1}(N)}_{\pi^{-1}(\varepsilon)}(\mathrm{ran}(\pi^{-1}(\sigma)))$, and $\pi^{-1}(\sigma)(a)$ belongs to the set on the right hand side of this equation. Hence, it belongs to the left hand side as well, and this means that there is an $m < \omega$, and $\tilde{\alpha} < \pi^{-1}(\varepsilon)$ and a $b \in \bar{N}$ such that $\pi^{-1}(\sigma)(a) = f^{\pi^{-1}(N)}(m, \langle \tilde{\alpha}, k(b)\rangle)$. So applying π to this fact gives

$$\sigma(a) = f^N(m, \langle \pi(\tilde{\alpha}), \sigma'(b)\rangle)$$

Substituting this into the equation above gives

$$c = f^N(n, \langle \alpha, f^N(m, \langle \pi(\tilde{\alpha}), \sigma'(b)\rangle)\rangle)$$

Since $\pi(\tilde{\alpha}) < \varepsilon$, this is in $C^N_\varepsilon(\mathrm{ran}(\sigma'))$, as wished. □

Corollary 2.8. *If $\mathbb{P} * \dot{\mathbb{Q}}$ is subcomplete, then $\mathbb{P}$ is $\delta(\mathbb{P} * \dot{\mathbb{Q}})$-subcomplete.*

[1] Following [Jen14], the basic axioms are the axioms of the form $\forall z \quad (z \in \underset{\sim}{x} \iff \bigvee_{y \in x} z = \underset{\sim}{y})$, for every $x \in M$. These axioms insure that any model of the resulting theory whose well-founded part is transitive will interpret $\underset{\sim}{x}$ as x.

Note that $\delta(\mathbb{P}) \leq \delta(\mathbb{P} * \dot{\mathbb{Q}})$. So this corollary does not quite show that factors of subcomplete forcings are subcomplete, but it does show that they are forcing equivalent to subcomplete forcings (namely, $\delta(\mathbb{P} * \dot{\mathbb{Q}}) \cdot \mathbb{P}$ is subcomplete). The following summarizes what can be said about factors and quotients of essentially subcomplete forcings.

Theorem 2.9. *Let $\mathbb{P}$ be a poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset.*

1. *If $\mathbb{P}$ is (essentially) subcomplete and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is (essentially) subcomplete, then $\mathbb{P} * \dot{\mathbb{Q}}$ is (essentially) subcomplete.*
2. *If $\mathbb{P} * \dot{\mathbb{Q}}$ is (essentially) subcomplete, then*
   (a) *$\mathbb{P}$ is essentially subcomplete, but*
   (b) *it is not necessarily true that $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is essentially subcomplete.*

Proof. 1. is the two step iteration theorem for subcomplete forcing, see [Jen14, pp. 136]. I will explore versions of this theorem for ε-subcomplete forcings further in Section 3. 2.(a) is Theorem 2.7. 2.(b) was also observed in [Min17]. Consider the following situation. Let $\mathbb{P}$ be the forcing from [Kun78] to add a homogeneous Souslin tree with countable conditions. $\mathbb{P}$ is countably closed, and hence subcomplete. Let $\dot{\mathbb{Q}}$ be the $\mathbb{P}$-name for the generic object (i.e., the homogeneous Souslin tree added by $\mathbb{P}$). Then $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is not essentially subcomplete, because subcomplete forcing preserves Souslin trees. But $\mathbb{P} * \dot{\mathbb{Q}}$ is forcing equivalent to $\mathrm{Add}(\omega_1, 1)$, and is thus subcomplete. $\square$

3 Two step theorems, $\mathcal{E}(\mathbb{P})$ and $\mathcal{F}(\mathbb{P})$

Next, I want to explore versions of the two step theorem for ε-subcompleteness. In the context of subcomplete forcing, $\delta(\mathbb{P})$ plays a double role: on the one hand, it is an upper bound for the possible sizes of maximal antichains in $\mathbb{P}$, and on the other hand, it specifies how closely the embedding σ' in the definition of subcompleteness can be made to resemble the originally given σ. By considering ε-subcompleteness, these two aspects get separated: the second role is played by ε, and it turns out that there is a way to loosen the restriction given by $\delta(\mathbb{P})$, by replacing it with another measurement of the "complexity" of $\mathbb{P}$.

Definition 3.1. For a poset $\mathbb{P}$, let $\mathcal{E}(\mathbb{P}) = \sup\{\mathrm{card}(A) \mid A \text{ is an antichain in } \mathbb{P}\}$.

Note that $\mathcal{E}(\mathbb{P}) \leq \delta(\mathbb{P})$, and $\mathcal{E}(\mathbb{P})$ may be strictly smaller than $\delta(\mathbb{P})$ (for example, if $\mathbb{P}$ is Příkrý forcing with respect to a normal measure on κ, then $\mathcal{E}(\mathbb{P}) = \kappa < 2^\kappa = \delta(\mathbb{P})$. Another example is when $\mathbb{P}$ is a κ^+-Souslin tree: in that case, $\mathcal{E}(\mathbb{P}) = \kappa < \kappa^+ = \delta(\mathbb{P})$). Note also that $\mathcal{E}(\mathbb{P})$ is closely related to the saturation of $\mathbb{P}$, or $\mathsf{Sat}(\mathbb{P})$, which is sometimes also denoted by $\mathsf{c.c.}(\mathbb{P})$, the least κ such that $\mathbb{P}$ satisfies the κ-c.c.

Theorem 3.2. *Suppose $\mathbb{P}$ is ε_0-subcomplete, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$, and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}_1$-subcomplete. Let $\varepsilon = \varepsilon_0 \cup \varepsilon_1 \cup \mathcal{E}(\mathbb{P})$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete.*

Note: Another way to say this is that if $\varepsilon \geq \mathcal{E}(\mathbb{P})$ is such that $\mathbb{P}$ is ε-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}$-subcomplete, then $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete. This is the form in which I will prove the theorem.

Proof. The argument works mostly as the proof of [Jen14, Thm. 1, p.136 ff.], with a key modification in the proofs of statements (A) and (B) below. Actually, that proof shows the statement of the present lemma, with $\delta(\mathbb{P})$ in place of $\mathcal{E}(\mathbb{P})$.

We have that $\mathbb{P}$ is ε-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}$-subcomplete, and that $\mathcal{E}(\mathbb{P}) \le \varepsilon$. To prove the lemma, let θ be large enough that it verifies the ε-subcompleteness of $\mathbb{P}$ and so that $\Vdash_{\mathbb{P}} \check{\theta}$ verifies the $\check{\varepsilon}$-subcompleteness of $\dot{\mathbb{Q}}$. I claim that θ verifies the ε-subcompleteness of $\mathbb{P} * \dot{\mathbb{Q}}$.

To see this, let $\tau > \theta$, $N = L^A_\tau \supseteq H_\theta$, $\sigma : \bar{N} \prec N$ be countable and full, with $S = \langle \mathbb{P}, \dot{\mathbb{Q}}, \theta, s\rangle \in \operatorname{ran}(\sigma)$ (where s is some fixed set). Let $\bar{S} = \langle \bar{\mathbb{P}}, \bar{\dot{\mathbb{Q}}}, \bar{\theta}, \bar{s}\rangle = \sigma^{-1}(S)$. Further, let $\bar{G} * \bar{H}$ be $\bar{\mathbb{P}} * \bar{\dot{\mathbb{Q}}}$-generic over $\bar{N}$. We have to show that there is a $\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ such that whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V with $\langle p, \dot{q}\rangle \in G * H$, then there is in $\mathrm{V}[G][H]$ an elementary embedding $\sigma' : \bar{N} \prec N$ such that $\sigma'(\bar{S}) = S$, $\sigma'``(\bar{G} * \bar{H}) \subseteq G * H$ and

(0) $C^N_\varepsilon(\operatorname{ran}(\sigma')) = C^N_\varepsilon(\operatorname{ran}(\sigma))$

Since $\mathbb{P}$ is ε-subcomplete, there is a $p \in \mathbb{P}$ such that, letting G be any filter that's $\mathbb{P}$-generic over V with $p \in G$, there is a $\sigma_0 : \bar{N} \prec N$ in $\mathrm{V}[G]$ such that $\sigma_0(\bar{S}) = S$, $\sigma_0``\bar{G} \subseteq G$ and

(1) $C^N_\varepsilon(\operatorname{ran}(\sigma_0)) = C^N_\varepsilon(\operatorname{ran}(\sigma))$

Then σ_0 lifts to an embedding $\sigma_0^* : \bar{N}[\bar{G}] \prec N[G]$ such that $\sigma_0(\bar{G}) = G$. $\bar{N}[\bar{G}]$ is then still full, and since θ verifies the ε-subcompleteness of $\dot{\mathbb{Q}}^G$ in $\mathrm{V}[G]$, it follows that there is a $q \in \dot{\mathbb{Q}}^G$ such that, letting H be any filter that's $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$ with $q \in H$, there is in $\mathrm{V}[G][H]$ a $\sigma_1 : \bar{N}[\bar{G}] \prec N[G]$ such that $\sigma_1(\bar{S}) = S$, and in addition $\sigma_1(\bar{G}) = G$, $\sigma_1``\bar{H} \subseteq H$ and

(2) $C^{N[G]}_\varepsilon(\operatorname{ran}(\sigma_1)) = C^{N[G]}_\varepsilon(\operatorname{ran}(\sigma_0^*))$

Let $\sigma' = \sigma_1 \restriction \bar{N}$. I want to show that σ' is as wished. Clearly, $\sigma' : \bar{N} \prec N$, $\sigma'(\bar{S}) = S$ and $\sigma'``\bar{G} * \bar{H} \subseteq G * H$. The crucial missing property is (0). To prove that (0) holds, it suffices to show

(3) $C^N_\varepsilon(\operatorname{ran}(\sigma')) = C^N_\varepsilon(\operatorname{ran}(\sigma_0))$

because (3), together with (1), immediately implies (0). To show (3), in turn, it suffices to show that the following hold:

(A) $N \cap C^{N[G]}_\varepsilon(\operatorname{ran}(\sigma_1)) = C^N_\varepsilon(\operatorname{ran}(\sigma'))$

(B) $N \cap C^{N[G]}_\varepsilon(\operatorname{ran}(\sigma_0^*)) = C^N_\varepsilon(\operatorname{ran}(\sigma_0))$

For (2)+(A)+(B) implies (3).

To see that (A) holds, note that the direction from right to left is obvious, because N is definable in $N[G]$ using the predicate A. For the converse, let $x \in N$, and suppose that in $N[G]$, x is the unique z such that $\varphi(z, \sigma_1(a), \alpha)$ holds, for some formula φ, where $a \in \bar{N}[\bar{G}]$, $\alpha < \varepsilon$. Let $a = \dot{a}^{\bar{G}}$, so that $\sigma_1(a) = \sigma'(\dot{a})^G$. Then there is a $p' \in G$ such that in N, p' forces that $\check{x}$ is the unique z such that $\varphi(z, \sigma'(\dot{a}), \check{\alpha})$ holds. So, letting

$$C = \{y \in N \mid \exists r \in \mathbb{P} \quad r \Vdash_{\mathbb{P}} \forall z \quad (z = \check{y} \iff \varphi(z, \sigma'(\dot{a}), \check{\alpha}))\}$$

it follows that $x \in C$. But note that the cardinality of C is at most $\mathcal{E}(\mathbb{P})$, because for each $y \in C$, we can pick an $r_y \in \mathbb{P}$ witnessing this, and clearly, for $y_0, y_1 \in C$ with $y_0 \ne y_1$, it follows that $r_{y_0} \perp r_{y_1}$, so that $\{r_y \mid y \in C\}$ is an antichain in $\mathbb{P}$ of the same cardinality as C. Let g be the $<_{L^A_\tau}$-least surjection from $\operatorname{card}(C)$ to C. Then g is definable in N from the parameters $\mathbb{P}, \sigma'(\dot{a}), \alpha$, and since $x = g(\gamma)$ for some $\gamma < \operatorname{card}(C) \le \mathcal{E}(\mathbb{P}) \le \varepsilon$, it follows that $x \in C^N_\varepsilon(\operatorname{ran}(\sigma'))$, as wished. The proof of (B) is almost identical to the proof of (A).

So, we have found a σ' in $\mathrm{V}[G][H]$ with the properties required by ε-subcompleteness of $\mathbb{P} * \dot{\mathbb{Q}}$, and the existence of such an embedding is then forced by some condition in $G * H$, completing the proof. $\square$

The assumption in the previous theorem can still be weakened.

Definition 3.3. Let $\mathbb{P}$ be a partial order. For $p \in \mathbb{P}$, let $\mathbb{P}_{\le p}$ be the restriction of $\mathbb{P}$ to conditions $q \le p$. Let $\mathcal{F}(\mathbb{P})$ be the least ε such that $\{p \mid \mathcal{E}(\mathbb{P}_{\le p}) \le \varepsilon\}$ is dense in $\mathbb{P}$.

Clearly, $\mathcal{F}(\mathbb{P}) \le \mathcal{E}(\mathbb{P})$. It's easy to construct examples where $\mathcal{F}(\mathbb{P})$ is much smaller than $\mathcal{E}(\mathbb{P})$, for example, using the fact that $\mathcal{F}(\mathbb{P}) = \mathcal{F}(\kappa \cdot \mathbb{P})$ while $\mathcal{E}(\kappa \cdot \mathbb{P}) \ge \kappa$. The point is that the previous theorem goes through with $\mathcal{F}(\mathbb{P})$ in place of $\mathcal{E}(\mathbb{P})$.

Theorem 3.4. *Suppose $\mathbb{P}$ is ε_0-subcomplete, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$, and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}_1$-subcomplete. Let $\varepsilon = \varepsilon_0 \cup \varepsilon_1 \cup \mathcal{F}(\mathbb{P})$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete.*

Note: Again, this can be expressed by saying that if $\varepsilon \ge \mathcal{F}(\mathbb{P})$ is such that $\mathbb{P}$ is ε-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}$-subcomplete, then $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete. This is the form in which I will prove the theorem.

Proof. Running the proof of Theorem 3.2 under the assumption that $\varepsilon \ge \mathcal{F}(\mathbb{P})$ rather than $\varepsilon \ge \mathcal{E}(\mathbb{P})$, and using the same notation, the only places where changes have to be made are in the proofs of the claims

(A) $N \cap C_\varepsilon^{N[G]}(\mathrm{ran}(\sigma_1)) = C_\varepsilon^N(\mathrm{ran}(\sigma'))$

(B) $N \cap C_\varepsilon^{N[G]}(\mathrm{ran}(\sigma_0^*)) = C_\varepsilon^N(\mathrm{ran}(\sigma_0))$

As before, the direction from right to left is obvious in both cases. To see that the substantial inclusion of (A) holds, let $x \in N$, and suppose that in $N[G]$, x is the unique z such that $\varphi(z, \sigma_1(a), \alpha)$ holds, for some formula φ, where $a \in \bar{N}[\bar{G}]$, $\alpha < \varepsilon$. Let $a = \dot{a}^{\bar{G}}$, so that $\sigma_1(a) = \sigma'(\dot{a})^G$. In $\bar{N}$, the set of $\bar{p} \in \bar{\mathbb{P}}$ such that $\mathcal{E}(\bar{\mathbb{P}}_{\le \bar{p}}) \le \mathcal{F}(\bar{\mathbb{P}})$ is dense, so there is a $\bar{p}_0 \in \bar{G}$ belonging to this set. By elementarity, then, $p_0 = \sigma'(\bar{p}_0)$ has $\mathcal{E}(\mathbb{P}_{\le p_0}) \le \mathcal{F}(\mathbb{P})$, and $p_0 \in G$. There is now a $p' \in G$, with $p' \le p_0$, such that in N, p' forces that $\check{x}$ is the unique z such that $\varphi(z, \sigma'(\dot{a}), \check{\alpha})$ holds. So, letting

$$C = \{y \in N \mid \exists r \le_{\mathbb{P}} p_0 \quad r \Vdash_{\mathbb{P}} \forall z \quad (z = \check{y} \iff \varphi(z, \sigma'(\dot{a}), \check{\alpha}))\}$$

it follows that $x \in C$. But note that the cardinality of C is at most $\mathcal{E}(\mathbb{P}_{\le p_0}) \le \mathcal{F}(\mathbb{P}) \le \varepsilon$, because for each $y \in C$, we can pick an $r_y \in \mathbb{P}_{\le p_0}$ witnessing this, and if $y_0, y_1 \in C$ are distinct, then $r_{y_0} \perp r_{y_1}$, so that $\{r_y \mid y \in C\}$ is an antichain in $\mathbb{P}_{\le p_0}$ of the same cardinality as C. Let g be the $<_{L_\tau^A}$-least surjection from $\mathrm{card}(C)$ to C. Then g is definable in N from the parameters $\mathbb{P}, p_0, \sigma'(\dot{a}), \alpha$, and since $x = g(\gamma)$ for some $\gamma < \mathrm{card}(C) \le \mathcal{E}(\mathbb{P}) \le \varepsilon$, it follows that $x \in C_\varepsilon^N(\mathrm{ran}(\sigma'))$ (noting that $p_0 \in \mathrm{ran}(\sigma')$), as wished. The proof of (B) is almost identical to the proof of (A), and the remainder of the proof goes through without modification. $\square$

4 Closure properties of $\mathcal{E}$- and $\mathcal{F}$-subcompleteness

The following lemma shows that $\mathcal{F}(\mathbb{P})$ is maybe a more natural measurement of a poset than $\delta(\mathbb{P})$ or $\mathcal{E}(\mathbb{P})$.

Notation 4.1. If $\mathbb{P}$ is a forcing notion, then I write $\dot{G}_{\mathbb{P}}$ for the canonical name for the $\mathbb{P}$-generic filter.

Lemma 4.2. *If $\mathbb{P}$ is forcing equivalent to $\mathbb{Q}$, then $\mathcal{F}(\mathbb{P}) = \mathcal{F}(\mathbb{Q})$.*

Proof. We show that if for every $G_{\mathbb{Q}}$ which is $\mathbb{Q}$-generic over V, there is a $G_{\mathbb{P}} \in \mathrm{V}[G_{\mathbb{Q}}]$ which is $\mathbb{P}$-generic over V, such that $\mathrm{V}[G_{\mathbb{P}}] = \mathrm{V}[G_{\mathbb{Q}}]$, then $\mathcal{F}(\mathbb{Q}) \leq \mathcal{F}(\mathbb{P})$.

Let $\kappa = \mathcal{F}(\mathbb{P})$, $\lambda = \mathcal{F}(\mathbb{Q})$, and assume (towards a contradiction) that $\kappa < \lambda$. It follows that $\{q \in \mathbb{Q} \mid \mathcal{E}(\mathbb{Q}_{\leq q}) \leq \kappa\}$ is not dense in $\mathbb{Q}$. So there is a $q_0 \in \mathbb{Q}$ such that for every $q \leq q_0$, $\mathcal{E}(\mathbb{Q}_{\leq q}) > \kappa$. So, for every $q \leq q_0$, there is a maximal antichain $A_q \subseteq \mathbb{Q}_{\leq q}$ with $\mathrm{card}(A_q) > \kappa$.

Let $G_{\mathbb{Q}} \ni q_0$ be generic for $\mathbb{Q}$ over V. Let $\dot{H}$ be a $\mathbb{Q}$-name for a $\mathbb{P}$-generic filter such that if we let $H = \dot{H}^{G_{\mathbb{Q}}}$, then $\mathrm{V}[G_{\mathbb{Q}}] = \mathrm{V}[H]$. In particular, $G_{\mathbb{Q}} \in \mathrm{V}[H]$, which means that there is a $\mathbb{P}$-name $\dot{I}$ in V such that $\dot{I}^H = G_{\mathbb{Q}}$. Since H is $\mathbb{P}$-generic and $D = \{p \in \mathbb{P} \mid \mathcal{E}(\mathbb{P}_{\leq p}) \leq \kappa\}$ is dense in $\mathbb{P}$, we may fix a condition $p_0 \in D \cap H$. It follows that there is a condition $q_1 \leq q_0$ in $G_{\mathbb{Q}}$ which forces that $\dot{H}$ is $\check{\mathbb{P}}$-generic over $\check{\mathrm{V}}$, that $\check{\dot{I}}^{\dot{H}} = \dot{G}_{\mathbb{Q}}$ is $\mathbb{Q}$-generic over V, and that $\check{p}_0 \in \dot{H}$. Now, for every $q \in A_{q_1}$, let G_q be $\mathbb{Q}$-generic with $q \in G_q$. Then, we have

$$\dot{I}^{(\dot{H}^{G_q})} = G_q$$

So, there is a $p_q \in (\dot{H}^{G_q})$ such that $p_q \Vdash_{\mathbb{P}}$ "$\dot{I} \cap \check{A}_{q_1} = \{\check{q}\}$ and $\dot{I}$ is a filter in $\check{\mathbb{Q}}$". Since $q \leq q_1$, we know that $p_0 \in \dot{H}^{G_q}$, and so, we may pick p_q with the additional property that $p_q \leq p_0$.

Of course, the function $q \mapsto p_q$ may not exist in V, because it depends on $(\dot{H}_{\mathbb{Q}}^{G_q})$. But still, in V, there is a function $f : A_{q_1} \longrightarrow \mathbb{P}_{\leq p_0}$ such that $f(q) \Vdash_{\mathbb{P}} \check{A}_{q_1} \cap \dot{I} = \{\check{q}\}$. Then clearly, f has to be injective, and $\mathrm{ran}(f)$ is an antichain in $\mathbb{P}_{\leq p_0}$ of greater size than $\kappa = \mathcal{E}(\mathbb{P}_{\leq p_0})$, a contradiction.

This shows that $\mathcal{F}(\mathbb{Q}) \leq \mathcal{F}(\mathbb{P})$. The other direction follows by exchanging the roles of $\mathbb{P}$ and $\mathbb{Q}$ in the above argument. □

Definition 4.3. A forcing $\mathbb{P}$ is $\mathcal{E}$-subcomplete if it is $\mathcal{E}(\mathbb{P})$-subcomplete. It is $\mathcal{F}$-subcomplete if it is $\mathcal{F}(\mathbb{P})$-subcomplete.

The following corollary shows the naturalness of $\mathcal{F}(\mathbb{P})$ as a measurement of a poset $\mathbb{P}$, the closure of the class of $\mathcal{F}$-subcomplete forcings under forcing equivalence.

Corollary 4.4. *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing equivalent posets. Then $\mathbb{P}$ is $\mathcal{F}$-subcomplete iff $\mathbb{Q}$ is $\mathcal{F}$-subcomplete.*

Proof. This follows from Lemma 4.2, which shows that $\varepsilon := \mathcal{F}(\mathbb{P}) = \mathcal{F}(\mathbb{Q})$, together with Lemma 2.3, which shows that $\mathbb{P}$ is ε-subcomplete iff $\mathbb{Q}$ is ε-subcomplete. □

It was shown in [Jen14, §4, Thm. 1] that if $\mathbb{P}$ is subcomplete and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is subcomplete, then $\mathbb{P}*\dot{\mathbb{Q}}$ is subcomplete. The next corollary says that the classes of $\mathcal{E}$- and $\mathcal{F}$-subcomplete forcings enjoy the corresponding closure properties as well.

Lemma 4.5. *Let $\mathbb{P}$ be a notion of forcing and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing.*

1. *If $\mathbb{P}$ is $\mathcal{E}$-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\mathcal{E}$-subcomplete, then $\mathbb{P} * \dot{\mathbb{Q}}$ is $\mathcal{E}$-subcomplete.*
2. *If $\mathbb{P}$ is $\mathcal{F}$-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\mathcal{F}$-subcomplete, then $\mathbb{P} * \dot{\mathbb{Q}}$ is $\mathcal{F}$-subcomplete.*

Proof. For 1, let $\varepsilon = \mathcal{E}(\mathbb{P} * \dot{\mathbb{Q}})$. Then $\mathcal{E}(\mathbb{P}) \leq \varepsilon$, so $\mathbb{P}$ is ε-subcomplete. Moreover, $\Vdash_{\mathbb{P}} \mathcal{E}(\dot{\mathbb{Q}}) \leq \mathrm{card}(\check{\varepsilon})$: suppose otherwise. Let G be $\mathbb{P}$-generic such that in $\mathrm{V}[G]$, $\mathcal{E}(\dot{\mathbb{Q}}^G) \geq \varepsilon^+$. Note that $\mathbb{P}$ satisfies the ε^+-c.c., so ε^+ is the same in V and in $\mathrm{V}[G]$. Since ε^+ is a successor cardinal, there is in $\mathrm{V}[G]$ an antichain in $\dot{\mathbb{Q}}^G$ of size ε^+. In $\mathrm{V}[G]$, let $f : \varepsilon^+ \longrightarrow \dot{\mathbb{Q}}^G$ be an (injective) enumeration of such an antichain. Let $\dot{f}$ be a name for f, and let $p \in G$ force that it behaves as described. Then, for every $\alpha < \varepsilon^+$, $p \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) \in \dot{\mathbb{Q}}$, and thus, there is a τ_α such that $p \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \tau_\alpha$ and

$\langle p, \tau_\alpha \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. Then $\{\langle p, \tau_\alpha \rangle \mid \alpha < \varepsilon^+\}$ is an antichain in $\mathbb{P} * \dot{\mathbb{Q}}$ of size ε^+, contradicting that $\varepsilon = \mathcal{E}(\mathbb{P} * \dot{\mathbb{Q}})$.

Hence, $\mathbb{P}$ is ε-subcomplete and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\check{\varepsilon}$-subcomplete, and $\varepsilon \geq \mathcal{E}(\mathbb{P})$, which implies, by Theorem 3.2, that $\mathbb{P} * \dot{\mathbb{Q}}$ is ε-subcomplete, as claimed.

For 2, let $\varepsilon = \mathcal{F}(\mathbb{P} * \dot{\mathbb{Q}})$. Then $\mathcal{F}(\mathbb{P}) \leq \varepsilon$, because if $\xi < \mathcal{F}(\mathbb{P})$, then there is a $p \in \mathbb{P}$ such that for every $p' \leq p$, $\mathcal{E}(\mathbb{P}_{\leq p'}) > \xi$. It follows then that for every $\langle p', \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ with $\langle p', \dot{q} \rangle \leq \langle p, \mathbb{1}_{\mathbb{Q}} \rangle$, $\mathcal{E}((\mathbb{P} * \dot{\mathbb{Q}})_{\leq \langle p', \dot{q} \rangle}) > \xi)$. This is because there is an an antichain A in $\mathbb{P}_{\leq p'}$ with $\mathrm{card}(A) > \xi$, and then, $A \times \{\dot{q}\}$ is an antichain in $(\mathbb{P} * \dot{\mathbb{Q}})_{\leq \langle p', \dot{q} \rangle}$ of the same size. So $\mathbb{P}$ is ε-subcomplete.

Similarly, $\Vdash_{\mathbb{P}} \mathcal{F}(\dot{\mathbb{Q}}) \leq \mathrm{card}(\check{\varepsilon})$: suppose otherwise. Let G be $\mathbb{P}$-generic such that in $\mathrm{V}[G]$, letting $\mathbb{Q} = \dot{\mathbb{Q}}^G$, $\mathcal{F}(\mathbb{Q}) \geq \varepsilon^+$. Since the set of $p \in \mathbb{P}$ such that $\mathbb{P}_{\leq p}$ satisfies the ε^+-c.c. is dense in $\mathbb{P}$, forcing with $\mathbb{P}$ preserves ε^+ as a cardinal. Since in $\mathrm{V}[G]$, $\mathcal{F}(\mathbb{Q}) \geq \varepsilon^+$, there is a $q \in \mathbb{Q}$ such that for all $q' \leq q$, $\mathcal{E}(\mathbb{Q}_{\leq q'}) > \mathrm{card}(\varepsilon)^{\mathrm{V}[G]}$, i.e., $\mathcal{E}(\mathbb{Q}_{\leq q'}) \geq \varepsilon^+$. Let $p \in G$ force this, and let p force that $\dot{q}$ is a witness, i.e., $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$ and for all $q' \leq_{\dot{\mathbb{Q}}}$, $\mathcal{E}(\dot{\mathbb{Q}}_{\leq_{\dot{\mathbb{Q}}} q'}) \geq \varepsilon^+$. Choose $\dot{q}$ in so that $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. Let $q = \dot{q}^G$. The set $D = \{\langle r, \dot{s} \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid \mathcal{E}((\mathbb{P} * \dot{\mathbb{Q}})_{\leq \langle r, \dot{s} \rangle}) \leq \varepsilon\}$ is dense and open in $\mathbb{P} * \dot{\mathbb{Q}}$. Let $q \in H$, H being $\mathbb{Q}$-generic over $\mathrm{V}[G]$, with $q \in H$. Then $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, and so, there is a condition $\langle p_1, \dot{q}_1 \rangle \in D \cap G * H$, where we may assume that $p_1 \leq p$ and $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \leq \dot{q}$. Let $q_1 = \dot{q}_1^G$. So in $\mathrm{V}[G]$, $\mathcal{E}(Q_{\leq q_1}) \geq \varepsilon^+$. So let $f : \varepsilon^+ \longrightarrow \mathbb{Q}_{\leq q_1}$ be a one-to-one enumeration of an antichain in $\mathbb{Q}_{\leq q_1}$, $f \in \mathrm{V}[G]$. Let $\dot{f}$ be a $\mathbb{P}$-name for f, and let $p_2 \in G$, $p_2 \leq p_1$, be such that p_2 forces that $\dot{f}$ is as described. For each $\alpha < \varepsilon^+$, let τ_α be such that $\langle p_2, \tau_\alpha \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and $p_2 \Vdash \tau_\alpha = \dot{f}(\check{\alpha})$. Then $\{\langle p_2, \tau_\alpha \rangle \mid \alpha < \varepsilon^+\}$ is an antichain of size ε^+ in $(\mathbb{P} * \dot{\mathbb{Q}})_{\leq \langle p_1, \dot{q}_1 \rangle}$, while $\langle p_1, \dot{q}_1 \rangle \in D$, which means that $\mathcal{E}((\mathbb{P} * \dot{\mathbb{Q}})_{\leq \langle p_1, \dot{q}_1 \rangle}) \leq \varepsilon$, a contradiction.

The claim now follows from Theorem 3.4, because $\mathbb{P}$ is ε-subcomplete, $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is ε-subcomplete, and $\varepsilon \geq \mathcal{F}(\mathbb{P})$. $\square$

As mentioned, assuming CH, Namba forcing $\mathbb{N}$ is ω_2-subcomplete, which implies that it is $\mathcal{F}$-subcomplete, since it necessarily collapses ω_2, and so, $\mathcal{F}(\mathbb{N}) \geq \omega_2$.

5 Iterating ε-subcomplete forcings

The next goal is to establish an iteration theorem for ε-subcomplete forcings that generalizes the one for subcomplete forcings (see [Jen14, Thm. 3, pp. 142ff.]), which says that if an RCS iteration $\langle \mathbb{B}_i \mid i < \alpha \rangle$ satisfies that whenever $i+1 < \alpha$, $\Vdash_{\mathbb{B}_i} \check{\mathbb{B}}_{i+1}/\dot{G}_{\mathbb{B}_i}$ is subcomplete and $\Vdash_{\mathbb{B}_{i+1}} \delta(\check{\mathbb{B}}_i) \leq \omega_1$, then every $\mathbb{B}_i$ is subcomplete. The value $\delta(\mathbb{B}_i)$ plays a double role here - on the one hand, $\mathbb{B}_i$ is subcomplete if it is $\delta(\mathbb{B}_i)$-subcomplete, i.e., it can be viewed as a measure of the closeness required between an originally given full elementary embedding and its subcomplete lift, and on the other hand, $\delta(\mathbb{B}_i)$ is an upper bound on the size of maximal antichains in $\mathbb{B}_i$, which is required to be collapsed in iterations so that it is insured that at limit stages of cofinality greater than ω_1, direct limits are formed. The iteration theorem for ε-subcomplete forcings separates these two roles: the first role will be taken over by the parameter ε_i such that $\mathbb{B}_i$ is ε_i-subcomplete, and $\delta(\mathbb{B}_i)$ will be taken over by $\mathcal{F}(\mathbb{B}_i)$.

Some technical facts will be needed. The following is a corollary of the proof of Lemma 4.5.

Corollary 5.1. *$\mathcal{E}(\mathbb{P}) \leq \mathcal{E}(\mathbb{P} * \dot{\mathbb{Q}})$ and $\mathcal{F}(\mathbb{P}) \leq \mathcal{F}(\mathbb{P} * \dot{\mathbb{Q}})$. Moreover, if $\varepsilon = \mathcal{E}(\mathbb{P} * \dot{\mathbb{Q}})$, then $\Vdash_{\mathbb{P}} \mathcal{E}(\dot{\mathbb{Q}}) \leq \mathrm{card}(\check{\varepsilon})$, and if $\varepsilon' = \mathcal{F}(\mathbb{P} * \dot{\mathbb{Q}})$, then $\Vdash_{\mathbb{P}} \mathcal{F}(\dot{\mathbb{Q}}) \leq \mathrm{card}(\check{\varepsilon}')$.*

Following Jensen, I will employ complete Boolean algebras when iterating, which is why it's worth noting a version of the previous corollary for this context.

Corollary 5.2. *If $\mathbb{A}$ and $\mathbb{B}$ are complete Boolean algebras such that $\mathbb{A} \subseteq \mathbb{B}$, i.e., $\mathbb{A}$ is completely contained in $\mathbb{B}$, then $\mathcal{E}(\mathbb{A}) \leq \mathcal{E}(\mathbb{B})$ and $\mathcal{F}(\mathbb{A}) \leq \mathcal{F}(\mathbb{B})$. Of course, for a Boolean algebra $\mathbb{C}$, $\mathcal{E}(\mathbb{C})$ is defined to be $\mathcal{E}(\langle \mathbb{C} \setminus \{0\}, \leq_{\mathbb{C}} \rangle)$, and similarly for $\mathcal{F}(\mathbb{C})$.*

Moreover, if G is $\mathbb{A}$-generic, then $\mathcal{E}^{\mathrm{V}[G]}(\mathbb{B}/G) \leq \mathcal{E}(\mathbb{B})$ and $\mathcal{F}^{\mathrm{V}[G]}(\mathbb{B}/G) \leq \mathcal{F}(\mathbb{B})$.

Let's relate the measurements $\mathcal{E}(\cdot)$ and $\mathcal{F}(\cdot)$ of a poset and its Boolean completion.

Observation 5.3. *Let $\mathbb{P}$ be separative, and let $\mathbb{B}$ be the complete Boolean algebra of $\mathbb{P}$. Then $\mathcal{E}(\mathbb{P}) = \mathcal{E}(\mathbb{B})$ and $\mathcal{F}(\mathbb{P}) = \mathcal{F}(\mathbb{B})$.*

Proof. Sice $\mathbb{P}$ and $\mathbb{B}$ are forcing equivalent, it follows from Lemma 4.2 that $\mathcal{F}(\mathbb{P}) = \mathcal{F}(\mathbb{B})$. To see that $\mathcal{E}(\mathbb{P}) = \mathcal{E}(\mathbb{B})$, let $i : \mathbb{P} \longrightarrow \mathbb{B}$ be the dense embedding.

$\leq$: if $A \subseteq \mathbb{P}$ is an antichain, then $i``A \subseteq \mathbb{Q}$ is an antichain.

$\geq$: let $A \subseteq \mathbb{B}$ be an antichain. For each $a \in A$, pick $\bar{a} \in \mathbb{P}$ with $i(\bar{a}) \leq a$. Then $\{\bar{a} \mid a \in A\} \subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$ of the same size as A. $\square$

Towards proving a general iteration theorem for parametric subcompleteness, let's isolate a technical corollary from the proofs of Theorems 3.2 and 3.4. If $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{A} \subseteq \mathbb{B}$ is a complete subalgebra of $\mathbb{B}$, then the retraction $h_{\mathbb{B},\mathbb{A}} : \mathbb{B} \longrightarrow \mathbb{A}$ is defined by

$$h_{\mathbb{B},\mathbb{A}}(b) = \bigwedge \{a \in \mathbb{A} \mid b \leq_{\mathbb{B}} a\}$$

Corollary 5.4. *Let $\mathbb{B}$ be a complete Boolean algebra, and let $\mathbb{A} \subseteq \mathbb{B}$ be a complete subalgebra of $\mathbb{B}$ such that $\Vdash_{\mathbb{A}}$ "$\check{\mathbb{B}}/\dot{G}_{\mathbb{A}}$ is $\check{\varepsilon}$-subcomplete, as verified by $\check{\theta}$," where $\mathbb{B} \in H_\theta$ and $\varepsilon \geq \mathcal{F}(\mathbb{A})$. Let $N = L_\tau^A$ be a ZFC^- model with $H_\theta \subseteq N$ and $\theta < \tau$, and let $\bar{N}$ be countable and full. Let $s \in N$ and $\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{s} \in \bar{N}$, where $\bar{\mathbb{B}}$ is a complete Boolean algebra in $\bar{N}$ and $\bar{\mathbb{A}} \subseteq \bar{\mathbb{B}}$ is a complete subalgebra. Let $\bar{I}$ be $\bar{\mathbb{B}}$-generic over $\bar{N}$, and let $\bar{G} = \bar{I} \cap \bar{\mathbb{A}}$. Set $\bar{S} = \langle \bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{s} \rangle$ and $S = \langle \theta, \mathbb{A}, \mathbb{B}, s \rangle$. Let $\dot{\sigma}_0$ and $\dot{t}$ be $\mathbb{A}$-names, and let $a \in \mathbb{A}$ be a condition that forces: $\dot{\sigma}_0 : \check{\bar{N}} \prec \check{N}$, $\dot{\sigma}_0(\check{\bar{S}}) = \check{S}$, $\dot{\sigma}_0``\check{\bar{G}} \subseteq \dot{G}_{\mathbb{A}}$ and $\dot{t} \in \check{\bar{N}}$. Let $h = h_{\mathbb{A}} : \mathbb{B} \longrightarrow \mathbb{A}$ be the retraction, defined above.*

Then there is a condition $b \in \mathbb{B}$ such that $h(b) = a$ and a $\mathbb{B}$-name $\dot{\sigma}'$ such that whenever I is $\mathbb{B}$-generic with $b \in I$, letting $\sigma' = (\dot{\sigma}')^I$, $G = I \cap \mathbb{A}$ and $\sigma_0 = \dot{\sigma}_0^G$, then $\sigma' : \bar{N} \prec N$, $\sigma'(\bar{S}) = S$, $\sigma'``\bar{I} \subseteq I$, $\sigma'(\dot{t}^G) = \sigma_0(\dot{t}^G)$, and $C_\varepsilon^N(\mathrm{ran}(\sigma')) = C_\varepsilon^N(\mathrm{ran}(\sigma_0))$.

Proof. I will follow the proof (including the notation) of Theorem 3.2. The argument is as outlined in [Jen14, p. 138-140]. In the situation described, let G^* be any $\mathbb{A}$-generic filter with $a \in G^*$. Then in $\mathrm{V}[G^*]$, $\mathbb{B}/G^*$ is ε-subcomplete, as verified by θ. let $t = \dot{t}^{G^*}$ and $\sigma_0 = \dot{\sigma}_0^{G^*}$. Then $t \in \bar{N}$ $\sigma_0 : \bar{N} \prec N$ and $\sigma_0``\bar{G} \subseteq G^*$, which means that σ_0 extends to an elementary embedding $\sigma_0^* : \bar{N}[\bar{G}] \prec N[G^*]$. Moreover, $\sigma_0(\bar{S}) = S$.

Let $\bar{H} = \bar{I}/\bar{G}$. We have that $H_\theta^{V[G^*]} = H_\theta[G^*] \subseteq N[G^*]$, so, since $\mathbb{B}/G^*$ is ε-subcomplete in $\mathrm{V}[G^*]$, there is a condition $b \in \mathbb{B}/G^*$ such that whenever H is generic for $\mathbb{B}/G^*$ over $\mathrm{V}[G^*]$, then in $\mathrm{V}[G^*][H]$, there is an elementary embedding $\sigma_1 : \bar{N}[\bar{G}] \prec N[G^*]$ with $\sigma_1(\bar{S}) = S$, $\sigma_1(t) = \sigma_0(t)$, $\sigma_1``(\bar{H}) \subseteq H$ and $C_\varepsilon^{N[G]}(\mathrm{ran}(\sigma_1)) = C_\varepsilon^{N[G]}(\mathrm{ran}(\sigma_0^*))$.

Let $\sigma' = \sigma_1 {\restriction} \bar{N}$. It then follows as in the proof of Theorem 3.4 that $C_\varepsilon^N(\mathrm{ran}(\sigma')) = C_\varepsilon^N(\sigma')$, by proving the statements (A) and (B) of that proof, using that $\varepsilon \geq \mathcal{F}(\mathbb{A})$. So there is a name $\dot{\sigma}'$ in $\mathrm{V}[G^*]^{\mathbb{B}/G^*}$ with $\sigma' = {\dot{\sigma}'}^H$, and such that b forces that $\dot{\sigma}'$ has the properties listed.

Now, all of this is true in $\mathrm{V}[G^*]$ whenever G^* is $\mathbb{A}$-generic over V, with $a \in G^*$, and so, there are names $\dot{b}, \pi \in \mathrm{V}^{\mathbb{A}}$ such that $b = \dot{b}^{G^*}$ and $\dot{\sigma}' = \pi^{G^*}$, and a forces the situation described.

We may choose the name $\dot{b}$ in such a way that $\Vdash_{\mathbb{A}} \dot{b} \in \check{\mathbb{B}}/\dot{G}_{\mathbb{A}}$ and $a = [\![\dot{b} \neq 0]\!]_{\mathbb{A}}$. Namely, given the original $\dot{b}$ such that a forces that $\dot{b} \in \check{B}/\dot{G}_{\mathbb{A}}$ and all the other statements listed above, there are two cases: if $a = \mathbb{1}_{\mathbb{A}}$, then since $a \leq [\![\dot{b} \neq 0]\!]$, it already follows that $a = [\![\dot{b} \neq 0]\!]$ and

$\Vdash_{\mathbb{A}} \dot{b} \in \check{\mathbb{B}}/\dot{G}_{\mathbb{A}}$. If $a < \mathbb{1}_{\mathbb{A}}$, then let $\dot{d} \in \mathrm{V}^{\mathbb{A}}$ be a name such that $\Vdash_{\mathbb{A}} \dot{d} = 0_{\check{\mathbb{B}}/\dot{G}_{\mathbb{A}}}$, and mix the names $\dot{b}$ and $\dot{d}$ to get a name $\dot{c}$ such that $a \Vdash_{\mathbb{A}} \dot{c} = \dot{b}$ and $\neg a \Vdash_{\mathbb{A}} \dot{c} = \dot{d}$. Then $\dot{c}$ is as desired. Clearly, $\Vdash_{\mathbb{A}} \dot{c} \in \check{\mathbb{B}}/\dot{G}_{\mathbb{A}}$, since $a \Vdash_{\mathbb{A}} \dot{c} = \dot{b}$, it follows that $a \leq [\![\dot{c} \neq 0]\!]$, and since $\neg a \Vdash_{\mathbb{A}} \dot{c} = \dot{d}$, it follows that $\neg a \leq [\![\dot{c} = 0]\!] = \neg[\![\dot{c} \neq 0]\!]$, so $[\![\dot{c} \neq 0]\!] \leq a$. So we could replace $\dot{b}$ with $\dot{c}$.

Then, by [Jen14, §0, Fact 4],[2] there is a unique $b \in \mathbb{B}$ such that $\Vdash_{\mathbb{A}} \check{b}/\dot{G}_{\mathbb{A}} = \dot{b}$, and it follows by [Jen14, §0, Fact 3] that

$$h(b) = [\![\check{b}/\dot{G}_{\mathbb{A}} \neq 0]\!]_{\mathbb{A}} = [\![\dot{b} \neq 0]\!]_{\mathbb{A}} = a$$

as wished. □

The import of the retraction function $h_{\mathbb{B},\mathbb{A}}$ is that it occurs in the formation of revised countable support limits of iterations. In Jensen's treatment of forcing iterations, an iteration of length α is a sequence $\langle \mathbb{B}_i \mid i < \alpha \rangle$ of complete Boolean algebras such that for $i \leq j < \alpha$, $\mathbb{B}_i$ is completely contained in $\mathbb{B}_j$, meaning that the of suprema and infima of subsets of $\mathbb{B}_i$ are the same as computed in $\mathbb{B}_i$ and in $\mathbb{B}_j$, and such that if $\lambda < \alpha$ is a limit ordinal, then $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_i$, meaning that $\mathbb{B}_\lambda$ is the completion of the collection of all infima and suprema of subsets of $\bigcup_{i<\lambda} \mathbb{B}_i$. In this setting, $\vec{b} = \langle b_i \mid i < \lambda \rangle$ is a thread in $\vec{\mathbb{B}}{\restriction}\lambda$ if for every $i \leq j < \alpha$, $b_i = h_{\mathbb{B}_j,\mathbb{B}_i}(b_j)$ and $b_j \neq 0$. $\mathbb{B}_\lambda$ is an inverse limit of $\vec{\mathbb{B}}{\restriction}\lambda$ if for every thread $\vec{b}$ in $\vec{\mathbb{B}}{\restriction}\lambda$, $b^* := \bigwedge^{\mathbb{B}_\lambda}_{i<\lambda} b_i \neq 0$, and if the set of such b^* is dense in $\mathbb{B}_\lambda$. This characterizes $\mathbb{B}_\lambda$ up to isomorphism. Following Donder, the RCS limit is defined as the inverse limit, except that only RCS threads $\vec{b}$ are used: $\vec{b} = \langle b_i \mid i < \lambda \rangle$ is an RCS thread in $\vec{\mathbb{B}}{\restriction}\lambda$ if it is a thread in $\vec{\mathbb{B}}{\restriction}\lambda$ and there is an $i < \lambda$ such that either, for all $j < \lambda$ with $i \leq j$, $b_i = b_j$, or $b_i \Vdash_{\mathbb{B}_i} \mathrm{cf}(\check{\lambda}) = \check{\omega}$. $\vec{\mathbb{B}}$ is then an RCS iteration if for every limit λ, $\mathbb{B}_\lambda$ is the RCS limit of $\vec{\mathbb{B}}{\restriction}\lambda$.

In the context of a given iteration $\vec{\mathbb{B}}$ as above, if $i < \alpha$ and $b \in \mathbb{B}_j$, for some $j < \alpha$, I'll just write $h_i(b)$ for $h_{\mathbb{B}_j,\mathbb{B}_i}(b)$. I'll write $\mathrm{lh}(\vec{\mathbb{B}}) = \alpha$, the length of the iteration. I'll use the following fact in the proof of the next theorem.

Fact 5.5 ([Jen14, P. 142])**.** *Let $\vec{\mathbb{B}} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS iteration.*

1. *If $\lambda < \alpha$ and $\mathrm{cf}(\lambda) = \omega$, then $\mathbb{B}_\lambda$ is the inverse limit of $\vec{\mathbb{B}}{\restriction}\lambda$.*
2. *If $\lambda < \alpha$ and for every $i < \lambda$, $\Vdash_{\mathbb{B}_i} \mathrm{cf}(\check{\lambda}) > \omega$, then $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$ (that is, $\mathbb{B}_\lambda$ is formed using only eventually constant threads, making it the direct limit).*
3. *If $i < \lambda$ and G is $\mathbb{B}_i$-generic, then the above are true in $\mathrm{V}[G]$ about the iteration $\langle \mathbb{B}_{i+j}/G \mid j < \alpha - i \rangle$.*

Theorem 5.6. *Let $\langle \mathbb{B}_i \mid i < \alpha \rangle$ be a revised countable support iteration of complete Boolean algebras, and let $\langle \varepsilon_i \mid i < \alpha \rangle$ be a sequence of ordinals such that*

1. *$\mathbb{B}_0 = 2$, $\varepsilon_0 \geq 1$,*
2. *for $i + 1 < \mathrm{lh}(\vec{\mathbb{B}})$, $\Vdash_{\mathbb{B}_i} \check{\mathbb{B}}_{i+1}/\dot{G}_{\mathbb{B}_i}$ is $\check{\varepsilon}_{i+1}$-subcomplete,*
3. *for $i + 1 < \mathrm{lh}(\vec{\mathbb{B}})$, $\Vdash_{\mathbb{B}_{i+1}} \mathrm{card}(\check{\varepsilon}_i) \leq \omega_1$,*
4. *for $i + 1 < \mathrm{lh}(\vec{\mathbb{B}})$, $\varepsilon_{i+1} \geq \mathcal{F}(\mathbb{B}_i)$,*

[2] There is a slightly confusing misprint in the statement of that fact. It should read: "Let $\mathbb{A} \subseteq \mathbb{B}$, and let $\Vdash_{\mathbb{A}} \dot{b} \in \check{\mathbb{B}}/\dot{G}_{\mathbb{A}}$, where $\dot{b} \in \mathrm{V}^{\mathbb{A}}$. There is a unique $b \in \mathbb{B}$ such that $\Vdash_{\mathbb{A}} \dot{b} = \check{b}/\dot{G}_{\mathbb{A}}$." That's what the proof given there shows.

5. *for* $i \leq j < \mathrm{lh}(\vec{\mathbb{B}})$, $\varepsilon_i \leq \varepsilon_j$.

Then each $\mathbb{B}_i$ *is* ε_i*-subcomplete.*

Note: I want to emphasize that the proof of this theorem is due to Jensen, and that the theorem is more or less the expected translation of the original iteration theorem for subcomplete forcing to the context of ε-subcompleteness, with the new ingredient of the function $\mathcal{F}$. Most of the changes made to the proof are straightforward modifications necessary for the setting of ε-subcompleteness, given the Two Step Theorem. The main reason why I describe the argument in some detail is that I want to show why it is required that $\mathcal{F}(\mathbb{B}_i) \leq \varepsilon_{i+1}$ and ε_{i+1} is collapsed to ω_1 by $\mathbb{B}_{i+2}$ - in the original setting of subcompleteness, there was just $\delta(\mathbb{B}_i)$. In the present setting, these two roles are separated more clearly.

Proof. (Jensen) I follow the proof of [Jen14, Thm. 3, p. 142ff.]. Most of the steps go through, using Theorem 3.4/Corollary 5.4 in place of [Jen14, Thm. 1, p. 136ff./Lemma 1.1, p. 140f.].

Before delving into the proof, let me make a simple yet crucial observation.

(+) *Suppose* $h < \alpha$ *and* G_h *is* $\mathbb{B}_h$*-generic over* V. *In* $\mathrm{V}[G_h]$, *define, for* $h + i < \alpha$,

$$\mathbb{B}'_i = \mathbb{B}_{h+i}/G_h \text{ and } \varepsilon'_i = \varepsilon_{h+i}$$

Then, in $\mathrm{V}[G_h]$, $\vec{\mathbb{B}}'$ *is an RCS iteration, and* $\vec{\mathbb{B}}'$, $\vec{\varepsilon}'$ *satisfies 1.-5. of the statement of the theorem.*

Proof of (+). 1. is true because $\mathbb{B}_h/G_h = 2$ and $\varepsilon'_0 = \varepsilon_h \geq \varepsilon_0 \geq 1$. For 2., we have to show that if $i+1 < \alpha - h$ and H_i is $\mathbb{B}'_i$-generic over $\mathrm{V}[G_h]$, then in $\mathrm{V}[G_h][H_i]$, $\mathbb{B}'_{i+1}/H_i$ is ε'_{i+1}-subcomplete. This is because $G_{h+i} := G_h * H_i$ is $\mathbb{B}_{h+i}$-generic, $\mathbb{B}_{h+i+1}/G_{h+i}$ is ε_{h+i+1}-subcomplete in $\mathrm{V}[G_{h+i}]$ (since $\vec{\mathbb{B}}$, $\vec{\varepsilon}$ satisfies 2.), and since $\mathbb{B}_{h+i+1}/G_{h+i}$ is isomorphic to $\mathbb{B}'_{i+1}/\tilde{G}$, $\mathrm{V}[G_{h+i}] = \mathrm{V}[G_h][H_i]$ and $\varepsilon_{h+i+1} = \varepsilon'_{i+1}$. 3. is obvious, because again, if $h + i + 1 < \alpha$ and if H_i is $\mathbb{B}'_i$-generic over $\mathrm{V}[G_h]$, then $G_h * H_i$ is $\mathbb{B}_{h+i}$-generic over V, so since $\vec{\mathbb{B}}$, $\vec{\varepsilon}$ satisfies 3., it follows that in $\mathrm{V}[G_h][H_i]$, $\mathrm{card}(\varepsilon_{h+i}) \leq \omega_1$, and $\varepsilon_{h+i} = \varepsilon'_i$. For 4., suppose $h + i < \alpha$. Since $\vec{\mathbb{B}}$, $\vec{\varepsilon}$ satisfies 4., and by Corollary 5.2, it follows that

$$\mathcal{F}^{\mathrm{V}[G_h]}(\mathbb{B}'_i) = \mathcal{F}^{\mathrm{V}[G_h]}(\mathbb{B}_{h+i}/G_h) \leq \mathcal{F}(\mathbb{B}_{h+i+1}) \leq \varepsilon_{h+i+1} = \varepsilon'_{i+1}$$

and 5. is trivial. $\square_{(+)}$

In the following, if $i < \alpha$, I will write $\dot{G}_i$ for $\dot{G}_{\mathbb{B}_i}$ (see Notation 4.1), and I'll use similar abbreviations throughout. The following claim will be proven by induction on $\lambda < \alpha$:

$(*)_\lambda$ *For every* $h < \lambda$, *if* G_h *is generic for* $\mathbb{B}_h$ *over* V, *then in* $\mathrm{V}[G_h]$, $\mathbb{B}_\lambda/G_h$ *is* ε_λ*-subcomplete.*

This is trivial if $\lambda = 0$.

Case 0: $\lambda = j + 1$ *is a successor.*

Let $h < \lambda$ be given, and let G_h be $\mathbb{B}_h$-generic over V. Inductively, we know that $\mathrm{V}[G_h] \models \mathbb{B}_j/G_h$ is ε_j-subcomplete, and since $\varepsilon_j \leq \varepsilon_\lambda$, by 5., it follows that

(1) $\mathrm{V}[G_h] \models$ $\mathbb{B}_j/G_h$ *is* ε_λ*-subcomplete.*

Moreover, by 2., $\Vdash_{\mathbb{B}_j} \check{\mathbb{B}}_{j+1}/\dot{G}_j$ is $\check{\lambda}$-subcomplete, so if we let $\tilde{G}$ be $\mathbb{B}_j/G_h$-generic over $\mathrm{V}[G_h]$, then since $G_h * \tilde{G}$ is $\mathbb{B}_j$-generic over V and $(\mathbb{B}_\lambda/G_h)/\tilde{G} \cong \mathbb{B}_\lambda/(G_h * \tilde{G})$, the latter is ε_λ-subcomplete in $\mathrm{V}[G_h][\tilde{G}]$. Since $\tilde{G}$ was arbitrary, this shows that

(2) $\mathrm{V}[G_h] \models$ ($\Vdash_{\mathbb{B}_j/G_h}$ $(\mathbb{B}_\lambda \check{/} G_h)/\dot{G}_{\mathbb{B}_j \check{/} G_h}$ *is* $\check{\varepsilon}_\lambda$*-subcomplete*)

It follows from Corollary 5.2 and conditions 4. and 5. that $\mathcal{F}(\mathbb{B}_j/G_h)^{\mathrm{V}[G_h]} \le \mathcal{F}(\mathbb{B}_j) \le \varepsilon_{j+1} \le \varepsilon_\lambda$. But then, Theorem 3.4, applied in $\mathrm{V}[G_h]$, shows that (1) and (2), taken together, imply

(3) $\mathrm{V}[G_h] \models (\mathbb{B}_j/G_h) * (\mathbb{B}_\lambda \check{/} G_h)/\dot{G}_{\mathbb{B}_j \check{/} G_h}$ *is* $\check{\varepsilon}_\lambda$*-subcomplete*

Since in $\mathrm{V}[G_h]$, $(\mathbb{B}_j/G_h) * (\mathbb{B}_\lambda \check{/} G_h)/\dot{G}_{\mathbb{B}_j \check{/} G_h} \cong \mathbb{B}_\lambda/G_h$, this means that in $\mathrm{V}[G_h]$, $\mathbb{B}_\lambda/G_h$ is ε_λ-subcomplete, as wished.

Case 1: λ is a limit ordinal.

I'll distinguish two subcases.

Case 1.1: $\mathrm{cf}(\lambda) \le \varepsilon_i$, for some $i < \lambda$.

Then, if $i < \lambda$ is such that $\mathrm{cf}(\lambda) \le \varepsilon_i$, it follows that for every h with $i < h < \lambda$, whenever G_h is $\mathbb{B}_h$-generic over V, then in $\mathrm{V}[G_h]$, the cofinality of λ is at most ω_1. It suffices to prove $(*)_\lambda$ for such h, because we can then use the Two Step Theorem together with the induction hypothesis in order to prove it for $h \le i$, just as in the successor case. We could now argue in $\mathrm{V}[G_h]$, using that $\mathrm{cf}^{\mathrm{V}[G_h]}(\lambda) \le \omega_1$, to prove $(*)_\lambda$ for h, and by (+), if we argue in $\mathrm{V}[G_h]$, we're essentially in the same situation as in V, so to simplify notation a little bit, we'll pretend $\mathrm{V} = \mathrm{V}[G_h]$, and show:

(4) *If* $\mathrm{cf}(\lambda) \le \omega_1$*, then* $\mathbb{B}_\lambda$ *is* ε_λ*-subcomplete.*

Let $f : \omega_1 \longrightarrow \lambda$ be cofinal, with $f(0) = 0$, and let $\theta > \lambda$ be large enough that for every $i < j < \lambda$,

$$\Vdash_{\mathbb{B}_i} \check{\theta} \text{ verifies the } \check{\varepsilon}_j\text{-subcompleteness of } \check{\mathbb{B}}_j/\dot{G}_{\mathbb{B}_i}$$

It will be shown that θ verifies the ε_λ-subcompleteness of $\mathbb{B}_\lambda$. As usual, let $N = L^A_\tau \supseteq H_\theta$ be a ZFC^--model, let $\sigma : \bar{N} \prec N$ countable and full, let $S = \langle \theta, \mathbb{B}, \lambda, f, s, \vec{\varepsilon} \rangle \in \mathrm{ran}(\sigma)$, let $\bar{S} = \langle \bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{f}, \bar{s}, \bar{\vec{\varepsilon}} \rangle = \sigma^{-1}(S)$, and let $\bar{G}$ be $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$. We have to find a condition $b \in \mathbb{B}_\lambda$ such that whenever $G \ni b$ is $\mathbb{B}_\lambda$-generic over V, then in $\mathrm{V}[G]$, there is an elementary embedding $\sigma' : \bar{N} \prec N$ with $\sigma'(\bar{S}) = \sigma(\bar{S}) = S$, $C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma')) = C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma))$ and $\sigma' “\bar{G} \subseteq G$.

Let $\tilde{\lambda} = \sup \sigma “\bar{\lambda}$, and let $\langle \nu_i \mid i < \omega \rangle$ be a sequence of ordinals less than $\omega_1^{\bar{N}}$ such that if we let $\bar{\xi}_i = \bar{f}(\nu_i)$, then $\bar{\xi}_0 = 0$ and $\langle \bar{\xi}_i \mid i < \omega \rangle$ is monotone and cofinal in $\bar{\lambda}$. Let $\xi_i = f(\nu_i)$. Then $\langle \xi_i \mid i < \omega \rangle$ is monotone and cofinal in $\tilde{\lambda}$, and whenever $\sigma' : \bar{N} \prec N$ with $\sigma'(\bar{f}) = f$, then $\sigma'(\bar{\xi}_i) = \sigma'(\bar{f}(\nu_i)) = \sigma'(\bar{f})(\sigma'(\nu_i)) = f(\nu_i) = \xi_i$, for all $i < \omega$. Fix an enumeration $\langle x_l \mid l < \omega \rangle$ of $\bar{N}$.

Now construct a sequence $\langle \langle b_i, \dot{\sigma}_i \rangle \mid i < \omega \rangle$ such that $b_i \in \mathbb{B}_{\xi_i}$, with the property that for every $i < \omega$, whenever $G_{\xi_i} \ni b_i$ is generic for $\mathbb{B}_{\xi_i}$ over V, then in $\mathrm{V}[G_{\xi_i}]$, if we let $\sigma_i = \dot{\sigma}_i^{G_{\xi_i}}$, then we have that (a) $\sigma_i : \bar{N} \prec N$, (b) $\sigma_i(\bar{S}) = S$, (c) $C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma_i)) = C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma))$ and (d) $\sigma_i “(\bar{G} \cap \bar{B}_{\bar{\xi}_i}) \subseteq G_{\xi_i}$. Moreover, we maintain that

(5) *for* $h \le i$*,* $\sigma_i(x_h) = \sigma_h(x_h)$*, where* $\sigma_h = \dot{\sigma}_h^{G_{\xi_i} \cap \mathbb{B}_{\xi_h}}$

If (c) above is satisfied at i, then it follows by Fact 1.5 that for every $x \in \bar{N}$, there is a $u \in \bar{N}$ such that $\mathrm{card}(u)^{\bar{N}} \le \bar{\varepsilon}_{\bar{\xi}_i}$ and $\sigma(x) \in \sigma_i(u)$, because

$$\sigma(x) \in C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma)) = C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma_i)) = \bigcup \{\sigma'(u) \mid u \in \bar{N} \ \wedge \ \mathrm{card}(u)^{\bar{N}} \le \bar{\varepsilon}_{\bar{\xi}_i}\}$$

So we may define $u_i =$ the $\bar{N}$-least u with $\sigma(x_i) \in \sigma_i(u)$ and $\mathrm{card}(u)^{\bar{N}} \le \bar{\varepsilon}_{\bar{\xi}_i}$, and require that

(6) *for* $h \leq i$, $\sigma_i(u_h) = \sigma_h(u_h)$

and that

(7) $h_{\xi_i}(b_{i+1}) = b_i$, *for all* $i < \omega$.

Following Jensen's setup, the natural injection from $\mathrm{V}^{\mathbb{B}_{\xi_h}}$ to $\mathrm{V}^{\mathbb{B}_{\xi_i}}$ is the identity, and so, $\sigma_h = \dot{\sigma}_h^{G_{\xi_i} \cap \mathbb{B}_{\xi_h}} = \dot{\sigma}_h^{G_{\xi_i}}$. We set $\dot{\sigma}_0 = \check{\sigma}$ and $b_0 = 1$. Given $\dot{\sigma}_i$ and b_i, Corollary 5.4 can be applied to $\mathbb{B}_{\xi_i} \subseteq \mathbb{B}_{\xi_{i+1}}$, $\dot{\sigma}_i$, b_i and a $\mathbb{B}_{\xi_i}$-name t such that $\Vdash_{\mathbb{B}_{\xi_i}} t = \langle \check{x}_0, \check{x}_1, \ldots, \check{x}_i, \dot{u}_0, \dot{u}_1, \ldots, \dot{u}_n \rangle$. We use the inductive assumption here, and we use that $\mathcal{F}(\mathbb{B}_{\xi_i}) \leq \varepsilon_{\xi_{i+1}}$. Then all the conditions are satisfied, and $\vec{b}$ is a thread through $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$, by (7). So $b = \bigwedge_{i<\omega} b_i \in \mathbb{B}_{\bar{\lambda}} \setminus \{0\}$.

Now let $G \ni b$ be $\mathbb{B}_\lambda$-generic. Let $G_{\xi_i} = G \cap \mathbb{B}_{\xi_i}$, and let $\sigma_i = \dot{\sigma}_i^{G_{\xi_i}} = \dot{\sigma}_i^G$, for $i < \omega$. Then (a)-(d), (5) and (6) hold for σ_i, for all i. By (5), the function $\sigma' : \bar{N} \longrightarrow N$ defined by

$$\sigma'(x_h) = \sigma_h(x_h)$$

is elementary. Moreover, it's obvious that $\sigma'(\bar{S}) = S$. Two more properties of σ' have to be shown now.

First, $C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma')) = C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma))$. For the inclusion from left to right, it suffices to show that $\mathrm{ran}(\sigma') \subseteq C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma))$. But $\sigma'(x_i) = \sigma_i(x_i) \in C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma)) \subseteq C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma))$, by (c). For the inclusion from right to left, it suffices to show that $\mathrm{ran}(\sigma) \subseteq C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma'))$. To see this, let $i < \omega$. Then $\sigma(x_i) \in \sigma_i(u_i) = \sigma'(u_i) \subseteq \bigcup\{\sigma'(u) \mid \mathrm{card}(u)^{\bar{N}} \leq \bar{\varepsilon}_{\bar{\xi}_i}\} = C^N_{\varepsilon_{\xi_i}}(\mathrm{ran}(\sigma')) \subseteq C^N_{\varepsilon_\lambda}(\mathrm{ran}(\sigma'))$, by (6) and Fact 1.5.

Finally, we have to show that $\sigma' “\bar{G} \subseteq G$. To this end, note that

(8) *for every* $i < \omega$, $\sigma_i “\bar{G}_{\bar{\xi}_i} \subseteq G$

because for $x \in \bar{G}_{\bar{\xi}_i}$, there is some $j \geq i$ such that $\sigma'(x) = \sigma_j(x)$, and for such a j, it follows that $x \in \bar{G}_{\bar{\xi}_j}$, and by (d), $\sigma'(x) = \sigma_j(x) \in G_{\xi_j} \subseteq G$. Given (8), there are now two cases to consider. The first case is that $\mathrm{cf}(\lambda) = \omega_1$. In this case, $\mathbb{B}_\lambda$ is the direct limit of $\bigcup_{i<\lambda} \mathbb{B}_i$, because it is formed using RCS threads, and no RCS thread f can have an $i < \lambda$ with $f(i) \Vdash_{\mathbb{B}_i} \mathrm{cf}(\lambda) = \omega$, since inductively, $\mathbb{B}_i$ is ε_i-subcomplete and hence cannot add reals. So every RCS thread is eventually constant. The same reasoning applies to $\bar{\mathbb{B}}_{\bar{\lambda}}$ in $\bar{N}$. We have that $\bar{G} \cap \bar{\mathbb{B}}_{\bar{\lambda}}$ is generated by $\bigcup_{n<\omega}(\bar{G}_{\bar{\xi}_n})$, and so, (8) implies that $\sigma' “\bar{G} \subseteq G$. The second case is that $\mathrm{cf}(\lambda) = \omega$. In this case, $\bar{\mathbb{B}}_{\bar{\lambda}}$ is the inverse limit of $\langle \bar{\mathbb{B}}_i \mid i < \bar{\lambda} \rangle$ in $\bar{N}$, since in $\bar{N}$, $\mathrm{cf}(\bar{\lambda}) = \omega$. Let $\bar{a} \in \bar{G}$. Let $\bar{a}' \leq_{\bar{\mathbb{B}}_{\bar{\lambda}}} \bar{a}$ be the infimum of a thread. We can then write $\bar{a}'$ as $\bar{a}' = \bigwedge_{i<\omega} a_i$, where $\vec{a} \in \bar{N}$, and each a_i is in $\bar{\mathbb{B}}_{\zeta_i}$, for some increasing sequence $\vec{\zeta} \in \bar{N} \cap {}^\omega\bar{\lambda}$. Clearly, each a_i is weaker than $\bar{a}'$, which is in $\bar{G}$, so each a_i belongs to some $\bar{G}_{\bar{\xi}_j}$, and so, $\sigma'(a_i) \in G$. Since G is V-generic, it follows that $\sigma'(\bar{a}') = \bigwedge_{i<n} \sigma'(a_i) \in G$, and since $\sigma'(\bar{a}') \leq \sigma'(\bar{a})$, it follows that $\sigma'(\bar{a}) \in G$. This completes the proof in case 1.1.

Case 1.2: For all $i < \lambda$, $\mathrm{cf}(\lambda) > \varepsilon_i$.

Let $h < \lambda$, and let G_h be $\mathbb{B}_h$-generic over V. If $\mathrm{cf}^{\mathrm{V}[G_h]}(\lambda) \leq \omega_1$, then we can argue in $\mathrm{V}[G_h]$ as in case 1.1. So let's assume that $\mathrm{cf}^{\mathrm{V}[G_h]}(\lambda) > \omega_1$. Since $\mathcal{F}(\mathbb{B}_h) \leq \varepsilon_{h+1} < \mathrm{cf}(\lambda)$, it follows that $\mathrm{cf}^{\mathrm{V}[G_h]}(\lambda) = \mathrm{cf}^{\mathrm{V}}(\lambda)$, because forcing with $\mathbb{B}_h$ is like forcing with a poset that has the $\mathcal{F}(\mathbb{B}_h)^+$-c.c. We have that in $\mathrm{V}[G_h]$, for every $i < \lambda$ with $h \leq i$, $\mathcal{F}(\mathbb{B}_i/G_h) \leq \varepsilon_{i+1} < \mathrm{cf}^{\mathrm{V}[G_h]}(\lambda)$, so by (+), we may pretend $\mathrm{V} = \mathrm{V}[G_h]$. It follows that $\mathbb{B}_\lambda$ is the direct limit of $\vec{\mathbb{B}}\restriction\lambda$. Namely, by definition, $\mathbb{B}_\lambda$ is the RCS limit of $\vec{\mathbb{B}}\restriction\lambda$. An RCS thread $f \in \prod_{i<\alpha} \mathbb{B}_i$ either is eventually constant, or there is some $i < \lambda$ such that $f(i) \Vdash_{\mathbb{B}_i} \mathrm{cf}(\check{\lambda}) = \omega$. But the latter is impossible, again because $\mathcal{F}(\mathbb{B}_i) < \mathrm{cf}(\lambda)$.

So every RCS thread is eventually constant, and this means that $\mathbb{B}_\lambda$ is the direct limit of $\vec{\mathbb{B}}\restriction\lambda$. It was crucial in this step that $\mathcal{F}(\mathbb{B}_i) < \varepsilon_{i+1} \le \varepsilon_\lambda$.

Let $\theta > \lambda$ be large enough that for every $i < j < \lambda$,

$$\Vdash_{\mathbb{B}_i} \check{\theta} \text{ verifies the } \check{\varepsilon}_j\text{-subcompleteness of } \check{\mathbb{B}}_j/\dot{G}_{\mathbb{B}_i}$$

Again, the claim is that θ verifies that $\mathbb{B}_\lambda$ is ε_λ-subcomplete. So let $N = L^A_\tau \supseteq H_\theta$ be a ZFC^--model, let $\sigma : \bar{N} \prec N$ be countable and full, let $S = \langle \theta, \mathbb{B}, \lambda, s, \vec{\varepsilon}\rangle \in \operatorname{ran}(\sigma)$, and let $\bar{S} = \langle \bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}, \vec{\bar{\varepsilon}}\rangle = \sigma^{-1}(S)$, and let $\bar{G}$ be $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$. We have to find a condition $c \in \mathbb{B}_\lambda$ such that

(9) *whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic over* V, *then in* V$[G]$, *there is an elementary embedding $\sigma' : \bar{N} \prec N$ with $\sigma'(\bar{S}) = \sigma(\bar{S}) = S$, $C^N_{\varepsilon_\lambda}(\operatorname{ran}(\sigma')) = C^N_{\varepsilon_\lambda}(\operatorname{ran}(\sigma))$ and $\sigma'``\bar{G} \subseteq G$.*

Let $\tilde{\lambda} = \sup \sigma``\bar{\lambda}$, and let $\langle \bar{\xi}_i \mid i < \omega\rangle$ be increasing and cofinal in $\bar{\lambda}$. Let $\xi_i = \sigma(\bar{\xi}_i)$. Then $\langle \xi_i \mid i < \omega\rangle$ is monotone and cofinal in $\tilde{\lambda}$. Fix an enumeration $\langle x_l \mid l < \omega\rangle$ of $\bar{N}$. We will construct sequences $\langle c_i \mid i < \omega\rangle$ and $\langle \dot{\sigma}_i \mid i < \omega\rangle$ with $c_i \in \mathbb{B}_{\xi_i}$ and $\dot{\sigma}_i \in \mathrm{V}^{\mathbb{B}_{\xi_i}}$ such that the following conditions are satisfied for each i:

(I) $c_0 = \mathbb{1}$, and $h_{\xi_{i-1}}(c_i) = c_{i-1}$ if $i > 0$.

(II) Let $G \ni \bar{\xi}_i$ be $\mathbb{B}_{\xi_i}$-generic over V. For $\eta \le \xi_i$, let $G_\eta = G \cap \mathbb{B}_\eta$, and for $\eta \le \bar{\xi}_i$, let $\bar{G}_\eta = \bar{G} \cap \bar{\mathbb{B}}_\eta$, and set, for $h \le i$, $\sigma_h = \dot{\sigma}_h^{G_{\xi_h}} = \dot{\sigma}_h^G$. Then

  (a) $\sigma_i : \bar{N} \prec N$,
  (b) $\sigma_i(\bar{S}) = S$,
  (c) $C^N_{\varepsilon_{\xi_i}}(\operatorname{ran}(\sigma_i)) = C^N_{\varepsilon_{\xi_i}}(\operatorname{ran}(\sigma))$,
  (d) if $\sigma_i(\bar{\xi}_m) \le \xi_i < \sigma_i(\bar{\xi}_{m+1})$, then $\sigma_i``\bar{G}_{\bar{\xi}_m} \subseteq G$,
  (e) if $h < i$, then $\sigma_i(x_h) = \sigma_h(x_h)$,
  (f) if $h < i$, then $\sigma_i(u_h) = \sigma_h(u_h)$, where u_h is the minimal u such that the $\bar{N}$-cardinality of u is at most $\varepsilon_{\bar{\xi}_h}$ and $\sigma(x_h) \in \sigma_h(u)$.
  (g) if $\sigma_h(\bar{\xi}_m) \le \xi_h < \xi_i < \sigma_h(\bar{\xi}_{m+1})$, then $\sigma_h = \sigma_i$.

Note that (a),(b),(c) imply that $\sup \sigma_i``\bar{\lambda} = \sup \sigma``\bar{\lambda} = \tilde{\lambda}$ (by Fact 1.6), and that (f) makes sense, by (c). Once we have constructed $\vec{c}$ and $\vec{\sigma}$, then we can set $c = \bigwedge_{i<\omega} c_i$. Since $\vec{c}$ is a thread, $c \neq 0$. If $G \ni c$, then let $\sigma_i = \dot{\sigma}_i^G$, and define $\sigma'(x_h) = \sigma_h(x_h)$. Then (9) is satisfied: it's obvious that $\sigma' : \bar{N} \prec N$ and $\sigma'(\bar{S}) = S$. Letting $\tilde{\varepsilon} = \sup_{i<\tilde{\lambda}} \varepsilon_i$, it then follows that $C^N_{\tilde{\varepsilon}}(\operatorname{ran}(\sigma')) = C^N_{\tilde{\varepsilon}}(\operatorname{ran}(\sigma))$. For the inclusion from left to right, it suffices to show that $\operatorname{ran}(\sigma') \subseteq C^N_{\tilde{\varepsilon}}(\operatorname{ran}(\sigma))$. But $\sigma'(x_i) = \sigma_i(x_i) \in C^N_{\varepsilon_{\xi_i}}(\operatorname{ran}(\sigma)) \subseteq C^N_{\tilde{\varepsilon}}(\operatorname{ran}(\sigma))$, by (c). For the inclusion from right to left, it suffices to show that $\operatorname{ran}(\sigma) \subseteq C^N_{\tilde{\varepsilon}}(\operatorname{ran}(\sigma'))$. To see this, let $i < \omega$. Then $\sigma(x_i) \in \sigma_i(u_i) = \sigma'(u_i) \subseteq \bigcup\{\sigma'(u) \mid \operatorname{card}(u)^{\bar{N}} \le \bar{\varepsilon}_{\bar{\xi}_i}\} = C^N_{\varepsilon_{\xi_i}}(\operatorname{ran}(\sigma')) \subseteq C^N_{\varepsilon_\lambda}(\operatorname{ran}(\sigma'))$.

Finally, we have to show that $\sigma'``\bar{G} \subseteq G$. Since $\bar{\mathbb{B}}_{\bar{\lambda}}$ is the direct limit of $\langle \bar{\mathbb{B}}_i \mid i < \bar{\lambda}\rangle$, it suffices to show that $\sigma'``\bar{G}_{\bar{\xi}_i} \subseteq G$, for every $i < \omega$.

(10) *for every $i < \omega$ and all sufficiently large natural numbers $j \ge i$,*

  $(*)$ *there is an $m \ge i$ such that $\sigma_j(\bar{\xi}_m) \le \xi_j < \sigma_j(\bar{\xi}_{m+1})$.*

Proof of (10). To see this, fix $i < \omega$. Note that for any $j < \omega$, since $\sup \sigma_j``\bar{\lambda} = \tilde{\lambda}$ (see the remark after (I) and (II)) and $\xi_j < \tilde{\lambda}$, there is a unique $m = m_j$ with $\sigma_j(\bar{\xi}_m) \le \xi_j < \sigma_j(\bar{\xi}_{m+1})$. What (10) says is that for all sufficiently large j, $m_j \ge i$. Clearly, this is trivial if $i = 0$. Suppose now that $i > 0$, and assume (10) fails for i. This means that for arbitrarily large j, $m_j < i$, and in particular, $\xi_j < \sigma_j(\bar{\xi}_i)$. But for sufficiently large j, $\sigma'(\xi_i) = \sigma_j(\xi_i)$, so we'd get that for arbitrarily large j, $\xi_j < \sigma'(\bar{\xi}_i)$. So $\tilde{\lambda} = \sup_{j<\omega} \xi_j \le \sigma'(\bar{\xi}_i)$, which is impossible, because then we'd get that for sufficiently large $j < \omega$, $\tilde{\lambda} \le \sigma'(\bar{\xi}_i) = \sigma_j(\bar{\xi}_i) < \sigma_j(\bar{\xi}_{i+1}) \le \sup \sigma_j``\bar{\lambda} = \tilde{\lambda}$. $\square_{(10)}$

Note that it was crucial for the argument for (10) that $\sup \sigma_j``\bar{\lambda} = \tilde{\lambda}$, and for this, (c) is instrumental, together with the fact that for $i < \lambda$, $\varepsilon_i < \mathrm{cf}(\lambda)$. This is why we required that $\mathbb{B}_{i+1}$ collapse ε_i to ω_1.

With (10) in hand, it is now not hard to prove that $\sigma'``\bar{G}_{\xi_i} \subseteq G$, for every $i < \omega$. For let $a \in \bar{G}_{\xi_i}$ be given. Let $j$ be sufficiently large that the unique $m$ with $\sigma_j(\bar{\xi}_m) \le \xi_j < \sigma_j(\bar{\xi}_{m+1})$ is at least $i$, and such that $\sigma'(a) = \sigma_j(a)$. Then, by (d), $\sigma_j``\bar{G}_{\bar{\xi}_m} \subseteq G$, but $a \in \bar{G}_{\xi_i} \subseteq \bar{G}_{\xi_m}$ because $i \le m$.

Thus, once we have constructed sequences $\vec{c}$ and $\vec{\dot{\sigma}}$ satisfying properties (I) and (II), the proof is complete. This is a rather lengthy construction, and most of the details work exactly as in Jensen's original proof. This is why I will omit the proofs in the following, and just describe the construction, to provide a more or less complete account.

The construction of $\vec{c}$ and $\vec{\dot{\sigma}}$ proceeds by induction. c_0 is given. If $\dot{\sigma}_{i-1}$ and c_{i-1} have been defined already, we will define $\dot{\sigma}_i$ and a condition b_i with the properties listed below, and then c_i will be defined. We require:

(III) *(a)* $b_0 = 1$, $\dot{\sigma}_0 = \check{\sigma}$,
(b) $h_{\xi_{i-1}}(b_i) = c_{i-1}$, *for* $i > 0$,
(c) (II)(a)-(g) hold whenever $b_i \in G$.

There is one more condition b_i has to satisfy. Namely, given $\nu < \tilde{\lambda}$, let i be least such that $\nu \le \xi_i$, and let $\xi_i < \mu < \tilde{\lambda}$. For $j < \omega$, set

$$a_i^{j\nu\mu} = a^{j\nu\mu} = b_i \wedge [\![\dot{\sigma}_i(\check{\bar{\xi}}_j) = \check{\nu} \wedge \dot{\sigma}_i(\check{\bar{\xi}}_{j+1}) = \check{\mu}]\!]_{\mathbb{B}_{\xi_i}}$$

It follows that if $\langle j, \nu, \mu\rangle \neq \langle j', \nu', \mu'\rangle$, then $a^{j\nu\mu}$ and $a^{j'\nu'\mu'}$ are incompatible. We demand:

(IV) *If* $\sup_{h<i} \xi_h < \nu \le \xi_i < \mu < \tilde{\lambda}$, *then for all* x *and* y, $a_i^{j\nu\mu} \wedge [\![\dot{\sigma}_i(\check{x}) = \check{y}]\!] \in \mathbb{B}_\nu$.

Define

$$A = A_i = \{a^{j\nu\mu} \neq 0 \mid \sup_{h<i} \xi_h < \nu \le \xi_i < \mu < \tilde{\lambda}\}$$

It follows from (IV) that for each $a = a^{j\nu\mu} \in A$, there is a $\dot{\sigma}_a \in \mathrm{V}^{\mathbb{B}_\nu}$ such that if $G \ni a$ is $\mathbb{B}_{\xi_i}$-generic, then $\dot{\sigma}_a^{G_\nu} = \dot{\sigma}_i^G$, where $G_\nu = G \cap \mathbb{B}_\nu$. But if $G \ni a$ is $\mathbb{B}_\nu$-generic, then G extends to a $\mathbb{B}_{\xi_i}$-generic filter G' and it follows that $\dot{\sigma}_a^G = \dot{\sigma}_a^{G'} = \dot{\sigma}_i^{G'}$. It follows then that (II) holds with $\sigma_a = \dot{\sigma}_a^G$ in place of σ_i and $\sigma_h = \dot{\sigma}_h^G = \dot{\sigma}_h^{G \cap \mathbb{B}_{\xi_h}}$, for $h < i$. Using Corollary 5.4, one obtains:

(11) *Fix* $a \in A_i$, $a = a^{j\nu\mu} = a_i^{j\nu\mu}$. *There are* $\tilde{a} \in \mathbb{B}_\mu$ *and* $\dot{\sigma}'_a \in \mathrm{V}^{\mathbb{B}_\mu}$ *such that* $h_\nu(\tilde{a}) = a$ *and such that whenever* $G \ni \tilde{a}$ *is* $\mathbb{B}_\mu$*-generic,* $\sigma_a = \dot{\sigma}_a^G$ *and* $\sigma'_a = (\dot{\sigma}')_a^G$, *then*

(a) $\sigma'_a : \bar{N} \prec N$,
(b) $\sigma'_a(\bar{s}) = s$,
(c) $C^N_{\varepsilon_{\xi_{j+1}}}(\mathrm{ran}(\sigma'_a)) = C^N_{\varepsilon_{\xi_{j+1}}}(\mathrm{ran}(\sigma_a))$,

(d) $\sigma'``\bar{G}_{\bar{\xi}_{j+1}} \subseteq G$,

(e) *if r is least such that $\mu \leq \xi_r$, then for all $h < r$, $\sigma'_a(x_h) = \sigma_a(x_h)$ and $\sigma'_a(u^a_h) = \sigma'_a(u^a_h)$, where u^a_h is the $\bar{N}$-least $u \in \bar{N}$ with* $\mathrm{card}(u)^{\bar{N}} < \bar{\varepsilon}_{\bar{\xi}_{j+1}}$,

(f) *for all $l \leq j+1$, $\sigma'_a(\bar{\xi}_l) = \sigma_a(\bar{\xi}_l)$.*

Fixing such objects $\tilde{a}$ and $\dot{\sigma}'_a$ for each $a \in A_i$, and assuming b_i is defined so that (III) and (IV) are satisfied, define c_i by setting

$$c_i = (b_i - \bigvee A_i) \vee \bigvee_{a \in A_i} h_{\xi_i}(\tilde{a})$$

Now one can repeat the inductive argument from [Jen14, p. 148 ff]: assuming (I)-(IV) holds below i and (III)-(IV) holds at at i, this definition of c_i yields (I)-(II) at i.

Thus, it remains to show that if $i = j+1$ and (I)-(IV) hold up to j, then one can define b_i and $\dot{\sigma}_i$ so that (III)-(IV) are satisfied. Letting

$$\hat{A}_j = \{a = a^{h\nu\mu} \in \bigcup_{l<i} A_l \mid \xi_j < \mu\}$$

one shows that if $\langle h\nu\mu\rangle \neq \langle h'\nu'\mu'\rangle$, then $a^{h\nu\mu} \wedge a^{h'\nu'\mu'} = 0$ and defines

$$b_i = \bigvee \{h_{\xi_i}(\tilde{a}) \mid a \in \hat{A}_j\}$$

It remains to define $\dot{\sigma}_i$. To do this, let

$$\tilde{A} = \{a^{i\nu\mu} \in A_j \mid \mu \leq \xi_i\}$$

Then define $\dot{\sigma}_i$ to be a $\mathbb{B}_{\xi_i}$-name such that for $a \in \tilde{A}$, $[\![\dot{\sigma}_i = \dot{\sigma}'_a]\!] = \tilde{a}$, and such that $[\![\dot{\sigma}_j = \dot{\sigma}_i]\!] \wedge b_i = b_i - \bigvee \tilde{A}$. The verification that these definitions ensure that (III) and (IV) hold at i is as in [Jen14, p. 149 ff]. □

6 ε-subcompleteness from distributivity

As an application of the concept of ε-subcompleteness, I show in this section that, for example, every ω_2-distributive forcing is essentially subcomplete. To make this more precise, I need a definition.

Definition 6.1. Let $\mathbb{P}$ be a notion of forcing. A cardinal ε *captures $\mathbb{P}$-genericity* if there is a cardinal θ such that the following holds: whenever $N = L_\tau[A]$, X, N_0 and k_0 are such that $H_\theta \in N$, $\varepsilon \cup \{\mathbb{P}\} \subseteq X$, $N|X \prec N$ and $k_0 : N_0 \longrightarrow N|X$ is the inverse of the Mostowski-isomorphism, then whenever G_0 is $k_0^{-1}(\mathbb{P})$-generic over N_0, then $k_0``G_0$ generates a filter $G = \{p \in \mathbb{P} \mid \exists q \in G_0 \quad k_0(q) \leq_{\mathbb{P}} p\}$ which is $\mathbb{P}$-generic over V.

It is easy to see that $2^{\mathrm{card}(\mathbb{P})}$ captures $\mathbb{P}$-genericity, but there may be smaller cardinals that do this. Also, if ε captures $\mathbb{P}$-genericity and $\varepsilon' \geq \varepsilon$, then ε' captures $\mathbb{P}$-genericity as well.

Recall that a forcing $\mathbb{P}$ is subcomplete above a cardinal ζ if whenever $\sigma : \bar{N} \longrightarrow N$ is as usual, with $\zeta \in \mathrm{ran}(\sigma)$, then there is a subcompleteness embedding $\sigma' : \bar{N} \longrightarrow N$ in $\mathrm{V}^{\mathbb{P}}$ such that, letting $\bar{\zeta} = \sigma^{-1}(\zeta)$, it follows that $\sigma{\upharpoonright}\bar{\zeta} = \sigma'{\upharpoonright}\bar{\zeta}$. Let's say that $\mathbb{P}$ is ε-subcomplete above ζ if $\mathbb{P}$ satisfies the definition of subcompleteness above ζ, with $\delta(\mathbb{P})$ replaced by ε, and with the assumption that $\varepsilon \in \mathrm{ran}(\sigma)$.

Theorem 6.2. *Suppose $\mathbb{P}$ is a κ-distributive notion of forcing, where $\kappa > \omega_1$ is a cardinal. Let $\varepsilon \geq \kappa$ be a cardinal that captures $\mathbb{P}$-genericity. Then $\mathbb{P}$ is ε-subcomplete. Moreover, if η is such that $\eta^\omega < \kappa$, then $\mathbb{P}$ is ε-subcomplete above η.*

Note: In particular, every ω_2-distributive notion of forcing is essentially subcomplete.

Proof. Let $\theta = 2^{\varepsilon+\kappa+\omega}$, and let $\sigma : \bar{N} \prec N = L_\tau^A \ni H_\theta$ be as usual, $\bar{G} \subseteq \bar{\mathbb{P}}$ generic over $\bar{\mathbb{P}}$, let s be a set in $\mathrm{ran}(\sigma)$, and let $\kappa, \eta \in \mathrm{ran}(\sigma)$. Let $\bar{\mathbb{P}}, \bar{\kappa}, \bar{\eta} = \sigma^{-1}(\mathbb{P}, \kappa, \eta)$.

Let $k_0 : N_0 \longrightarrow C := C_\varepsilon^N(\mathrm{ran}(\sigma))$ be an isomorphism, N_0 transitive, $\sigma_0 : \bar{N} \prec N_0$ defined by $\sigma_0 = k_0^{-1} \circ \sigma$. Let $\mathbb{P}_0 = k_0^{-1}(\mathbb{P})$, $\kappa_0 = k_0^{-1}(\kappa)$, etc. Let

$$\sigma_1 : \bar{N} \longrightarrow_{\sigma_0 \restriction (H_{\bar{\kappa}})^{\bar{N}}} N_1$$

be the liftup, and let $k_1 : N_1 \prec N_0$ be such that $k_1 \circ \sigma_1 = \sigma_0$. So, we have that N_1 is transitive, $\sigma_1 \restriction (H_{\bar{\kappa}})^{\bar{N}} = \sigma_0 \restriction (H_{\bar{\kappa}})^{\bar{N}}$, and σ_1 is $\bar{\kappa}$-cofinal (i.e., for every $b \in N_1$, there is an $a \in \bar{N}$ such that $\mathrm{card}(a)^{\bar{N}} < \bar{\kappa}$ and $b \in \sigma_1(a)$). See [Jen14], [Fuc16] for basic facts and notations around liftups.

Let $\mathbb{P}_1 = \sigma_1(\bar{\mathbb{P}})$, $\kappa_1 = \sigma_1(\bar{\kappa})$, etc. Let

$$G_1 = \{q \in \mathbb{P}_1 \mid \exists p \in \bar{G} \quad \sigma_1(p) \leq_{\mathbb{P}_1} q\}$$

Claim: G_1 is $\mathbb{P}_1$-generic over N_1.

Proof of claim. It's obvious that G_1 is a filter in $\mathbb{P}_1$. To see genericity, let $D \subseteq \mathbb{P}_1$ be dense and open. Since σ_1 is $\bar{\kappa}$-cofinal, there is a set $\mathcal{D} \in \bar{N}$ of dense and open subsets of $\bar{\mathbb{P}}$ such that $\mathrm{card}(\mathcal{D})^{\bar{N}} < \bar{\kappa}$ and $D \in \sigma_1(\mathcal{D})$. Since $\bar{\mathbb{P}}$ is $\bar{\kappa}$-distributive in $\bar{N}$, $D^* = \bigcap \mathcal{D}$ is a dense and open subset of $\bar{\mathbb{P}}$. Since $\bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$, there is $\bar{p} \in \bar{G} \cap D^*$. Then $\sigma_1(\bar{p}) \in \sigma_1(D^*) = \bigcap \sigma_1(\mathcal{D}) \subseteq D$, because $D \in \sigma_1(\mathcal{D})$. $\square_{\mathrm{Claim}}$

Note that $\sigma \restriction \bar{\eta} \in N_1$: by elementarity, since $\eta^\omega < \kappa$, it follows that $(\bar{\eta}^\omega)^{\bar{N}} < \bar{\kappa}$, so $\{({}^\omega\bar{\eta})^{\bar{N}}\} \in H_{\bar{\kappa}}^{\bar{N}}$, and hence, $\sigma_1(({}^\omega\bar{\eta})^{\bar{N}}) \in \tilde{H} = \bigcup \sigma_1``(H_{\bar{\kappa}}^{\bar{N}})$. Since $k_1 \restriction \tilde{H} = \mathrm{id}$ (see [Fuc16, Lemma 2.10]), it follows that $\sigma_1(({}^\omega\bar{\eta})^{\bar{N}}) = \sigma_0(({}^\omega\bar{\eta})^{\bar{N}})$. But since k_0 is the inverse of the Mostowski collapse of $C_\varepsilon^N(\mathrm{ran}(\sigma))$ and $\varepsilon \geq \kappa$, it follows that $\sigma_0(({}^\omega\bar{\eta})^{\bar{N}}) = \sigma(({}^\omega\bar{\eta})^{\bar{N}})$. And since $H_\theta \subseteq N$, and $\theta > \kappa$, it follows that $\sigma(({}^\omega\bar{\eta})^{\bar{N}}) = {}^\omega\eta$. Thus, ${}^\omega\eta \in N_1$, so $\sigma``\bar{\eta} \in N_1$, and hence, $\sigma \restriction \bar{\eta} \in N_1$.

As a result, we can use $\sigma \restriction \bar{\eta}$ as a constant in the infinitary theory $\mathcal{L}_1$ over $L_{\alpha(N_1)}(N_1)$ defined as follows.[3] It contains constants $\underline{x}$ for every $x \in N_1$ and extra constant symbols $\dot{\sigma}$, $\dot{G}$, and consists of the ZFC^- axioms, plus the sentences expressing:

1. $\dot{\sigma} : \underline{\bar{N}} \prec \underline{N_1}$ $\bar{\kappa}$-cofinally,
2. $\dot{\sigma}``\underline{\bar{G}} \subseteq \dot{G}$,
3. $\dot{\sigma} \restriction \underline{\bar{\eta}} = \underline{\sigma \restriction \bar{\eta}}$,
4. $\dot{G}$ is $\underline{\mathbb{P}_1}$-generic over $\underline{N_1}$.

Clearly, $\mathcal{L}_1$ is consistent, as witnessed by σ_1 and G_1.

Since $k_1 : N_1 \prec N_0$ is cofinal, the infinitary theory $\mathcal{L}_0$ over $L_{\alpha(N_0)}(N_0)$ which is defined like $\mathcal{L}_1$, with the parameters moved by k_1, and with N_0 in place of N_1, is also consistent, since $\mathcal{L}_1$ is. Note that k_1 doesn't move $\bar{N}$ or elements of it, and $k_1(\sigma \restriction \bar{\eta}) = \sigma \restriction \bar{\eta}$, since $\sigma``\bar{\eta} < \sup \sigma_1``\bar{\kappa} = \mathrm{crit}(k_1)$.

Let F be $\mathrm{Col}(\omega, \theta')$-generic, for some sufficiently large θ', so that the theory $\mathcal{L}_0$ is countable in $\mathrm{V}[F]$. Then there is a solid model $\mathcal{A}$ for $\mathcal{L}_0$ in $\mathrm{V}[F]$. In $\mathrm{V}[F]$, let $\tilde{\sigma}_0 = \dot{\sigma}^{\mathcal{A}}$ and $G_0 = \dot{G}^{\mathcal{A}}$ witness this, so that

[3] I will use Barwise theory, as outlined in [Fuc16], where the notation $\alpha(N_1)$ for the least ordinal γ such that $L_\gamma(N)$ is a model of KP_ω is used. A more detailed account of this theory can be found in [Jen14].

1. $\tilde{\sigma}_0 : \bar{N} \prec N_0$ $\bar{\kappa}$-cofinally
2. $\tilde{\sigma}_0``\bar{G} \subseteq G_0$
3. $\tilde{\sigma}_0 \upharpoonright \bar{\eta} = \sigma \upharpoonright \bar{\eta}$
4. G_0 is $\mathbb{P}_0$-generic over N_0.

Let $\tilde{\sigma} = k_0 \circ \tilde{\sigma}_0$, and let $G = \{p \in \mathbb{P} \mid \exists q \in G_0 \quad k_0(q) \leq_{\mathbb{P}} p\}$. Then:

1. $\tilde{\sigma} : \bar{N} \prec N$
2. $\tilde{\sigma}(\bar{\mathbb{P}}) = \mathbb{P}$, etc.
3. $C_\varepsilon^N(\text{ran}(\tilde{\sigma})) = C_\varepsilon^N(\text{ran}(\sigma))$
4. $\tilde{\sigma} \upharpoonright \bar{\eta} = \sigma \upharpoonright \bar{\eta}$
5. $\tilde{\sigma}``\bar{G} \subseteq G$
6. G is $\mathbb{P}$-generic over V.

The first two points here are obvious. Point 4. is because $k_0 \upharpoonright \kappa = \text{id}$. Point 3 follows as in the proof of [Jen14, p. 131 (9)(d)]. First, note that $N_0 = C_\varepsilon^{N_0}(\text{ran}(\tilde{\sigma}_0))$, because $\tilde{\sigma}_0$ is $\bar{\kappa}$-cofinal and $\varepsilon \geq \kappa = \tilde{\sigma}_0(\bar{\kappa})$. It then follows that

$$C_\varepsilon^N(\text{ran}(\sigma)) = k_0``N_0 = C_\varepsilon^N(\text{ran}(k_0 \circ \tilde{\sigma}_0)))$$

Point 5. is again obvious, and point 6. follows because ε captures $\mathbb{P}$-genericity.

Finally, a standard argument shows that there is an embedding σ' in V$[G]$ satisfying the points 1.-6. above. Working in V$[G]$, let μ be regular with $N \in H_\mu$. Let $M = \langle H_\mu, \in, N, G, \theta, \mathbb{P}, s, \sigma, \varepsilon \rangle$. Let $\mathcal{L}_2$ be the infinitary theory on M with constants $\dot{\sigma}$ and $\underline{x}$ for $x \in M$ and predicate symbol $\in$, consisting of ZFC^- and the basic axioms, together with the axioms expressing that $\dot{\sigma} : \bar{N} \prec N$, $\dot{\sigma}(\bar{\mathbb{P}}) = \mathbb{P}$ etc., $C_{\underline{\varepsilon}}^{\underline{N}}(\text{ran}(\dot{\sigma})) = C_{\underline{\varepsilon}}^{\underline{N}}(\text{ran}(\underline{\sigma}))$, $\dot{\sigma} \upharpoonright \underline{\bar{\eta}} = \underline{\sigma \upharpoonright \bar{\eta}}$ and $\dot{\sigma}``\underline{\bar{G}} \subseteq \underline{G}$. By the points 1.-6. above, $\mathcal{L}_2$ is consistent, as witnessed by $\langle M, \tilde{\sigma} \rangle$ in V$[G]$.

Now let $\pi : \tilde{M} \prec M$ where $\tilde{M}$ is countable and transitive. Let $\tilde{\mathcal{L}}_2$ be the infinitary theory on $\tilde{M}$ with the same definition as $\mathcal{L}_2$, and the parameters moved by π^{-1}. Since $\tilde{M}$ is countable, it has a solid model $\tilde{\mathcal{A}}$ (even in V - note that $\tilde{M} \in$ V, as $\mathbb{P}$ is κ-distributive). Let $\sigma' = \pi \circ \dot{\sigma}^{\tilde{\mathcal{A}}}$. It follows that σ' has all the properties listed in 1-6. The verifications can be done as in the end of the proof of Theorem 2.7. □

Lemma 6.3. *If $\kappa > \omega_1$ is a cardinal and T is a κ-Souslin tree, then T, viewed as a notion of forcing (with the order reversed), is subcomplete. Moreover, if $\eta^\omega < \kappa$, then T is subcomplete above η.*

Proof. T is κ-distributive and κ-c.c., and as a result, it is well-known that any cofinal branch through T is T-generic. It is then clear that κ captures T-genericity, in the sense of Definition 6.1. So by Theorem 6.2, T is κ-subcomplete, which, since $\delta(T) = \kappa$, means that T is subcomplete. □

Thus, for example, if $2^\omega = \omega_1$ and T is an ω_2-Souslin tree, then T is subcomplete above every $\eta < \omega_2$, and in particular, ω_2-Souslin trees are not preserved by such subcomplete forcings, even if they are cardinal preserving. This is in contrast to the situation at ω_1, where every subcomplete forcing preserves ω_1-Souslin trees. Note also that it is consistent that $2^\omega \geq \omega_2$ and that there is an ω_2-Souslin tree T. It's easy to see that in this situation, forcing with T is not equivalent to a countably closed forcing, in fact, no countably closed forcing can add a cofinal branch to T. But by the previous result, it is nevertheless subcomplete.

References

[Cum10] James Cummings. Iterated forcing and elementary embeddings. In M. Foreman and M. Magidor, editors, *Handbook of Set Theory*, volume 2, pages 775–883. Springer, 2010.

[Fuc16] Gunter Fuchs. The subcompleteness of Magidor Forcing. *Submitted to the Archive for Mathematical Logic*, 2016. Preprint available at http://www.math.csi.cuny.edu/∼fuchs/.

[Jen09] Ronald B. Jensen. Subproper and subcomplete forcing. *Handwritten notes, available at http://www.mathematik.hu-berlin.de/∼raesch/org/jensen.html.*, 2009. Handwritten notes, available at http://www.mathematik.hu-berlin.de/∼raesch/org/jensen.html.

[Jen14] Ronald B. Jensen. Subcomplete forcing and $\mathcal{L}$-forcing. In Chitat Chong, Qi Feng, Theodore A. Slaman, W. Hugh Woodin, and Yue Yang, editors, *E-recursion, forcing and C^*-algebras*, volume 27 of *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*, pages 83–182. World Scientific, 2014.

[Kun78] Kenneth Kunen. Saturated ideals. *Journal of Symbolic Logic*, 43(1):65–76, 1978.

[Min17] Kaethe Minden. *On subcomplete forcing.* PhD thesis, The CUNY Graduate Center, 2017.