DIAGONAL REFLECTIONS ON SQUARES

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Abstract. The effects of (bounded versions of) the forcing axioms SCFA, PFA and MM on the failure of weak threaded square principles of the form $\square(\lambda,\kappa)$ are analyzed. To this end, a diagonal reflection principle, $\mathsf{DSR}({<}\kappa, S)$ is introduced. It is shown that SCFA implies $\mathsf{DSR}(\omega_1, S^\lambda_\omega)$, for all regular $\lambda \geq \omega_2$, and that $\mathsf{DSR}(\omega_1, S^\lambda_\omega)$ implies the failure of $\square(\lambda,\omega_1)$ if $\lambda > \omega_2$, and it implies the failure of $\square(\lambda,\omega)$ if $\lambda = \omega_2$. It is also shown that this result is sharp. It is noted that MM/PFA imply the failure of $\square(\lambda,\omega_1)$, for every regular $\lambda > \omega_1$, and that this result is sharp as well.

1. Introduction

This paper is concerned with the effects of (bounded) forcing axioms on the failure of weak threaded square principles, the focus being on the bounded forcing axioms for the class of subcomplete forcings.

Definition 1.1. Let Γ be a class of forcings, and let λ be a cardinal. Then the *λ-bounded forcing axiom for Γ,* denoted $\mathsf{BFA}(\Gamma, {\leq}\lambda)$, is the statement that if $\mathbb{P}$ is a forcing in Γ, $\mathbb{B}$ is its complete Boolean algebra, and $\mathcal{A}$ is a collection of at most ω_1 many maximal antichains in $\mathbb{B}$, each of which has size at most λ, then there is a $\mathcal{A}$-generic filter in $\mathbb{B}$, that is, a filter that intersects each antichain in $\mathcal{A}$. If Γ is the class of proper, stationary set preserving or subcomplete forcings, then I write $\mathsf{BPFA}({\leq}\lambda)$, $\mathsf{BMM}({\leq}\lambda)$, $\mathsf{BSCFA}({\leq}\lambda)$ (respectively) for $\mathsf{BFA}(\Gamma, {\leq}\lambda)$. BPFA, BMM, BSCFA are short for $\mathsf{BPFA}({\leq}\omega_1)$, $\mathsf{BMM}({\leq}\omega_1)$, $\mathsf{BSCFA}({\leq}\omega_1)$. If the cardinality restriction on the antichains in $\mathcal{A}$ is dropped, then the resulting unbounded principles are called PFA, MM and SCFA.

In [4], I analyzed this hierarchy of forcing axioms, as well as its "weak" variant, for the class of subcomplete forcings, in terms of consistency strength. The motivation for analyzing the forcing axioms for this class is that they are at the same time very different from the more well-known forcing axioms PFA and MM, in that they don't imply the failure of CH, and also very similar, in that they have many of the more striking consequences in terms of the failure of square principles, for example. Subcomplete forcings are stationary set preserving, but do not add reals. Every countably closed forcing is subcomplete, but no nontrivial ccc. forcing is (see [16]). Subcomplete forcing may change the cofinality of an uncountable cardinal to ω, and thus may be nonproper, for example, under CH, Namba forcing is subcomplete. So the class of subcomplete forcings is a very different part of the class of stationary set preserving forcings than the class of proper forcings is. But subcomplete forcing is iterable with revised countable support. The concept was introduced by Jensen in [9], see also [10] for a great overview article. No knowledge of subcomplete forcing

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is required to follow the material in the present paper. However, it relies on some material in [4] which does.

Jensen showed that the unbounded forcing axiom SCFA can be forced from a supercompact cardinal, see [8], where it was also shown that SCFA implies the failure of Jensen's square principle $\square_\kappa$, for every uncountable cardinal κ. He also showed that if one carries out the version of the Baumgartner iteration for subcomplete forcing, the resulting model will satisfy SCFA, CH (a consequence of the fact that subcomplete forcing does not add reals) and even Jensen's $\diamondsuit$ principle.

In [4], inspired by [3], where this was done in the context of MM, a more detailed analysis of the effects of SCFA on the failure of *weak* square principles of the form $\square_{\kappa,\lambda}$ was carried out.

Definition 1.2. Let κ be a cardinal, and let $\lambda \le \kappa$. A $\square_{\kappa,\lambda}$-sequence is a sequence $\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each $\mathcal{C}_\alpha$ has size at most λ, and each $C \in \mathcal{C}_\alpha$ is club in α, has order-type at most κ, and satisfies the coherency condition that if β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$. Again, $\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence. $\square_{\kappa,\kappa}$ is known as weak square, denoted by $\square^*_\kappa$. $\square_{\kappa,<\lambda}$ is defined like $\square_{\kappa,\lambda}$, except that each $\mathcal{C}_\alpha$ is required to have size less than λ.

These weak square principles were introduced by Schimmerling. Similar weakenings of the threaded square principles of the form $\square(\kappa)$ were treated in [5], [21], [12] and [6]. I call these the *weak threaded square principles.*

Definition 1.3. Let λ be a limit of limit ordinals. A sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ is *coherent* if for every limit $\alpha < \lambda$, $\mathcal{C}_\alpha \neq \emptyset$ and for every $C \in \mathcal{C}_\alpha$, C is club in α, and for every limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$. A *thread* through $\vec{\mathcal{C}}$ is a club subset T of λ that coheres with $\vec{\mathcal{C}}$, that is, for every limit point β of T with $\beta < \kappa$, it follows that $T \cap \beta \in \mathcal{C}_\beta$. If every $\mathcal{C}_\alpha$ has size less than κ, then $\vec{\mathcal{C}}$ is said to have *width* $<\kappa$. The *length* of $\vec{\mathcal{C}}$ is λ.

If κ is a cardinal, $\vec{\mathcal{C}}$ has width $<\kappa$, and $\vec{\mathcal{C}}$ does not have a thread, then $\vec{\mathcal{C}}$ is called a $\square(\lambda, <\kappa)$ sequence. The principle $\square(\lambda, <\kappa)$ says that there is a $\square(\lambda, <\kappa)$ sequence.

In place of $\square(\lambda, <\kappa^+)$, I may write $\square(\lambda, \kappa)$.

$\square(\lambda, 1)$ is known as $\square(\lambda)$, and $\square(\lambda, <\kappa)$ becomes easier to satisfy as κ increases. It's also clear that every $\square_{\lambda,<\kappa}$-sequence is a $\square(\lambda^+, <\kappa)$-sequence.

It was pointed out in [4] that $\mathsf{BSCFA}(\le\lambda)$ implies the failure of $\square(\lambda)$, for regular $\lambda > \omega_1$, and it follows from work in [6] that this can be improved to get the failure of $\square(\lambda, <\omega)$. The question I want to investigate in the present paper is how strong a failure of these square principles can be derived from SCFA, or, more locally, from $\mathsf{BSCFA}(\le\lambda)$. It will turn out that the effects of SCFA on the failure of the weak threaded square principles are almost exactly the same as those of PFA and MM, except for the status of $\square(\omega_2, \omega_1)$, the failure of which follows from PFA/MM but not from SCFA.

Most of the known results on the failure of weak square principles are consequences of principles of simultaneous stationary reflection which, in turn, follow from the particular forcing axiom under consideration. I here introduce a new form of simultaneous stationary reflection which I call *diagonal stationary reflection*, appropriate versions of which follow from SCFA and imply the desired failures of weak threaded square principles.

The paper is organized as follows. Section 2 contains some basic information on weak threaded square principles. I show that the weak threaded square principles form a hierarchy, that is, that the implications stated after Definition 1.3 cannot be reversed in general. The proof here is a straightforward adaptation of an argument of Magidor. I also say something about $\square(\lambda,{<}\mu)$ when λ is singular and $\mu<\kappa=\mathrm{cf}(\lambda)$, namely that $\square(\lambda,{<}\mu)$ implies $\square(\kappa,{<}\mu)$. The special case where $\mu=2$ is due to Abraham and Schimmerling ([18]). From that point on, I focus on the case that λ is regular. In Section 3, I introduce the diagonal reflection principles and show that they imply the failure of weak threaded square principles. Then, in Section 4, I show that bounded subcomplete forcing axioms imply diagonal reflection, and I conclude that $\mathsf{BSCFA}({\leq}\lambda)$ implies the failure of $\square(\lambda,\omega_1)$ if $\lambda>\omega_2$ is regular, and the failure of $\square(\lambda,\omega)$ if $\lambda=\omega_2$. Finally, in Section 5, I show that this result is sharp, and I provide a sharp result on the effects of PFA and MM on the failure of weak threaded square as well (but this could have been done without having diagonal reflection at one's disposal, by using Todorčević's original argument). The main result of the paper is the following theorem, proved in that section as Theorem 5.6.

Theorem. *Assume* SCFA*. Let* λ *be a limit ordinal.*

(1) *If* $\mathrm{cf}(\lambda)=\omega_2$*, then* $\square(\lambda,\omega)$ *fails.*
(2) *If* $\mathrm{cf}(\lambda)\geq\omega_3$*, then* $\square(\lambda,\omega_1)$ *fails.*

Moreover, these results are sharp, in the sense that if the existence of a supercompact cardinal is consistent, then it is consistent that SCFA *holds, and for every limit ordinal* λ*:*

(3) *If* $\mathrm{cf}(\lambda)=\omega_2$*, then* $\square(\lambda,\omega_1)$ *holds.*
(4) *If* $\mathrm{cf}(\lambda)\geq\omega_3$*, then* $\square(\lambda,\omega_2)$ *holds.*

A similar statement is made about the effects of MM and PFA in Theorem 5.7. I also show in Theorem 5.9 that strong diagonal reflection principles at $\aleph_{\omega+1}$ do not imply the failure of $\square^*_{\aleph_\omega}$, as one might have hoped, and I end with a couple of open questions.

2. Separation and basic properties of weak threaded squares

Before proving the abovementioned results on the failure of $\square(\lambda,\kappa)$ from SCFA, I would like to make some observations. First, these principles actually form a hierarchy, that is, increasing κ makes them strictly weaker. Arguments, originally due to Jensen, used to separate weak square principles, can be used to separate the principles under consideration here as well, at least in the case that λ is a successor of a regular cardinal.

Jensen proved in [7] the version of the following theorem where the assumption is weakened to λ being Mahlo, $\kappa=2$, and the conclusion is weakened to saying that $\square_{\rho,1}$ fails, showing that $\square_{\rho,2}$ does not imply $\square_{\rho,1}$. Since the consistency strength of the failure of $\square_\rho$, for a regular cardinal ρ, is a Mahlo cardinal, the assumption in Jensen's theorem is optimal. In [15, Thm. 4.7], Magidor proves the version of the theorem where λ is assumed to be measurable, $\kappa=2$, and the conclusion is that $\square_\rho$ fails. So at first sight, Magidor's version of the theorem makes a stronger assumption and gives the same conclusion. But the proof generalizes to any cardinal κ with $2\leq\kappa<\rho$ (as is pointed out after the proof in [15, Thm. 4.7]), and close inspection of the proof reveals that it actually shows not only that $\square_\rho$ fails, but that even $\square(\lambda)$ fails. Moreover, the assumption that λ is measurable can be reduced to λ just being weakly compact. The consistency strength of the failure of $\square(\lambda)$ at a regular λ is a weakly compact cardinal, by Veličković [20, Thm. 5] and Jensen (unpublished, but see [20, Thm. 3]). Hence, the assumption of the following theorem is optimal. The proof is basically the one given in [15, Thm. 4.7], due to Magidor.

Theorem 2.1. *Let $\omega_1 \leq \rho < \lambda$ be regular cardinals, where λ is weakly compact, and let $1 < \kappa < \rho$ be a cardinal. Then there is a forcing extension in which $\rho^+ = \lambda$, all cardinals up to ρ are preserved, $\square_{\rho,\kappa}$ holds and $\square(\lambda, {<}\kappa)$ fails.*

Note: In particular, in the forcing extension, $\square(\lambda, \kappa)$ holds but $\square(\lambda, {<}\kappa)$ fails.

Proof. I follow [15, Proof of Thm. 4.7]. The key point is the observation that the proof of [15, Lemma 4.5] actually shows the following slightly stronger statement.

Lemma 2.2 ([5, Lemma 3.18])**.** *Let ρ be a regular uncountable cardinal. Then a ${<}\rho$-closed forcing cannot add a new thread (i.e., one that doesn't exist in* V*) to a coherent sequence of length ρ^+ and width ${<}\rho$.*

Let's now prove Theorem 2.1. Let $\mathbb{P} = \mathrm{Col}(\rho, {<}\lambda)$, and let G be generic for $\mathbb{P}$. In $\mathrm{V}[G]$, let $\mathbb{S} = \mathbb{S}_{\rho,\kappa}$ be the canonical forcing to add a $\square_{\rho,\kappa}$-sequence with initial segments whose length is a successor ordinal. Let H be $\mathbb{S}$-generic over $\mathrm{V}[G]$. $\mathrm{V}[G][H]$ is the forcing extension of V that will have all the desired properties.

Let $\vec{\mathcal{C}}$ be the generic sequence $\square_{\rho,\kappa}$-sequence added by H. To see that $\mathrm{V}[G][H]$, we have to show that there is no $\square(\lambda, {<}\kappa)$-sequence in $\mathrm{V}[G][H]$. To see this, assume the contrary. Let $\vec{\mathcal{D}} = \dot{\vec{\mathcal{D}}}^{G*H}$ be a $\square(\lambda, {<}\kappa)$-sequence in $\mathrm{V}[G][H]$, where $\dot{X} \in H_{\lambda^+}$.

In $\mathrm{V}[G][H]$, let $\mathbb{T} = \mathbb{T}_\rho(\mathcal{C})$ be the canonical forcing to add a thread to $\vec{\mathcal{C}}$, consisting of closed bounded subsets of λ of order type less than ρ that are potential initial segments of a thread, ordered by end-extension. Let $\dot{\mathbb{T}} \in \mathrm{V}[G]$ be the canonical $\mathbb{S}$-name for $\mathbb{T}$. By [15, Lemma 4.2], in $\mathrm{V}[G]$, $\mathbb{S} * \dot{\mathbb{T}}$ has a ${<}\rho$-closed dense subset. Let I be generic for $\mathbb{T}$ over $\mathrm{V}[G][H]$.

I will use the following well-known characterization of the weak compactness of λ: For any transitive model $N \ni \lambda$ of size λ, there is a transitive model M of size λ and an elementary embedding $j : M \prec N$ with critical point λ. In V, using a standard construction, let $\dot{\vec{\mathcal{D}}} \in N \prec H_{\lambda^+}$ be such a transitive model of size λ, with ${}^{<\lambda}N \subseteq N$, and let $j : N \prec M$, where M is transitive. Again, using a standard argument, we may assume that ${}^{<\lambda}M \subseteq M$.

$\mathbb{P}$, $\dot{\mathbb{S}}$ and $\dot{\mathbb{T}}$ can be coded as subsets of λ, definable in H_{λ^+}, and hence are in N, and are defined by the same formula there. Since $\mathcal{P}(\lambda)^N \subseteq \mathcal{P}(\lambda)^M$, they are also available in M. In V, $\mathbb{P} * \dot{\mathbb{S}} * \dot{\mathbb{T}}$ is equivalent to a ${<}\rho$-closed forcing, and it follows that the same is true in M. By [15, Lemma 4.3], applied inside M, it follows that if we let $\theta = j(\lambda)$, then $\mathbb{P} * \dot{\mathbb{S}} * \dot{\mathbb{T}}$ can be absorbed by $\mathrm{Col}(\rho, {<}\theta)$, and the quotient forcing $\mathrm{Col}(\rho, {<}\theta)/\mathbb{P} * \dot{\mathbb{S}} * \dot{\mathbb{T}}$ is equivalent to a ${<}\rho$-closed forcing in $M^{\mathbb{P}*\dot{\mathbb{S}}*\dot{\mathbb{T}}}$. So we can let J be generic for the quotient forcing over $\mathrm{V}[G][H][I]$, and it follows that j lifts to an elementary embedding

$$j : N[G] \prec M[G][H][I][J]$$

Letting T be the thread added by I, $S = \vec{\mathcal{C}} \cup \{\langle \lambda, \{T\} \rangle\}$ is a condition in $j(\mathbb{S})$. Let K be V-generic for $j(\mathbb{S})$ with $S \in K$. Then j lifts to

$$j : N[G][H] \prec M[G][H][I][J][K]$$

Since $\dot{\vec{\mathcal{D}}} \in N$, it follows that $\vec{D} = \dot{\vec{\mathcal{D}}}^{G*H} \in N[G][H]$. We want to derive a contradiction by showing that there is a thread for $\vec{\mathcal{D}}$ in $\mathrm{V}[G][H]$. To this end, let $F \in j(\vec{D})_\lambda$. Then F is a thread for $\vec{D}$, and $F \in M[G][H][I][J][K]$. But $j(\mathbb{S})$ is $<j(\lambda)$-distributive in $M[G][H][I][J]$, so $F \in M[G][H][I][J]$. The quotient $j(\mathbb{P})/G * H * I$ is $<\rho$-closed in $M[G][H][I]$, so by Lemma 2.2, it follows that $F \in M[G][H][I]$. The key claim is now that $F \in \mathrm{V}[G][H]$. To see this, I follow the proof of Claim 4.9 in [15, proof of Thm. 4.7].

Assume that $F \notin \mathrm{V}[G][H]$. Fix a name $\dot{F}$ for F, a name $\dot{\vec{\mathcal{D}}}$ for $\vec{\mathcal{D}}$ and a condition $\langle s, t\rangle \in \mathbb{S} * \dot{\mathbb{T}}$ that forces (over $\mathrm{V}[G]$, with respect to $\mathbb{S} * \dot{\mathbb{T}}$) that $\dot{F}$ is a thread for $\dot{\vec{\mathcal{D}}}$, and that $\dot{F} \notin \mathrm{V}[G][\Gamma_{\mathbb{S}}]$, where $\Gamma_{\mathbb{S}}$ is the canonical $\mathbb{S} * \dot{\mathbb{T}}$-name for the $\mathbb{S}$-generic filter.

In a first step, working in $\mathrm{V}[G]$, find a condition $s' \leq s$ in $\mathbb{S}$ and construct a sequence $\langle \delta_i \mid i < \kappa\rangle$ of ordinals and a sequence $\langle t_i \mid i < \kappa\rangle$ such that for every $i < \kappa$, $\langle s', t_i\rangle \leq \langle s, t\rangle$, and such that for every pair $i < j < \kappa$, there is a $k = f(i,j) < \kappa$ such that $\langle s', t_i\rangle$ and $\langle s', t_j\rangle$ decide "$\check{\delta}_k \in \dot{F}$" in different ways. Let's take it for granted, for now, that such sequences can be constructed. Let $\alpha = \sup_{i<\kappa} \delta_i$. Clearly, $\alpha < \lambda$.

Still working in $\mathrm{V}[G]$, let $\Delta \subseteq \mathbb{S} * \dot{\mathbb{T}}$ be the set of $\langle s, \dot{t}\rangle \in \mathbb{S} * \dot{\mathbb{T}}$ such that there is a t such that $s \Vdash_{\mathbb{S}} \dot{t} = \check{t}$ and such that $\max(\mathrm{dom}(s)) = \max(t)$ (in this situation, I'll identify $\langle s, \dot{t}\rangle$ with $\langle s, t\rangle$). Δ is dense in $\mathbb{S} * \dot{\mathbb{T}}$ and the restriction of the ordering of $\mathbb{S} * \dot{\mathbb{T}}$ is $<\rho$-closed, see [15, Lemma 4.2]. Construct $\langle\langle s^i_j, t^i_j, \alpha^i_j\rangle \mid i < \omega, j < \kappa\rangle$ such that each $\langle s^i_j, t^i_j\rangle$ belongs to Δ, and such that the following conditions hold.

(1) $s^0_0 \leq s'$ and for all $i < \omega$ and all $j < j' < \kappa$, $s^{i+1}_0 \leq s^i_{j'} \leq s^i_j$
(2) $\alpha < \alpha^0_0$, and if $\langle i, j\rangle <_{\mathrm{lex}} \langle i', j'\rangle$, then $\alpha^i_j < \alpha^{i'}_{j'}$
(3) for each $j < \kappa$, $\langle s^0_j, t^0_j\rangle \leq \langle s', t_j\rangle$, and for all $i < \omega$, $\langle s^{i+1}_j, t^{i+1}_j\rangle \leq \langle s^i_j, t^i_j\rangle$
(4) for each $i < \omega$ and $j < \kappa$, $\langle s^i_j, t^i_j\rangle \Vdash \dot{\alpha}^i_j \in \dot{F}$.
(5) if $\langle i, j\rangle <_{\mathrm{lex}} \langle i', j'\rangle$, then $\alpha^i_j \in \mathrm{dom}(s^{i'}_{j'})$.

The construction is straightforward, given step 1. In order to satisfy (1), we use a strategy witnessing that $\mathbb{S}$ is $(\rho + 1)$-strategically closed (see [2, Lemma 6.7]). Let $\alpha^* = \sup_{i<\omega} \alpha_{i,0} = \sup_{i<\omega, j<\kappa} \alpha^i_j$. So α^* is an ordinal less than λ of countable cofinality. For every $j < \kappa$, let $t^*_j = \bigcup_{i<\omega} t^i_j \cup \{\alpha^*\}$, and let $s^* = (\bigcup_{i<\omega, j<\kappa} s^i_j) \cup \{\langle \alpha^*, \{t^*_j \cap \alpha^* \mid j < \kappa\}\rangle\}$. It follows that $\langle s^*, t_j\rangle \in \mathbb{S} * \dot{\mathbb{T}}$, for all $j < \kappa$. Find $\tilde{s} \leq s^*$ such that $\tilde{s}$ decides the value of $\dot{\vec{\mathcal{D}}}_{\alpha^*}$, say to be equal to $P \in \mathrm{V}[G]$. So P is a collection of subsets of α^* of size less than κ (it has size less than κ in $\mathrm{V}[G][H]$, but H preserves cardinals less than or equal to ρ, so P has size less than κ in $\mathrm{V}[G]$ as well). For each $j < \kappa$, $\langle \tilde{s}, t^*_j\rangle$ forces that $\dot{F} \cap \alpha^* \in \check{P}$. So we can choose, for every $j < \kappa$, a condition $\langle \tilde{s}'_j, \tilde{t}_j\rangle \leq \langle \tilde{s}, t^*_j\rangle$ and a set $p_j \in P$ such that $\langle \tilde{s}'_j, \tilde{t}_j\rangle \Vdash \dot{F} \cap \check{\alpha}^* = \check{p}_j$. Now, since P has size less than κ, this means that there must be $j_0 < j_1 < \kappa$ such that $p_{j_0} = p_{j_1}$. But if we let $\delta = f(j_0, j_1)$. Then it follows that $\langle \tilde{s}'_{j_0}, \tilde{t}_{j_0}\rangle$ and $\langle \tilde{s}'_{j_1}, \tilde{t}_{j_1}\rangle$ decide the statement "$\check{\delta} \in \dot{F}$" in opposite ways, because these conditions are strengthenings of $\langle s', t_{j_0}\rangle$ and $\langle s', t_{j_1}\rangle$, respectively. This is a contradiction, because $\delta < \alpha < \alpha^*$.

To fill in the details of the construction in step one, recall that we wanted to find a condition $s' \leq s$ in $\mathbb{S}$, a sequence $\langle \delta_i \mid i < \kappa\rangle$ of ordinals and a sequence $\langle t_i \mid i < \kappa\rangle$ such that for every $i < \kappa$, $\langle s', t_i\rangle \leq \langle s, t\rangle$, and such that for every pair $i < j < \kappa$, there is a $k = f(i, j) < \kappa$ such that $\langle s', t_i\rangle$ and $\langle s', t_j\rangle$ decide "$\check{\delta}_k \in \dot{F}$" in different ways. To do this, we first observe that for any condition $\langle \tilde{s}, \tilde{t}\rangle \le \langle s, t\rangle$, there are an ordinal α and conditions $\langle \tilde{s}', t_0\rangle, \langle \tilde{s}', t_1\rangle \le \langle \tilde{s}, \tilde{t}\rangle$ such that $\langle \tilde{s}', t_0\rangle$ and $\langle \tilde{s}', t_1\rangle$ decide "$\check{\alpha} \in \dot{F}$" in opposite ways, since $F \in \mathrm{V}[G][H][I] \setminus \mathrm{V}[G][H]$. For if $\langle \tilde{s}, \tilde{t}\rangle \le \langle s, t\rangle$ were a counterexample, then let H' be $\mathbb{S}$-generic over $\mathrm{V}[G]$ with $\tilde{s} \in H'$, and let I', I'' be mutually $\dot{\mathbb{T}}^{H'}$-generic over $\mathrm{V}[G][H']$, with $\tilde{t}^{H'} \in I' \cap I''$. By assumption then,

$$\dot{F}^{H' * I'} = \dot{F}^{H' * I''} \in \mathrm{V}[G][H'][I'] \cap \mathrm{V}[G][H'][I''] \subseteq \mathrm{V}[G][H']$$

But this contradicts that $\langle s, t\rangle$ forces that $\dot{F} \notin \mathrm{V}[G][\Gamma_{\mathbb{S}}]$.

This fact can be strengthened slightly. In order to do this, let's make a simple observation: if $\langle \tilde{s}, \tilde{t}_0\rangle$, $\langle \tilde{s}, \tilde{t}_1\rangle$ are conditions in $\mathbb{S} * \dot{\mathbb{T}}$, then there are $\tilde{s}', \tilde{t}'_0, \tilde{t}'_1$ such that $\langle \tilde{s}', \tilde{t}'_0\rangle \le \langle \tilde{s}, \tilde{t}_0\rangle$, $\langle \tilde{s}', \tilde{t}'_1\rangle \le \langle \tilde{s}, \tilde{t}_1\rangle$ and $\langle \tilde{s}', \tilde{t}'_0\rangle, \langle \tilde{s}', \tilde{t}'_1\rangle \in \Delta$. For one can first strengthen $\tilde{s}$ to $\tilde{s}'$, so as to decide the values of both $\dot{\tilde{t}}_0$ and $\dot{\tilde{t}}_1$, to be $\tilde{t}_0$ and $\tilde{t}_1$, respectively, say. We may assume that $\mu = \max(\mathrm{dom}(\tilde{s}')) > \max(\tilde{t}_0) \cup \max(\tilde{t}_1)$. We can then let $\tilde{t}'_h = \tilde{t}_h \cup \{\mu\}$.

Using this observation, it follows that for any condition $\langle \tilde{s}, \tilde{t}\rangle \le \langle s, t\rangle$ in $\mathbb{S} * \dot{\mathbb{T}}$, there are an ordinal α and conditions $\langle \tilde{s}', t_0\rangle, \langle \tilde{s}', t_1\rangle \in \Delta$, $\langle \tilde{s}', t_0\rangle, \langle \tilde{s}', t_1\rangle \le \langle \tilde{s}, \tilde{t}\rangle$ such that $\langle \tilde{s}', t_0\rangle$ and $\langle \tilde{s}', t_1\rangle$ decide "$\check{\alpha} \in \dot{F}$" in opposite ways.

By recursion on α, construct a sequence $\langle\langle s_\alpha, t^0_\alpha, t^1_\alpha, \delta_\alpha\rangle \mid \alpha < \kappa\rangle$ with the following properties:

(1) $\langle s_\alpha, t^0_\alpha\rangle$ and $\langle s_\alpha, t^1_\alpha\rangle$ are conditions in Δ that decide "$\check{\delta}_\alpha \in \dot{F}$" in different ways
(2) for all $\alpha < \beta < \kappa$, $\langle s_\alpha, t^0_\alpha\rangle \le \langle s_\beta, t^0_\beta\rangle, \langle s_\beta, t^1_\beta\rangle$

For the construction we use the previous observation, together with the fact that Δ is $<\rho$-closed. Since $\kappa < \rho$, we can find a condition $\langle s', t'\rangle \in \Delta$ such that for all $\alpha < \kappa$, $\langle s', t'\rangle \le \langle s_\alpha, t^0_\alpha\rangle$. We can choose s' so that $\mu = \max(\mathrm{dom}(s)) > \max(t^h_\alpha)$, for all $\alpha < \kappa$ and $h < 2$. Letting $t_\alpha = t^1_\alpha \cup \{\mu\}$, for all $\alpha < \kappa$, it follows that

(1) $\langle s', t_\alpha\rangle \in \Delta$
(2) $\langle s', t_\alpha\rangle \le \langle s, t\rangle$
(3) for all $\alpha < \kappa$, $\langle s', t_\alpha\rangle \le \langle s_\alpha, t^1_\alpha\rangle$
(4) for all $\alpha < \beta < \kappa$, $\langle s', t_\beta\rangle \le \langle s_\alpha, t^0_\alpha\rangle$

To see the latter, note that by 3., $\langle s', t_\beta\rangle \le \langle s_\beta, t^1_\beta\rangle \le \langle s_\alpha, t^0_\alpha\rangle$. It follows that the conditions $\langle\langle s', t_\alpha\rangle \mid \alpha < \kappa\rangle$ are as wished, for if $\alpha < \beta < \kappa$, then $\langle s_\alpha, t^0_\alpha\rangle$ and $\langle s_\alpha, t^1_\alpha\rangle$ decide $\Phi(\delta_\alpha) =$"$\check{\delta}_\alpha \in \dot{F}$" in different ways, but $\langle s', t_\beta\rangle \le \langle s_\alpha, t^0_\alpha\rangle$, so $\langle s', t_\beta\rangle$ and $\langle s_\alpha, t^0_\alpha\rangle$ decide $\Phi(\delta_\alpha)$ in the same way, and $\langle s', t_\alpha\rangle \le \langle s_\alpha, t^1_\alpha\rangle$, so $\langle s', t_\alpha\rangle$ and $\langle s_\alpha, t^1_\alpha\rangle$ decide $\Phi(\delta_\alpha)$ in the same way. So $\langle s', t_\alpha\rangle$ and $\langle s', t_\beta\rangle$ decide "$\check{\delta}_\alpha \in \dot{F}$" in different ways, as wished. □

I'll make a couple of observations concerning $\square(\lambda, <\kappa)$ in the case that λ is singular, in the remainder of this section. Of course, if $\mathrm{cf}(\lambda) = \omega$, then $\square(\lambda, <\kappa)$ fails, for every κ, and if $\mathrm{cf}(\lambda) = \omega_1$, then $\square(\lambda)$ holds.

A key feature of the original threaded square principles that's instrumental in deriving large cardinal strength from the simultaneous failure of $\square(\kappa)$ and $\square_\kappa$ in [18] is that if λ is a singular ordinal and $\kappa = \mathrm{cf}(\lambda)$, then $\square(\lambda)$ implies $\square(\kappa)$. A version of this remains true for the weak threaded square principles. In the proof, and at some other places as well, I'll use the following fact, due to Kurepa.

Fact 2.3 ([11])**.** *Let λ be a regular cardinal, and let $\kappa < \lambda$ be a cardinal. Then every tree of height λ all of whose levels have size less than κ has a cofinal branch.*

Observation 2.4. *Assume $\mu < \kappa = \mathrm{cf}(\lambda)$, where μ is a cardinal. Then*

$$\Box(\lambda, {<}\mu) \implies \Box(\kappa, {<}\mu)$$

Proof. The proof of [18, Lemma 2.1] goes through, with minor modifications. Let $e : \kappa \longrightarrow \lambda$ be a normal cofinal function, let $P = \mathrm{ran}(e)$, and let $\vec{\mathcal{C}}$ be a $\Box(\lambda, {<}\mu)$-sequence. Assume that $\kappa \geq \omega_1$ and $\mu > 0$, since otherwise, $\Box(\lambda, {<}\mu)$ fails, and so, the claim is vacuously true. Define a "projection" function $p : \lambda \longrightarrow \lambda$ by setting

$$p(\alpha) = \sup(P \cap \alpha)$$

So if $\alpha \in P'$, then $p(\alpha) = \alpha$, and otherwise, $p(\alpha) = \max(P \cap \alpha)$. In particular, $p : \lambda \longrightarrow P$. Define a sequence $\vec{\mathcal{B}} = \langle \mathcal{B}_\alpha \mid \alpha \in P' \rangle$ by setting

$$\mathcal{B}_\alpha = \{p``C \mid C \in \mathcal{C}_\alpha\}$$

for $\alpha \in P'$. Then, each $B \in \mathcal{B}_\beta$ is club in β, and it's easy to see that $\vec{\mathcal{B}}$ coheres, in the sense that if $\alpha \in B'$, where $B \in \mathcal{B}_\beta$, then $B \cap \alpha \in \mathcal{B}_\alpha$. Now, let us define, for limit $i < \kappa$,

$$\mathcal{A}_i = \{e^{-1}``B \mid B \in \mathcal{B}_{e(i)}\}$$

It follows that $\vec{\mathcal{A}}$ is a $\Box(\kappa, {<}\mu)$-sequence. Each $\mathcal{A}_i$ consists of clubs, for limit i, since then, $e(i)$ is a limit point of D. The coherency of this sequence follows from the coherency of $\vec{\mathcal{B}}$ just described. Obviously, each $\mathcal{A}_i$ has size less than μ. To see that $\vec{\mathcal{A}}$ does not have a thread, assume there were a thread $T \subseteq \kappa$. Let $E = e``T$. Then $E \subseteq \lambda$ is club.
Claim: Let α be a limit point of E. Then there is a $C \in \mathcal{C}_\alpha$ such that $E' \cap \alpha \subseteq C$.

This is because $E \cap \alpha \in \mathcal{B}_\alpha$, so $E \cap \alpha = p``C$, for some $C \in \mathcal{C}_\alpha$, but clearly, every limit point of $p``C$ is also a limit point of C, so it follows that $E' \cap \alpha \subseteq C$.

For $\alpha \in E'$, let $\mathcal{C}^*_\alpha$ consist of those $C \in \mathcal{C}_\alpha$ that have the property described in the claim, i.e., $E' \cap \alpha \subseteq C$.

Let's define a tree $\mathcal{T}$ as follows. The nodes are the members of $\bigcup_{\alpha \in E'} \mathcal{C}^*_\alpha$, and the ordering is by end-extension, i.e., $C \leq_{\mathcal{T}} D$ if $C = D \cap \sup C$. It follows by the claim that if $i < \kappa$ and α is the i-th member of P' in its monotone enumeration, then the members of $\mathcal{C}^*_\alpha$ form the i-th level of this tree. Thus, $\mathcal{T}$ has height κ, and each level of $\mathcal{T}$ has size less than μ. Since $\mu < \kappa$, it follows by Fact 2.3 that $\mathcal{T}$ has a cofinal branch b. But then, $\bigcup b$ is a thread through $\vec{\mathcal{C}}$, a contradiction. So $\vec{\mathcal{A}}$ is a $\Box(\kappa, {<}\mu)$-sequence. □

A the converse of the previous observation is also true. This was not noted in [18], which deals with principles of the form $\Box(\lambda)$, but is easy to see, and it holds in the more general context of $\Box(\lambda, \kappa)$ as well.

Observation 2.5. *Suppose λ is a singular ordinal of cofinality κ, and let μ be a cardinal. Then*

$$\Box(\kappa, {<}\mu) \implies \Box(\lambda, {<}\mu)$$

Proof. Let $\mu > 0$ and $\kappa > \omega$, since otherwise, $\Box(\kappa, {<}\mu)$ fails, and the claim is vacuously true. Let $\vec{\mathcal{C}}$ be a $\Box(\kappa, {<}\mu)$-sequence, let $e : \kappa \longrightarrow \lambda$ be normal and cofinal, and let $P = e``\kappa$. I will define a $\square(\lambda, {<}\mu)$-sequence $\vec{\mathcal{D}}$. First, I'll specify $\mathcal{D}_\alpha$ for $\alpha \in P'$. Namely, if $e(\bar{\alpha}) = \alpha$, then let

$$\mathcal{D}_\alpha = \{e``C \mid C \in \mathcal{C}_{\bar{\alpha}}\}$$

To define $\mathcal{D}_\alpha$ if $\alpha < \lambda$ is a limit ordinal not in P', let $p(\alpha) = \max(P' \cap \alpha)$ in this case, and set

$$\mathcal{D}_\alpha = \{[p(\alpha), \alpha)\}$$

It is routine to check that $\vec{\mathcal{D}}$ is coherent, and it cannot have a thread because if $T \subseteq \lambda$ were a thread, then for all $\alpha \in (P \cap T)'$, $T \cap \alpha \in \mathcal{D}_\alpha$, so letting $e(\bar{\alpha}) = \alpha$, $T \cap \alpha = e``C$, for some $C \in \mathcal{C}_{\bar{\alpha}}$. It would follow that $T \subseteq P$, and it is again easy to see that $e^{-1}``T$ would end up being a thread for $\vec{\mathcal{C}}$, a contradiction. $\square$

The previous observations explain why I am mostly focusing on the failure of the principles $\square(\lambda, {<}\mu)$, where λ is regular, in the following.

3. Diagonal reflection

The analysis of the effects of SCFA on the failure of weak square principles of the form $\square_{\lambda,\kappa}$, carried out in [4], relied almost exclusively on the fact that SCFA implies rather strong principles of simultaneous stationary reflection. These were introduced and studied by Cummings, Foreman and Magidor, and were exploited in a similar way in the analysis of the effects of MM on the failure of weak square principles by Cummings and Magidor [3].

Definition 3.1 ([2])**.** Let μ be a cardinal, let λ be an uncountable regular cardinal, and let $S \subseteq \lambda$ be stationary. The simultaneous reflection principle $\mathsf{Refl}(\mu, S)$ holds iff for every sequence $\langle T_i \mid i < \mu\rangle$ of stationary subsets of S, there exists an $\alpha < \kappa$ of uncountable cofinality such that for all $i < \mu$, $T_i \cap \alpha$ is stationary ("$\vec{T}$ reflects simultaneously at α").

The principle $\mathsf{Refl}({<}\mu, S)$ says that $\mathsf{Refl}(\bar{\mu}, S)$ holds, for every $\bar{\mu} < \mu$.

If $\kappa < \lambda$ is a regular cardinal, then I write S^λ_κ for the set of $\gamma < \lambda$ with $\mathrm{cf}(\gamma) = \kappa$.

It is easy to see that $\mathsf{Refl}({<}\mu, S)$ implies that the set of α as in the definition is stationary in λ.

The crucial fact for the abovementioned analysis was that for all regular $\lambda > \omega_1$, SCFA implies that $\mathsf{Refl}(\omega_1, S^\lambda_\omega)$ holds. For by results in [3], $\mathsf{Refl}(\omega_1, S^{\lambda^+}_\omega)$ implies, in the case that $\mathrm{cf}(\lambda) \le \omega_1$, that $\square_{\lambda,\mu}$ fails, for every $\mu < \lambda$, and in general that $\square_{\lambda,\mu}$ fails, for every $\mu < \mathrm{cf}(\lambda)$. A fact that's more relevant to the present work is that if $\mathsf{Refl}(2, S)$ holds, for some stationary $S \subseteq \lambda$, then $\square(\lambda)$ fails. By work of [6], this can be improved to yield the failure of $\square(\lambda, {<}\omega)$. It follows from a beautiful result in the same paper that for regular, infinite cardinals $\kappa < \lambda$, $\mathsf{Refl}({<}\kappa, S^\lambda_\omega)$ does not imply the failure of $\square(\lambda, \kappa)$ ([6, Thm. 4.11]), and it is stated as an open question there whether $\mathsf{Refl}({<}\kappa, \lambda)$ implies the failure of $\square(\lambda, {<}\kappa)$. In particular, it is unknown whether $\mathsf{Refl}(\omega_1, \lambda)$ implies the failure of $\square(\lambda, \omega_1)$. So in order to get this strong a conclusion from SCFA, a different form of reflection is needed.

I'll now introduce the class of reflection principles I'll be working with. They are generalizations of the principle OSR_{ω_2} of [13], adapted to cardinals larger than ω_2, and relativized to some stationary set.

Definition 3.2. Let λ be a regular cardinal, let $S \subseteq \lambda$ be stationary, and let $\kappa < \lambda$. The *diagonal reflection principle* $\mathsf{DSR}({<}\kappa, S)$ says that whenever $\langle S_{\alpha,i} \mid$

$\alpha < \lambda, i < j_\alpha\rangle$ is a sequence of stationary subsets of S, where $j_\alpha < \kappa$ for every $\alpha < \lambda$, then there is a $\gamma < \lambda$ of uncountable cofinality, and there is a club $F \subseteq \gamma$ such that for every $\alpha \in F$ and every $i < j_\alpha$, $S_{\alpha,i} \cap \gamma$ is stationary in γ. The version of the principle in which $j_\alpha \le \kappa$ is denoted $\mathsf{DSR}(\kappa, S)$.

It is again easy to see that $\mathsf{DSR}({<}\kappa, S)$ implies that the set of γ as in the definition is stationary in λ.

The point of diagonal reflection is that it is at the same time strong enough to imply substantial failures of weak threaded square principles and weak enough to follow from SCFA. I'll turn to the first aspect now, and will deal with the second one in the next section. I'll need the following technical concept in the proof of the next theorem. It is a variation of a concept from [6].

Definition 3.3. Let λ be a regular cardinal, $S \subseteq \lambda$ a stationary set, and $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in \lambda \cap \mathrm{Lim}\rangle$ a coherent sequence of any width.

Then let $A_{\vec{\mathcal{C}},S}$ be the set of $\alpha < \lambda$ such that there is a club $D_\alpha \subseteq \lambda$ such that for every $\beta \in D_\alpha \cap S$, $\alpha \in \bigcup_{C \in \mathcal{C}_\beta} C'$.

$\vec{\mathcal{C}}$ is *S-full* if $A_{\vec{\mathcal{C}},S} \cap S$ is stationary.

Theorem 3.4. *Let λ be regular, $\kappa < \lambda$ a cardinal, and assume that $\mathsf{DSR}({<}\kappa, S)$ holds, for some stationary $S \subseteq \lambda$. Then $\square(\lambda, {<}\kappa)$ fails.*

Proof. Assume, towards a contradiction, that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in \lambda \cap \mathrm{Lim}\rangle$ is a $\square(\lambda, {<}\kappa)$-sequence. Let $\mathcal{C}_\alpha = \{C_{\alpha,i} \mid i < j_\alpha\}$, for limit $\alpha < \lambda$, where $j_\alpha < \kappa$.

I will first show that it is not S-full.

Let's assume $\vec{\mathcal{C}}$ were S-full. Let $A \subseteq A_{\vec{\mathcal{C}},S} \cap S$ be stationary, such that all $\mathcal{C}_\alpha$ with $\alpha \in A$ have the same cardinality, say $j_\alpha = \mu < \kappa$. For $\alpha \in A$, let $D_\alpha \subseteq \lambda$ be club such that

$$\forall \beta \in D_\alpha \cap S \quad \alpha \in \bigcup_{C \in \mathcal{C}_\beta} C'$$

Set, for $\alpha \in A$ and $i < \mu$,

$$S_{\alpha,i} = \{\beta \in S \mid \forall C \in \mathcal{C}_\beta \quad C \cap \alpha \neq C_{\alpha,i}\}$$

The key point is the following fact.

(1) *For stationarily many $\alpha \in A$, we have that for all $i < \mu$, $S_{\alpha,i}$ is stationary.*

Proof of (1). Otherwise, there is a club $Z \subseteq \lambda$ such that, letting $B = A \cap Z$, we have that for all $\alpha \in B$, there is an $i_\alpha < \mu$ such that S_{α,i_α} is not stationary. The rest of the proof goes through as the proof of [6, Claim 2.19, Thm. 2.18], since $B \subseteq \lambda$ is unbounded. In detail, for $\alpha \in B$, let $E_\alpha \subseteq \lambda$ be a club disjoint from S_{α,i_α}. Define a tree $\mathcal{T}$ whose nodes are of the form C_{α,i_α} for some $\alpha \in B$, ordered by end-extension. Thus, $\mathcal{T}$ has size λ. $\mathcal{T}$ has no antichain of size κ. To see this, let $X = \{C_{\alpha_\xi,i_{\alpha_\xi}} \mid \xi < \kappa\} \subseteq \mathcal{T}$ be a subset of size κ. Let $\beta \in S \cap \bigcap_{\xi<\kappa} E_{\alpha_\xi}$. Then, for every $\xi < \kappa$, $\beta \notin S_{\alpha_\xi,i_{\alpha_\xi}}$, since $E_{\alpha_\xi} \cap S_{\alpha_\xi,i_{\alpha_\xi}} = \emptyset$. Since $\beta \in S$, it follows from the definition of $S_{\alpha_\xi,i_{\alpha_\xi}}$ that there is a $C \in \mathcal{C}_\beta$ such that $C_{\alpha_\xi,i_{\alpha_\xi}} = C \cap \alpha_\xi$. Since this is true for every $\xi < \kappa$, and since there are less than κ many choices for C, there are $\xi, \zeta < \kappa$ with $\alpha_\xi \le \alpha_\zeta$, such that there is a $C \in \mathcal{C}_\beta$ with $C_{\alpha_\xi,i_{\alpha_\xi}} = C \cap \alpha_\xi$ and $C_{\alpha_\zeta,i_{\alpha_\zeta}} = C \cap \alpha_\zeta$. It follows that $C_{\alpha_\xi,i_{\alpha_\xi}} <_{\mathcal{T}} C_{\alpha_\zeta,i_{\alpha_\zeta}}$. So X is not an antichain. It follows in particular that every level of $\mathcal{T}$ has size less than κ. Since $\kappa < \lambda$ and λ

is regular, this implies by a Fact 2.3 that there is a cofinal branch through $\mathcal{T}$. But then, $\bigcup b$ is a thread for $\vec{\mathcal{C}}$. $\square_{(1)}$

Let $T \subseteq A$ be the set shown to be stationary in (1), so for all $\alpha \in T$ and all $i < \mu$, $S_{\alpha,i}$ is stationary. By $\mathsf{DSR}({<}\kappa, S)$, let $\gamma < \lambda$ be such that $\mathrm{cf}(\gamma) \geq \omega_1$, $T \cap \gamma$ is stationary in γ, and such that for some club $F \subseteq \gamma$, we have that for all $\alpha \in F \cap T$ and all $i < \mu$, $S_{\alpha,i} \cap \gamma$ is stationary in γ. Pick $E \in \mathcal{C}_\gamma$ and $\alpha \in E' \cap F \cap T$. Let $E \cap \alpha = C_{\alpha,i^*}$. Since $\alpha \in F \cap T$, $S_{\alpha,i^*} \cap \gamma$ is stationary in γ. So we can pick $\beta \in (E' \setminus (\alpha+1)) \cap S_{\alpha,i^*}$. Then $\bar{E} = E \cap \beta \in \mathcal{C}_\beta$, and $\bar{E} \cap \alpha = C_{\alpha,i^*}$. But $\beta \in S_{\alpha,i^*}$, so $\bar{E} \cap \alpha \neq C_{\alpha,i^*}$, by definition of S_{α,i^*}, since $\bar{E} \in \mathcal{C}_\beta$. This contradiction shows that $\vec{\mathcal{C}}$ is not S-full.

Thus, $A_{\vec{\mathcal{C}},S} \cap S$ is not stationary. Let $C \subseteq \lambda$ be club such that $A_{\vec{\mathcal{C}},S} \cap S \cap C = \emptyset$. This means that for every $\alpha \in Y = S \cap C$, the set

$$S_\alpha = \{\beta \in S \mid \alpha \notin \bigcup_{C \in \mathcal{C}_\beta} C'\}$$

is stationary in λ, because otherwise there would be a club $D_\alpha \subseteq \lambda$ disjoint from S_α. But then, for all $\beta \in D_\alpha \cap S$, we'd have that $\alpha \in \bigcup_{C \in \mathcal{C}_\beta} C'$, i.e., $\alpha \in A_{\vec{\mathcal{C}},S}$.

By $\mathsf{DSR}({<}\kappa, S)$, noting that $Y \subseteq S$ is stationary, let $\gamma < \lambda$ have uncountable cofinality, such that $Y \cap \gamma$ is stationary in γ and such that there is a club $F \subseteq \gamma$ such that for every $\alpha \in F \cap Y$, $S_\alpha \cap \gamma$ is stationary in γ. Let $E \in \mathcal{C}_\gamma$, and let $\alpha \in E' \cap F \cap Y$. Then, since $\alpha \in F \cap Y$, $S_\alpha \cap \gamma$ is stationary in γ. Let $\beta \in S_\alpha \cap (E' \setminus (\alpha+1))$. Then $\alpha \notin \bigcup_{C \in \mathcal{C}_\beta} C'$, because $\beta \in S_\alpha$. But $\alpha, \beta \in E'$, so $\bar{E} = E \cap \beta \in \mathcal{C}_\beta$. So $\alpha \in \bar{E}'$, which shows that $\alpha \in \bigcup_{C \in \mathcal{C}_\beta} C'$, after all. This contradiction concludes the proof. $\square$

4. Effects of forcing axioms

I'll now deal with the second aspect of diagonal reflection, namely that it is weak enough to follow from SCFA. The forcing I will use to derive it is introduced in the following definition. It's the same generalization of Larson's forcing from [13] that was used in [4].

Definition 4.1. Let λ be regular. Let $\vec{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a partition of ω_1 into stationary sets, and let $\vec{T} = \langle T_\alpha \mid \alpha < \lambda \rangle$ be a sequence of stationary subsets of λ, each T_α consisting of ordinals of cofinality ω. The forcing $\mathbb{P}_{\vec{A},\vec{T}}$ consists of the pairs $\langle p, q \rangle$ such that

(1) p is a function with $\mathrm{dom}(p) \subseteq \omega_1$, $\mathrm{ran}(p) \subseteq \lambda$ and $\overline{\overline{p}} < \omega_1$,
(2) $q : \gamma + 1 \longrightarrow \omega_1$ is normal, for some $\gamma < \omega_1$,
(3) $\sup(\mathrm{ran}(q)) \subseteq \mathrm{dom}(p)$,
(4) for all $\xi \in \mathrm{dom}(q)$, if α is such that $q(\xi) \in A_\alpha$, then $\alpha \in \mathrm{dom}(p)$ and $\sup p``(q(\xi)) \in T_{p(\alpha)}$.

The ordering is by reverse inclusion in each component.

The following fact, which I proved in [4, Lemma 2.24], is crucial for the further development.

Fact 4.2. *The forcing $\mathbb{P} = \mathbb{P}_{\vec{A},\vec{T}}$ is subcomplete.*

I will use the following characterization of bounded forcing axioms, tracing back to [1, Thm. 1.3].

Fact 4.3. $\mathsf{BFA}(\{\mathbb{Q}\},\leq\kappa)$ *is equivalent to the following statement: if* $M = \langle |M|, \in, R_0, R_1, \ldots, R_i, \ldots\rangle_{i<\omega_1}$ *is a transitive model for the language of set theory with* ω_1 *many predicate symbols* $\langle \dot{R}_i \mid i < \omega_1\rangle$*, of size* κ*, and* $\varphi(x)$ *is a* Σ_1*-formula in the language of set theory, such that* $\Vdash_{\mathbb{Q}} \varphi(\check{M})$*, then there is in* V *a transitive* $\bar{M} = \langle |\bar{M}|, \in, \vec{\bar{R}}\rangle$ *for the same language, and an elementary embedding* $j : \bar{M} \prec M$ *such that* $\varphi(\bar{M})$ *holds.*

Now the connection between bounded subcomplete forcing axioms and diagonal reflection can be established.

Theorem 4.4. *Let* $\lambda > \omega_1$ *be a regular cardinal. Then* $\mathsf{BSCFA}(\leq\lambda)$ *implies* $\mathsf{DSR}(\omega_1, S^\lambda_\omega)$.

Proof. Let $S_{\alpha,i} \subseteq S^\lambda_\omega$ be stationary, for each $\alpha < \lambda$ and $i < \omega_1$. Let $c : \lambda \longrightarrow \lambda \times \omega_1$ be a bijection, and let $T_\alpha = S_{c(\alpha)}$, for $\alpha < \lambda$.

Let $\vec{A} = \langle A_\alpha \mid \alpha < \omega_1\rangle$ be a partition on ω_1 into disjoint stationary sets, and let $\mathbb{P} = \mathbb{P}_{\vec{S},\vec{T}}$. Let G be generic for $\mathbb{P}$. Let $M \prec H_{\lambda^+}$ with $\lambda \subseteq M$, so that M has size λ, with $\lambda, \vec{A}, \vec{T}, c \in M$. Let M also be equipped with constant symbols for the countable ordinals and the objects just mentioned. Let P be the union of the first components of conditions in G, and Q the union of the second components. Then in V$[G]$, the following Σ_1-statement about M holds: "there is a club $C \subseteq \omega_1^M$ and a function $g : \omega_1^M \longrightarrow \dot{\lambda}^M$ such that g is onto, and such that for every $\zeta \in C$, if α is such that $\zeta \in A_\alpha$, then $\sup g``\zeta \in T_{g(\alpha)}$." This is witnessed by $g = P$ and $C = \mathrm{ran}(Q)$.

So, by $\mathsf{BSCFA}(\leq\lambda)$, there is in V a model $\bar{M}$ such that the same Σ_1 statement is true of $\bar{M}$, and such that there is an elementary embedding $j : \bar{M} \prec M$. Let $\bar{C}, \bar{P}$ witness that the statement holds for $\bar{M}$. Let $\vec{\bar{A}} = j^{-1}(\vec{S})$, $\vec{\bar{T}} = j^{-1}(\vec{T})$, $\bar{c} = j^{-1}(c)$. Let $\bar{\gamma} = \dot{\lambda}^{\bar{M}}$. Note that $\omega_2^{\bar{M}}$ is the critical point of j and $\omega_1^{\bar{M}} = \omega_1$, so that for $\alpha < \omega_1$, $\bar{A}_\alpha = A_\alpha$. So we have that $\bar{P} : \omega_1 \longrightarrow \bar{\gamma}$ is onto, $\bar{C} \subseteq \omega_1$ is club, and for every $\zeta \in \bar{C}$, if α is such that $\zeta \in A_\alpha$, then $\sup \bar{P}``\zeta \in \bar{T}_{\bar{P}(\alpha)}$.

Let $e : \omega_1 \longrightarrow \bar{C}$ be the monotone enumeration of $\bar{C}$, and define $h : \omega_1 \longrightarrow \bar{\gamma}$ by $h(\xi) = \sup \bar{P}``e(\xi)$. Clearly, $h$ witnesses that the cofinality of $\gamma$ is $\omega_1$. Let $\gamma = \sup j``\bar{\gamma}$. Again, γ has cofinality ω_1, and so, $\gamma < \lambda$. Also, for every $\xi < \omega_1$, $h(\xi) \in \bar{T}_{\bar{P}(\alpha)}$, where $e(\xi) \in A_\alpha$. In particular, in $\bar{M}$, $h(\xi)$ has countable cofinality. Thus, $\mathrm{ran}(h) \subseteq \bar{\gamma}$ is a club consisting of ordinals which, inside $\bar{M}$, have countable cofinality. So, j is continuous on $\mathrm{ran}(h)$, and hence, letting $F = j``\mathrm{ran}(h)$, it follows that $F \subseteq \gamma$ is club.

Let's now show that if $\beta \in F$ and $i < \omega_1$, it follows that $S_{\beta,i} \cap \gamma$ is stationary in γ. To see this, let $d \subseteq \gamma$ be club, and show that $d \cap S_{\beta,i} \neq \emptyset$. Let $\bar{d} = h^{-1}``j^{-1}``(d \cap F)$. Then $\bar{d}$ is club in ω_1.

Let $j(\alpha) = \beta$, $\xi' = c^{-1}(\beta, i)$ and $\xi = \bar{c}^{-1}(\alpha, i)$. Then $j(\xi) = \xi'$. Let $\bar{P}(\bar{\xi}) = \xi$ and pick $\zeta \in \bar{d} \cap \bar{C} \cap A_{\bar{\xi}}$.

By the properties of $\bar{P}$ and $\bar{C}$, it follows that $h(\zeta) = \sup \bar{P}``\zeta \in \bar{T}_{\bar{P}(\bar{\xi})}$, and so, by elementarity of j,

$$j(h(\zeta)) \in j(\bar{T}_{\bar{P}(\bar{\xi})}) = T_{j(\bar{P}(\bar{\xi}))} = T_{j(\xi)} = T_{\xi'}$$

Since $\zeta \in \bar{d} = h^{-1}``j^{-1}``(d \cap F)$, $j(h(\zeta)) \in d \cap F$, and so we have shown that $d \cap T_{\xi'} \neq \emptyset$. Since d was an arbitrary club in γ, this shows that $T_{\xi'}$ is stationary. Since $T_{\xi'} = S_{c(\xi')} = S_{\beta,i}$, this completes the proof. □

Combining the results from the present and the previous sections results in the following lemma.

Lemma 4.5. *The following implications hold.*

1. $\mathsf{BSCFA}(\le\omega_2)$ *implies the failure of* $\square(\omega_2,\omega)$ *but is consistent with* $\square(\omega_2,\omega_1)$.
2. *If* $\lambda > \omega_2$ *is a regular cardinal, then* $\mathsf{BSCFA}(\le\lambda)$ *implies the failure of* $\square(\lambda,\omega_1)$.

Proof. For 1., $\mathsf{BSCFA}(\le\omega_2)$ implies $\mathsf{DSR}(\omega_1, S^{\omega_2}_\omega)$, by Theorem 4.4, and in particular, it implies the weaker principle $\mathsf{DSR}({<}\omega_1, S^{\omega_2}_\omega)$. Then, Theorem 3.4 yields the failure of $\square(\omega_2,\omega)$. For the last part of 1., BSCFA is consistent with CH, CH implies $\square^*_{\omega_1}$, and this implies $\square(\omega_2,\omega_1)$.

Part 2. follows in the same way. $\mathsf{BSCFA}(\le\lambda)$ implies $\mathsf{DSR}({<}\omega_2, S^\lambda_\omega)$, by Theorem 4.4, and by Theorem 3.4 this implies the failure of $\square(\lambda,\omega_1)$, since $\omega_2 < \lambda$. □

5. Maximizing square, and a limitation

The questions I want to address in this section are whether the results of the previous lemma are optimal, and whether diagonal reflection can be used to settle the status of $\square^*_{\aleph_\omega}$ under SCFA. I will answer the first question affirmatively, and unfortunately, the answer to the second question will turn out to be no. The methods used owe much to the work of Cummings, Foreman and Magidor.

Turning to the first question, the strategy is similar to the one employed in [3]: maximize the extent of the relevant square principles over a model with a supercompact cardinal, preserving supercompactness, and then force the forcing axiom at hand, preserving as much of the square princples as possible. Maximizing weak threaded square principles while preserving supercompactness will be achieved by working in a model in which the supercompact cardinal κ is indestructible under ${<}\kappa$-directed closed forcing. Thus, we need a way to maximize the relevant weak threaded square principles by ${<}\kappa$-directed closed forcing. It was shown in [2, proof of Thm. 16] that if λ is a singular cardinal, then there is a ${<}\mathrm{cf}(\lambda)$-directed closed forcing that's also ${<}\lambda$-strategically closed, that adds a $\square_{\lambda,\mathrm{cf}(\lambda)}$-sequence. The authors then introduced the concept of an indexed square sequence, $\square^{\mathrm{ind}}_{\lambda,\mathrm{cf}(\lambda)}$, and observed that their ${<}\mathrm{cf}(\lambda)$-directed closed forcing actually adds such a sequence. As the notation suggests, their indexed square principles involve restrictions on the order types of the clubs in the sequence, and so, in order to be applicable to the present context, these restrictions need to be dropped. The resulting principles, $\square^{\mathrm{ind}}(\lambda,\kappa)$, have been introduced and studied in by Lambie-Hanson, in [12, Section 6].

Definition 5.1. Let $\kappa < \lambda$ be regular. Then $\vec{\mathcal{C}} = \langle C_{\alpha,i} \alpha < \lambda,\ \alpha \text{ limit},\ i(\alpha) \le i < \kappa\rangle$ is a $\square^{\mathrm{ind}}(\lambda,\kappa)$-sequence if the following hold, where $\alpha < \lambda$ is a limit ordinal:

1. $i(\alpha) < \kappa$.
2. For all $i \in [i(\alpha),\kappa)$, $C_{\alpha,i} \subseteq \alpha$ is club.
3. For all i,j with $i(\alpha) \le i < j < \kappa$, $C_{\alpha,i} \subseteq C_{\alpha,j}$.
4. If $\beta \in C'_{\alpha,i}$, then $i(\beta) \le i$ and $C_{\alpha,i} \cap \beta = C_{\beta,i}$.
5. For all limit $\beta \in (\alpha,\lambda)$, there is an $i \in [i(\beta),\kappa)$ such that $\alpha \in C'_{\beta,i}$.
6. $\vec{\mathcal{C}}$ has no thread, that is, there is no club $C \subseteq \lambda$ such that for every $\beta \in C'$, there is an $i \in [i(\beta),\kappa)$ with $C \cap \beta = C_{\beta,i}$.

Point 5 here is a crucial addition to the original concept of indexed square from [2]. The main utility of $\square^{\text{ind}}(\lambda,\kappa)$ to me is that it can be forced by benign forcing. As expected, the forcing in question, $\mathbb{P}(\lambda,\kappa)$, consists of sequences of length less than λ whose domain has a maximal element, and which satisfy points (1)-(5) of the previous definition. The ordering is by end-extension.

Fact 5.2 ([12, Section 7])**.** *Let $\kappa < \lambda$ be infinite regular cardinals. Then the forcing $\mathbb{P}(\lambda,\kappa)$ is $<\kappa$-directed closed, $<\lambda$-strategically closed and adds an $\square^{\text{ind}}(\lambda,\kappa)$-sequence.*

Corollary 5.3. *If the existence of a supercompact cardinal is consistent, then so is the existence of a supercompact cardinal κ such that for every regular cardinal $\lambda > \kappa$, the principle $\square^{\text{ind}}(\lambda,\kappa)$ holds.*

Proof. Starting in a model with a supercompact cardinal κ, use the Laver preparation of [14] to produce a model of set theory where κ is still supercompact and its supercompactness is preserved by any $<\kappa$-directed closed forcing. We may assume that GCH holds in that model above κ, since otherwise we could force it by an Easton support iteration that's $<\kappa$-directed closed. Working in that model, let's call it V, we can now form an Easton iteration $\langle\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\rangle \mid \alpha\in\mathrm{On}\rangle$ in which the only nontrivial stages of forcing are the stages $\lambda>\kappa$ where λ is a regular cardinal, and in that case, we let $\dot{\mathbb{Q}}_\lambda$ be a $\mathbb{P}_\lambda$-name for the forcing $\mathbb{P}(\lambda,\kappa)$ described above. Let $\mathbb{P}$ be the resulting class forcing. Since it is a forcing with increasing (strategic) closure, it gives rise to a ZFC-model. $\mathbb{P}$ preserves cardinals, because at every regular cardinal $\lambda\ge\kappa$, it splits as $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \dot{\mathbb{P}}_{\mathtt{tail}}$, where, by the GCH in V, and by Easton support, $\mathbb{P}_\lambda$ is λ-c.c., $\dot{\mathbb{Q}}_\lambda$ is $<\lambda$-strategically closed, and $\dot{\mathbb{P}}_{\mathtt{tail}}$ is $<\lambda^+$-strategically closed. Finally, letting G be $\mathbb{P}$-generic over V, it follows that $\square^{\text{ind}}(\lambda,\kappa)$ holds in $\mathrm{V}[G]$ for every regular $\lambda>\kappa$, because $\dot{\mathbb{Q}}_\lambda^{G_\lambda}$ forces $\square^{\text{ind}}(\lambda,\kappa)$ over $\mathrm{V}[G\restriction\lambda]$, and since the tail forcing is $<\lambda^+$-strategically closed in $\mathrm{V}[G\restriction(\lambda+1)]$, it does not add any new λ-sequences, and in particular, it cannot add a thread to the $\square^{\text{ind}}(\lambda,\kappa)$ sequence of $\mathrm{V}[G\restriction(\lambda+1)]$, and so, this principle continues to hold in $\mathrm{V}[G]$. $\square$

I want to make an observation which is a version of Lemma 4.5 for the bounded proper forcing axioms. The proof is based on Todorčević's [19], even though the original presentation is very different. The original theorem in that paper says that PFA implies that partial versions of $\square(\lambda)$ fail, for regular $\lambda>\omega_1$. It is noted in [3] that a close inspection of Todorčević's proof shows that PFA denies $\square_{\lambda,\omega_1}$, for all regular $\lambda\ge\omega_1$. Unsurprisingly, the proof actually shows that PFA denies $\square(\lambda,\omega_1)$, for all regular $\lambda>\omega_1$. I give the proof, because it is instructive to see that it hinges on the fact that PFA implies the failure of CH, which is of particular importance in the present context, where the focus is on SCFA, which is compatible with CH.

Note also that PFA does not imply $\mathsf{Refl}(\omega_1,S^\lambda_\omega)$, as MM and SCFA do (since PFA is compatible with $\square_{\kappa,\omega_2}$, for every $\kappa\ge\omega_2$; compare with the effects of simultaneous stationary reflection on the failure of weak squares in [3]), and in particular, it does not imply $\mathsf{DSR}(\omega_1,S^\lambda_\omega)$. So the argument using PFA necessarily has to be different.

Lemma 5.4. *Let $\lambda>\omega_1$ be a regular cardinal, and assume $\mathsf{BPFA}(\le\lambda)$. Then $\square(\lambda,\omega_1)$ fails.*

Proof. Assume towards a contradiction that there is a $\square(\lambda,\omega_1)$-sequence $\vec{\mathcal{C}}$. Let G be generic for $\mathbb{P}=\mathrm{Col}(\omega_1,\lambda)$. It follows that $\vec{\mathcal{C}}$ is still a $\square(\lambda,\omega_1)$-sequence in

$V[G]$, because if not, then let $\dot{b}$ be a $\mathbb{P}$-name such that some $p \in G$ forces wrt. $\mathbb{P}$ that $\dot{b}$ is a thread for $\vec{\mathcal{C}}$. One can now easily construct a sequence $\langle p_s, \alpha_s \mid s \in {}^{<\omega}2\rangle$ such that for $s \subsetneqq t$, $p_t \le p_s \le p$, $\alpha_s < \alpha_t$ and p_t decides the statement "$\check{\alpha}_s \in \dot{b}$", and moreover, $p_{s^\frown 0}$ and $p_{s^\frown 1}$ decide it in opposite ways. One can further arrange that for every $n < \omega$, there is a β_n such that $\beta_n \ge \alpha_s$, for every s with $|s| < n$, and such that for every t with $|t| \ge n$, $\alpha_t > \beta_n$, and every p_s forces that $\check{\beta}_{|s|} \in \dot{b}$. Since $\mathbb{P}$ is countably closed, for every $x \in {}^\omega\omega$, we can let p_x be a lower bound in $\mathbb{P}$ for $\langle p_{x\restriction n} \mid n < \omega\rangle$. Let $\beta = \sup_{n<\omega}\beta_n$. Then p_x forces that β is a limit point of $\dot{b}$, and hence that $\dot{b} \cap \check{\beta} \in \check{\mathcal{C}}_\beta$. Hence, there must be a $C = f(x) \in \mathcal{C}_\beta$ such that for every $n < \omega$, $\alpha_{x\restriction n} \in C$ iff p_x forces "$\check{\alpha}_{x\restriction n} \in \check{C}$". But for $x \ne y$, it has to be that $f(x) \ne f(y)$, because if n is least such that $x(n) \ne y(n)$, then, letting $\xi = \alpha_{x\restriction n} = \alpha_{y\restriction n}$, $p_x \Vdash \check{\xi} \in \dot{b}$ iff $p_y \Vdash \check{\xi} \notin \dot{b}$. So $f : {}^\omega 2 \longrightarrow \mathcal{C}_\beta$ is injective, but $\mathcal{C}_\beta$ has size ω_1, while under BPFA, $2^\omega = \omega_2$, by [17]. This contradiction shows that $\vec{\mathcal{C}}$ is still a $\square(\lambda, \omega_1)$-sequence in $V[G]$. Now, working in $V[G]$, the cofinality of λ is ω_1, so we can let $X \subseteq \lambda$ be a club of order type ω_1, consisting of ordinals of countable cofinality. Let $T = T_{\mathcal{C},X}$ be the tree with vertex set $\bigcup_{\alpha\in X}\mathcal{C}_\alpha$, ordered by end-extension. This is a tree of height ω_1 that does not have a cofinal branch, since a branch would give rise to a thread for $\vec{\mathcal{C}}$. Thus, the forcing $\mathbb{Q}$ to specialize T is c.c.c. Let H be generic for $\mathbb{Q}$ over $V[G]$. In V, let $Y \prec H^{V}_{\lambda^+}$ have size λ, with $\lambda \subseteq Y$ and $\mathcal{C} \in Y$. Let $M = \langle H, \in, \mathcal{C}, 0, 1, \ldots, \xi, \ldots\rangle_{\xi<\omega_1}$. We will apply Fact 4.3 to the structure M. In $V[G][H]$, the following Σ_1 statement $\varphi(M)$ is true: there are a club $Z \subseteq \lambda$ of order type ω_1^M, consisting of ordinals of countable M-cofinality, and a function $g : T_{\mathcal{C},Z} \longrightarrow \omega$ such that for all $C, D \in T_{\mathcal{C},Z}$, if $C \ne D$ and C,D are comparable, then $g(C) \ne g(D)$. This is witnessed by the club X and the specializing function for T added by H. Thus, by Fact 4.3, there are in V a transitive model $\bar{M}$ with $\varphi(\bar{M})$ and an elementary embedding $j : \bar{M} \prec M$. Let $\bar{X}$, f witness that $\varphi(\bar{M})$ holds. Let $\vec{\mathcal{D}} = j^{-1}(\vec{\mathcal{C}})$, $\bar{T} = T_{\vec{\mathcal{D}},\bar{X}}$, $\bar{\lambda} = j^{-1}(\lambda)$ and $\theta = \sup j``\bar{\lambda}$. Since $\mathrm{otp}(\bar{X}) = \omega_1$, $\mathrm{cf}(\theta) = \omega_1$, and so, $\theta < \lambda$. Since $\bar{X}$ consists of ordinals of countable $\bar{M}$-cofinality, $j\restriction\bar{X}$ is continuous, and hence, $j``\bar{X}$ is club in θ. But now, if $T \in \mathcal{C}_\theta$, then for every $\alpha \in T' \cap j``\bar{X}$, $T \cap \alpha \in \mathcal{C}_\alpha = j(\mathcal{D}_{\bar{\alpha}}) = j``\mathcal{D}_{\bar{\alpha}}$, where $j(\bar{\alpha}) = \alpha$, since $\mathcal{D}_{\bar{\alpha}}$ has size ω_1 in $\bar{M}$ and $\mathrm{crit}(j) > \omega_1 = \omega_1^{\bar{M}}$. Thus, the set $\{j^{-1}(T \cap \alpha) \mid \alpha \in T' \cap j``\bar{X}\}$ generates a cofinal branch b through the special tree $\bar{T}$, a contradiction. □

I am now ready to show that the results of Lemmas 4.5 and 5.4 are sharp.

Theorem 5.5. *Assume the consistency of the existence of a supercompact cardinal.*

(1) *It is consistent that*
  (a) SCFA + CH + $\diamondsuit$ *holds*
  (b) *for every regular* $\lambda \ge \omega_2$, $\square^{\mathrm{ind}}(\lambda, \omega_2)$ *holds.*

  In any model of SCFA+CH, *necessarily,* $\square(\lambda, \omega_1)$ *fails for all regular* $\lambda > \omega_2$, $\square(\omega_2, \omega)$ *fails, and* $\square(\omega_2, \omega_1)$ *holds.*

(2) *Similarly, it is consistent that*
  (a) MM *or* PFA *holds*
  (b) *for every regular* $\lambda \ge \omega_2$, $\square^{\mathrm{ind}}(\lambda, \omega_2)$ *holds.*

  In a model of (a), necessarily, $\square(\lambda, \omega_1)$ *fails, for every* $\lambda \ge \omega_2$.

Proof. For (1), by Corollary 5.3, we may start in a model of set theory V with a supercompact cardinal κ such that for every regular cardinal $\lambda > \kappa$, $\square^{\mathrm{ind}}(\lambda, \kappa)$

holds. Working in this model, let $\mathbb{P}$ be the standard RCS iteration of subcomplete forcings of length κ, see [10]. If G is $\mathbb{P}$-generic, then $V[G]$ satisfies $\mathsf{SCFA}+\mathsf{CH}+\diamondsuit$, and $\kappa=\omega_2^{V[G]}$. $\mathbb{P}$ is κ-c.c., which implies that for $\lambda>\kappa=\omega_2^{V[G]}$, the $\square^{\mathrm{ind}}(\lambda,\kappa)$-sequences of V still don't have a thread, since for every club set $C\subseteq\lambda$, there is a club $\bar{C}\subseteq C$ in V. Of course, $\square^{\mathrm{ind}}(\omega_2,\omega_2)$ holds trivially, since it is implied by "silly square", $\square_{\omega_1,\omega_2}$, which always holds. It follows from Lemma 4.5 that $\square(\lambda,\omega_1)$ fails, for all regular $\lambda>\omega_2$, and that $\square(\omega_2,\omega)$ fails. It follows from CH that $\square^*_{\omega_1}$, and thus $\square(\omega_2,\omega_1)$, holds.

The proof of (2) is very similar. Thus, starting in the model of Corollary 5.3, we use the supercompact cardinal κ to force MM (which implies PFA). Calling the resulting model $V[G]$, it follows as before that $\square^{\mathrm{ind}}(\lambda,\omega_2)$ holds, for every regular $\lambda\geq\omega_2$. The final claim follows from Lemma 5.4. □

Let's now summarize the effects of SCFA on $\square(\lambda,\kappa)$, where λ is any limit ordinal. Since in ZFC, it's provable that $\square(\lambda)$ fails if $\mathrm{cf}(\lambda)=\omega$ and it holds if $\mathrm{cf}(\lambda)=\omega_1$, it suffices to focus on the case $\mathrm{cf}(\lambda)>\omega_1$.

Theorem 5.6. *Assume* SCFA*. Let λ be a limit ordinal.*

(1) *If $\mathrm{cf}(\lambda)=\omega_2$, then $\square(\lambda,\omega)$ fails.*
(2) *If $\mathrm{cf}(\lambda)\geq\omega_3$, then $\square(\lambda,\omega_1)$ fails.*

Moreover, these results are sharp, in the sense that if the existence of a supercompact cardinal is consistent, then it is consistent that SCFA *holds, and for every limit ordinal λ:*

(3) *If $\mathrm{cf}(\lambda)=\omega_2$, then $\square(\lambda,\omega_1)$ holds.*
(4) *If $\mathrm{cf}(\lambda)\geq\omega_3$, then $\square(\lambda,\omega_2)$ holds.*

Proof. (1) and (2) follow from Lemma 4.5 and Observation 2.4. Note that it's close, because for $\mathrm{cf}(\lambda)=\omega_2$, we have $\neg\square(\omega_2,{<}\omega_1)$, and since $\omega_1<\omega_2$, this implies by Observation 2.4 that $\square(\lambda,{<}\omega_1)$ fails. For $\mathrm{cf}(\lambda)\geq\omega_3$, we have that $\square(\mathrm{cf}(\lambda),<\omega_2)$ fails, and the same observation gives us that $\square(\lambda,{<}\omega_2)$ fails. Points (3) and (4) follow from Theorem 5.5 and Observation 2.5. □

Theorem 5.7. *Assume* PFA *or* MM*. Then $\square(\lambda,\omega_1)$ fails for every ordinal λ with $\mathrm{cf}(\lambda)\geq\omega_2$, and this result is sharp, in the sense that if the existence of a supercompact cardinal is consistent, then it is consistent that* SCFA *holds, and for every limit ordinal λ with $\mathrm{cf}(\lambda)\geq\omega_2$, $\square(\lambda,\omega_2)$ holds.*

Proof. There is one subtlety here: if $\mathrm{cf}(\lambda)=\omega_2$, then one cannot use Observation 2.4 to conclude from the failure of $\square(\omega_2,\omega_1)$ that $\square(\lambda,\omega_1)$ fails. But instead, one can check that the proof of Lemma 5.4 goes through for such λ, assuming $\mathsf{BPFA}(\leq|\lambda|)$. □

In [4], I pointed out that the analysis of the extent of weak square principles under Martin's Maximum of [3] can be carried out in the context of SCFA with some success, making it possible to say exactly which failures of $\square_{\lambda,\kappa}$ are implied by SCFA and which are not, with one question remaining open: does SCFA imply the failure of $\square^*_\lambda$, when $\mathrm{cf}(\lambda)=\omega$? This is the case under MM, but the situation with SCFA is unclear.

It would be tempting to hope that diagonal reflection might settle this question. This turns out not to be the case. In fact, a model constructed in [2] shows that very strong forms of diagonal stationary reflection are consistent with $\square^*_{\aleph_\omega}$. To formulate the result succinctly, let me introduce another parameter in the diagonal stationary reflection principles.

Definition 5.8. Let λ be a regular cardinal, $S \subseteq \lambda$ a stationary set, κ a cardinal, and μ a regular cardinal. Then the principle $\mathsf{DSR}(<\kappa, S, \nu)$ says: let $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ be a sequence of stationary subsets of S, with $j_\alpha < \kappa$, for all $\alpha < \lambda$. Then there is a $\gamma < \lambda$ with $\mathrm{cf}(\gamma) = \nu$ and a club $F \subseteq \gamma$ such that for every $\alpha \in F$ and all $i < j_\alpha$, $S_{\alpha,i} \cap \gamma$ is stationary in γ. As before, the principle where $j_\alpha \leq \kappa$ is denoted $\mathsf{DSR}(\kappa, S, \nu)$.

Theorem 5.9. *Assuming the consistency of infinitely many supercompact cardinals, it is consistent that for every nonzero $n < \omega$, $\mathsf{DSR}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$ holds, and moreover, $\square^*_{\aleph_\omega}$ holds.*

Proof. An inspection of [2, Section 12] reveals that the model described in the proof of Theorem 21 of that paper has the desired properties. It is shown there that in the model, $\mathsf{Refl}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$ holds for every nonzero $n < \omega$, and I will sketch the part of the argument that needs to be modified slightly in order to improve this to $\mathsf{DSR}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$, taking the rest of the construction, which is very intricate, and fortunately does not need to be changed, for granted.

The starting point is a model V with an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals, in which GCH holds above $\kappa = \sup_{n<\omega} \kappa_n$. In a first step, Laver's forcing is iterated, and a $\diamondsuit^+_{\kappa^+}$-sequence is added, to reach a generic extension V' in which the supercompactness of each κ_n is indestructible under $<\kappa_n$-directed closed forcing and $\diamondsuit^+_{\kappa^+}$ holds. V_1 is a generic extension of V', resulting from collapsing cardinals so that in V_1, $\kappa_n = \aleph_{n+1}$, for $n < \omega$, $\kappa = \aleph_\omega$, $(\kappa^+)^{\mathrm{V}'} = \aleph_{\omega+1}$, and $\diamondsuit^+_{\aleph_{\omega+1}}$ holds. Finally, $\mathrm{V}_2 = \mathrm{V}_1[\vec{\mathcal{C}}]$ is a forcing extension of V_1 by a forcing $\mathbb{S}$ to add a $\square^*_{\aleph_\omega}$-sequence. This is not the standard forcing to do so. It preserves the $\aleph_n$'s, $\aleph_\omega$ and $\aleph_{\omega+1}$ - it adds no new $\aleph_\omega$-sequences of ordinals over V_1. V_2 will be the model in which the desired diagonal reflection properties hold, along with $\square^*_{\aleph_\omega}$. The forcing extensions V_1, V', V_2 have some other nice properties which I will state when they are needed in the argument to follow.

Fixing a nonzero $n < \omega$, in V_2, let $\langle S_{\alpha,i} \mid \alpha < \aleph_{\omega+1}, i < \aleph_n \rangle$ be given, each $S_{\alpha,i}$ being a subset of $S^{\aleph_{\omega+1}}_{<\aleph_n}$. There is a forcing called $\mathbb{T}_{n+1}$ which preserves the stationarity of each $S_{\alpha,i}$, and forces the cofinality of $\aleph_{\omega+1}$ to become $\aleph_{n+1}$. Furthermore, $\mathbb{S} * \mathbb{T}_{n+1}$ has a $<\aleph_{n+1}$-directed closed dense subset, and hence, it preserves cardinals up to $\aleph_{n+1}$. Let c be generic over V_2 for this forcing. $\aleph_{n+1}$ is "indestructibly generically supercompact" in V_2, and it follows from [2, Fact 6.10] that there is a generic extension V_3 of $\mathrm{V}_2[c]$ by a $<\aleph_n$-closed forcing such that in V_3, there is an elementary embedding $j : \mathrm{V}_2[c] \prec M$ with $\mathrm{crit}(j) = \aleph^{\mathrm{V}_2}_{n+1}$, $j{\restriction}\aleph^{\mathrm{V}_2}_{\omega+1} \in M$, $j(\aleph^{\mathrm{V}_2}_{n+1}) > \aleph^{\mathrm{V}_2}_{\omega+1}$, and $\mathrm{cf}(\aleph^{\mathrm{V}_2}_{\omega+1})^M = \aleph_n = \aleph^{\mathrm{V}_2}_n$.

It follows that each $S_{\alpha,i}$ is stationary in V_3: since $\mathrm{cf}(\aleph^{\mathrm{V}_1}_{\omega+1})^{\mathrm{V}_2[c]} = \aleph_{n+1}$, there is a normal, cofinal function $f : \omega_{n+1} \longrightarrow \omega^{\mathrm{V}_1}_{\omega+1}$ in $\mathrm{V}_2[c]$. Let $\bar{F} = \mathrm{ran}(f)$, and let $T_{\alpha,i} = f^{-1}``S_{\alpha,i}$. Then $T_{\alpha,i} \subseteq S^{\aleph_{n+1}}_{<\omega_n}$ in $\mathrm{V}_2[c]$, since $S_{\alpha,i} \subseteq S^{\aleph_{\omega+1}}_{<\aleph_n}$ in V_2. It suffices to show that $T_{\alpha,i}$ is stationary in V_3. Since $\mathrm{V}_2[c] \models \mathsf{GCH}$, $\square^*_{\aleph_n}$ holds in $\mathrm{V}_2[c]$, so AP_{ω_n} holds, so by a result of Shelah, $<\aleph_n$-closed forcing preserves stationary subsets of $S^{\aleph_n}_{<\aleph_n}$ in $\mathrm{V}_2[c]$.

Now let $\mu = \sup j``\aleph^{\mathrm{V}_2}_{\omega+1}$. In $\mathrm{V}_3$, each $j``S_{\alpha,i}$ is stationary in μ, since $\mathrm{crit}(j) = \aleph_{n+1}$, so j is continuous at points of cofinality less than ω_{n+1}. μ has cofinality

$\aleph_n$ in V_3, since this is true in M. Let $\vec{U} = j(\vec{S}) = \langle U_{\alpha,i} \mid \alpha < j(\omega_{\omega+1}^{\mathrm{V}_2}), i < \aleph_n\rangle$. Let $F = j``\bar{F}$. Since $\mathrm{otp}(\bar{F}) = \omega_{n+1}$ and j is continuous at limits of cofinality less than ω_{n+1}, it follows that F is club in μ, and obviously, $F \in M$. Since in M, μ has cofinality $\aleph_n$, one can intersect F in M with a club of order type $\aleph_n$, thus witnessing that

$$M \models \exists\mu'\exists F' \qquad (F' \subseteq \mu' \text{ is club}, \mathrm{cf}(\mu') = \aleph_n \\ \text{and } \forall\alpha \in F'\forall i < \aleph_n \quad U_{\alpha,i} \cap \mu' \text{ is stationary in } \mu')$$

The same statement is then true in $\mathrm{V}_2[c]$ about $\vec{S}$, by the elementarity of j. So let $F' \in \mathrm{V}_2[c]$, μ' be such that, in $\mathrm{V}_2[c]$, $\mathrm{cf}(\mu') = \aleph_n$, $F' \subseteq \mu'$ is club, $\mathrm{otp}(F') = \aleph_n$, and for all $\alpha \in F'$ and all $i < \aleph_n$, $S_{\alpha,i} \cap \mu'$ is stationary in μ'. Let $\mathbb{Q}$ be the $<\aleph_{n+1}$-directed closed dense suborder of $\mathbb{S} * \mathbb{T}_{n+1}$ mentioned above, and let g be $\mathbb{Q}$-generic over V_1 such that $\mathrm{V}_1[g] = \mathrm{V}_1[\vec{C}][c] = \mathrm{V}_2[c]$. Let $\dot{F}', \dot{\vec{S}} \in \mathrm{V}_1^{\mathbb{Q}}$ be such that $F' = (\dot{F}')^g$ and $\vec{S} = \dot{\vec{S}}^g$, and let $p \in g$ force with respect to $\mathbb{Q}$ that $\dot{F}'$ and $\check{\mu}'$ are as described. Working in V_1, let D be the set of q such that there is a club $E \subseteq \mu'$ such that q forces with respect to $\mathbb{Q}$ that for all $\alpha \in \check{E}$ and all $i < \aleph_n$, $\dot{S}_{\alpha,i}$ reflects at μ. It can be easily shown that X is dense below p, by forming a decreasing sequence of conditions in $\mathbb{Q}$ below any desired strengthening of p, deciding longer and longer initial segments of $\dot{F}'$. The order type of $\dot{F}'$ is forced to be $\aleph_n$, while $\mathbb{Q}$ is $<\aleph_{n+1}$-directed closed, so this process leads to a condition deciding all of $\dot{F}'$, and this gives rise to E. By genericity, there is a condition $q \in g \cap D$. Let $E \subseteq \mu'$, $E \in \mathrm{V}_1$ witness this. Then we have in $\mathrm{V}_1[g] = \mathrm{V}_2[c]$ that for every $\alpha \in E$ and every $i < \aleph_n$, $S_{\alpha,i}$ reflects to μ'. But both E and $\vec{S}$ are in V_2, and stationarity goes down to inner models, so the same statement is true in V_2, as desired. □

I would like to end with a couple of question that may guide future research. The first one summarizes what's left unresolved from the present paper and [4].

Question 5.10. Let κ be a cardinal of countable cofinality. Does SCFA, or BSCFA($\leq\kappa^+$) imply the failure of $\square^*_\kappa$, that there is no better scale on κ, or that there is no good scale on κ?

It was shown in [3] that MM has these consequences, but the situation with SCFA is unclear, and diagonal reflection does not help, by Theorem 5.9.

Question 5.11. Does SCFA + ¬CH imply $\neg\square(\omega_2, \omega_1)$?

PFA and MM each imply $\neg\square(\omega_2, \omega_1)$, while SCFA doesn't. This question asks whether this difference is solely attributable to the fact that SCFA is consistent with CH.

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