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Limit Theorems for Sums of Dependent Random Variables Occurring in Statistical Mechanics

II. Conditioning, Multiple Phases, and Metastability

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Summary. By the use of conditioning, we extend previously obtained results on the asymptotic behavior of partial sums for certain triangular arrays of dependent random variables, known as Curie-Weiss models. These models arise naturally in statistical mechanics. The relation of these results to multiple phases, metastable states, and other physical phenomena is explained.

I. Introduction and Physical Background

This paper continues the analysis, begun in [4], of the asymptotic behavior of sums S_n of certain dependent random variables which occur in statistical mechanics. These random variables are associated with the Curie-Weiss, or mean field model, a lattice model of ferromagnetism [1], [8; §3], [11; Ch. 6].

We briefly indicate how the results of the present paper extend those of its predecessor. The asymptotic behavior of S_n depends crucially upon the nature of the extremal points of a function G which we associate with the model. In [4], a weak law of large numbers-type result for S_n, involving global minima of G, was proved. This is restated here as Theorem 2.1. In addition, in the case that G has a unique global minimum a central limit theorem-type refinement was obtained. Only in special cases is the limit Gaussian. In general it has a density proportional to $\exp(-\lambda x^{2k}/(2k)!)$, $\lambda>0$, $k\in\{1,2,\ldots\}$.

One of the purposes of this paper is to prove such a result for arbitrary G (Theorem 2.2). Furthermore, through the use of conditioning we will see how to extend Theorems 2.1 and 2.2 to results involving *local* minima of G. Our results are stated in Sect. II and proven in Sect. III.

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We spend the next part of this section explaining in elementary terms some of the physical ideas underlying our results. Although these results could have been presented in a strictly mathematical context, we feel that the physical background provides a better setting. These physical ideas include the notions of phase transition, multiple phase, metastable state, and Gibbs free energy.

Let d be a fixed positive integer and Λ a finite subset of $\mathbb{Z}^d$. A ferromagnetic crystal can be described by random variables X_j^Λ which represent the spins, or magnetic moments, of the atoms at sites $j\in\Lambda$; Λ describes the macroscopic shape of the crystal. In the Curie-Weiss model, the joint distribution at fixed temperature $T>0$ of the spin random variables is given by

$$\frac{1}{Z_\Lambda(\beta)}\exp\left(\frac{\beta}{2|\Lambda|}\Big(\sum_{j\in\Lambda}x_j\Big)^2\right)\prod_{j\in\Lambda}d\rho(x_j). \tag{1.1}$$

In (1.1), $\beta=T^{-1}$, $Z_\Lambda(\beta)$ is a normalizing constant (also known as the partition function), $|\Lambda|$ is the cardinality of Λ, and ρ is the distribution of a single spin in the limit $\beta\to 0$. We assume that ρ is in the class $\mathscr{B}$ of non-degenerate Borel probability measures on $\mathbb{R}$ which satisfy

$$\int \exp\left(\frac{bx^2}{2}\right)d\rho(x)<\infty, \qquad \text{all } b>0. \tag{1.2}$$

Here and below all integrals extend over $\mathbb{R}$ unless otherwise stated. In the Curie-Weiss model (1.1), we will take (without loss of generality) $d=1$ and $\Lambda=\{1,\ldots,n\}$, where n is a positive integer. We write n instead of Λ in the notation for the spin random variables and the partition function.

We define $S_n(\beta)=\sum_{i=1}^{n}X_i^n$, which represents the total magnetization of Λ. We are interested in studying the asymptotic behavior of $S_n(\beta)$ in the so-called thermodynamic limit $n\to\infty$. This behavior can be described physically in terms of multiple phases and metastable states. We explain these concepts for a gas-liquid system rather than for a ferromagnetic system because they are easier to visualize in the former case. Afterwards, we return to the Curie-Weiss model. For additional background on the physics, see [3].

Consider a substance in a gaseous state occupying a container at a specified pressure. We assume that a movable piston is attached to the container so that the pressure may be varied at constant temperature with a consequent variation in the volume and density of the substance. The state of the substance is considered stable if it is stable under perturbations; e.g., if the container is shaken momentarily, then the substance will soon return to its original state. In general, there exists a critical temperature T_c such that the following behavior occurs. For $T\geqq T_c$, the stable state of the substance remains a gas, no matter how much it is compressed, the density d constantly increasing as the pressure P is increased. On the other hand, for $T<T_c$, the gas, upon being compressed to a certain pressure $P_0=P_0(T)$ and corresponding density $d_G=d_G(T)$, condenses at constant pressure to a liquid of higher density $d_L=d_L(T)$. The gas and the liquid are two pure phases of the substance and the phenomenon just described is a phase transition. For $d\in(d_G,d_L)$ the two phases

coexist as a mixture, or a multiple phase, which represents a stable state of the substance for the given values of $P = P_0$ and T [7; §2.1]. For later reference, we note the distribution function of the mass density (i.e., mass per unit volume) of the mixed phase. We denote by A_G the proportion of the gaseous pure phase present in the mixed phase and by A_L the analogous quantity for the liquid (so that $A_G + A_L = 1$ and $A_G d_G + A_L d_L = d$). The normalized distribution function of the mass density of the mixed phase is easily seen to be

$$A_G\, \delta(w - d_G) + A_L\, \delta(w - d_L), \tag{1.3}$$

where $\delta(w - m)$ denotes the unit point mass with support at m.

It turns out that for $T < T_c$, other behavior is possible. By exercising care one may compress the gas above the pressure P_0 and density d_G up to a pressure $P_S = P_S(T)$ and corresponding density $d_S = d_S(T)$, without any of the liquid phase appearing. This form of a substance is known as a supersaturated vapor. Now fix a value $P \in (P_0, P_S)$. As explained in [7; §2.2], effects such as surface tension prevent any appreciably large droplets of the liquid phase from spontaneously forming. However, droplets may be induced to form by perturbing the system in one of several ways; e.g., by introducing droplets from the outside (called seeding) or by vigorous shaking, which causes molecules to clump together as droplets. If droplets larger than a critical size, which depends on P and T, do form, then the substance will quickly go over into the stable liquid state corresponding to P and T. Thus, the supersaturated vapor state is unstable under the perturbations just described. It is called a metastable state.

The critical size of droplets (which determines the size of the perturbation) necessary to cause a metastable state to go over to a stable state decreases as P increases. If P is any pressure greater than $P_S(T)$, then the critical size is of the order of a molecular radius and droplets larger than the critical size form spontaneously. Supersaturation cannot be pushed beyond this point. This extreme metastable state corresponding to $P = P_S(T)$ is called a spinodal state. See [10] for additional information on this matter.

There is another possible metastable state of the substance known as a superheated liquid. With care, one may decompress the liquid phase below the pressure P_0 and density d_L down to a pressure $P_H = P_H(T)$ without any of the gaseous phase appearing. Superheating can be discussed in a manner similar to supersaturation if one considers bubbles instead of droplets.

In thermodynamics, these phenomena can be described in terms of the Gibbs free energy. We emphasize one relevant fact. For fixed P and T, a stable state is a state which minimizes the free energy over all possible states which the substance may assume [7; p. 24]. On the other hand, a metastable state minimizes the free energy over all states subject to an additional constraint; e.g., in the case of supersaturated vapor, this state minimizes the free energy over all states constrained so that no large driplets occur. One may say that a stable state is a global minimum of the free energy and a metastable state, a local minimum.

The goal of statistical mechanics is to derive the macroscopic properties of substances from the laws of molecular interactions. In classical statistical mechanics, a stable state of a substance is described by a probability distribution on phase space, the space of all possible configurations of the molecules constituting the substance. For the Curie-Weiss model, this distribution is given by (1.1). In [9], Penrose and Lebowitz address the problem of basing a rigorous theory of metastability upon classical statistical mechanics. As a key step in realizing metastable states, it is proposed there that the points in phase space be suitably restricted. For example, in the case of a supersaturated vapor, one should consider only configurations where no droplets occur. This restriction is implemented probabilistically by means of conditioning. For a rigorous treatment of metastable states in a statistical mechanical model of a gas-liquid system, see [9].

We return to the Curie-Weiss model. The analogue of the Gibbs free energy in this model is the canonical free energy $f=f_\rho(\beta)$, defined by the formula

$$f(\beta)=-\frac{1}{\beta}\lim_{n\to\infty}\frac{1}{n}\ln Z_n. \tag{1.4}$$

We define the functions $\phi=\phi_\rho(s)$ and $G=G_\rho(\beta,s)$ by the formulae

$$\phi_\rho(s)=\ln\int\exp(sx)\,d\rho(x) \tag{1.5}$$

and

$$G_\rho(\beta,s)=\frac{\beta s^2}{2}-\phi(\beta s). \tag{1.6}$$

It is not hard to show that

$$f_\rho(\beta)=\frac{1}{\beta}\inf\{G_\rho(\beta,s): s \text{ real}\}; \tag{1.7}$$

see [12; p. 100] for a special case. Until further notice, we fix a value of $\beta>0$. We shall drop β in the notation for S_n and G when there is no danger of confusion; similarly for ρ in the notation for f, g, and G. In [4; Lemma 3.1], G was proved to be real analytic and to tend to $+\infty$ as $|s|\to\infty$. Thus, G must have global minima, which can be only finite in number. Define $C=C_\rho$ as the discrete, non-empty set of real numbers m such that m is either a global minimum, a local minimum, or a point of inflection of G. If $m\in C$ is a minimum of G (global or local), then there exists a positive integer $k=k(m)$ and a positive real number $\lambda=\lambda(m)$ such that

$$G(s)=G(m)+\frac{\lambda(m)(s-m)^{2k}}{(2k)!}+O((s-m)^{2k+1}) \quad \text{as } s\to m. \tag{1.8}$$

If $m\in C$ is a point of inflection of G, then an expansion like (1.8) holds but with $k=k(m)$ a half-integer greater than or equal to 3/2 and $\lambda=\lambda(m)$ a non-zero real number. In either case, $k(m)$ and $\lambda(m)$ are called the type and strength, respectively, of the extremal point m. We define the maximal type k^* of G by the formula

$$k^*=\max\{k(m): m \text{ a global minimum of } G\}.$$

Note that if $k=1$,

$$\lambda(m)=\beta-\beta^2\phi''(\beta m), \tag{1.9}$$

while if $k>1$, $\lambda(m)=-\beta^{2k}\phi^{(2k)}(\beta m)$.

Although the function G has no apparent physical significance, we will soon see a direct analogy with the situation for a liquid-gas system. Global minima of G of maximal type will correspond to stable states (with multiple global minima representing a mixed phase and a unique global minimum a pure phase), local minima of G will correspond to metastable states, and points of inflection of G will correspond to spinodal states.

First suppose that $G(\beta,\cdot)$ has θ global minima of maximal type at $m_1,\ldots,m_\theta$, $\theta\in\{1,2,\ldots\}$ (possibly together with other global minima of non-maximal type). It was proved in [4] and is restated in Theorem 2.1 below that

$$\frac{S_n(\beta)}{n} \xrightarrow{\mathscr{D}} \sum_{j=1}^{\theta} b_j\,\delta(w-m_j), \tag{1.10}$$

where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution as $n\to\infty$ and the strictly positive weights b_j can be written explicitly in terms of the types $k(m_j)$ and the strengths $\lambda(m_j)$. The right-hand side of (1.10) clearly resembles the distribution function (1.3) which describes a gas-liquid multiple phase. In order to reinforce the connection, we point out that the notion of mass density for a gas-liquid system is replaced by magnetization per site for a ferromagnetic system. Since $n^{-1}S_n$ is the (random) magnetization per site for a finite system of size n, we may interpret (1.10) as saying that each global minimum m_j of maximal type corresponds to a pure phase of the Curie-Weiss system in the thermodynamic limit; the magnetization per site of this pure phase is m_j. If $\theta=1$, then we have a unique pure phase. If $\theta>1$, then we have multiple phases, each occurring in proportion to b_j. Thus, S_n obeys a weak law of large numbers if and only if the Curie-Weiss model in the thermodynamic limit has a unique pure phase. We note that only the global minima of maximal type contribute to the limit distribution in (1.10). (Global minima of non-maximal type correspond to stable states which do not contribute to this limit distribution and which should thus be thought of as being absent from the mixed phase.)

At the end of this introduction we describe a class of measures $\rho\in\mathscr{B}$ with the following properties: there exists a value β_c of β, the inverse critical temperature, such that G_ρ has a unique global minimum at the origin for $0<\beta\leqq\beta_c$ and exactly two global minima, of equal type, for $\beta>\beta_c$. This means that as T decreases past $T_c=\beta_c^{-1}$, the stable state of our system switches from a pure phase with zero magnetization per site to a mixture of two pure phases each possessing a non-zero magnetization per site. Thus, the system exhibits a phase transition at T_c.

Returning to the general case of $\rho\in\mathscr{B}$, we can go further and analyze the fluctuations of $S_n(\beta)/n$ around its magnetization in each pure phase. Given Q_n a sequence of random variables, F a probability distribution on $\mathbb{R}$, and $b\in[0,1]$, we write $Q_n \xrightarrow{v} bF$ to mean that for any continuous function f on $\mathbb{R}$ vanishing at $\pm\infty$,

$$\lim_{n\to\infty} E(f(Q_n)) = b\int f(w)\,dF(w);$$

this sense of convergence is known as vague convergence [2; §4.3]. We show in Theorem 2.2 that

$$\frac{S_n(\beta)-n m_j}{n^{1-1/2k(m_j)}} \xrightarrow{v} b_j F_{k(m_j),\lambda(m_j),\beta}, \tag{1.11}$$

where F is an explicitly determined probability measure depending on $k(m_j)$, $\lambda(m_j)$ and β. F is Gaussian if and only if $k(m_j)=1$. The number b_j, which is the same number appearing in (1.10), is the proportion of our system in the phase with magnetization m_j. This result was proven in [4] in the special case of a unique pure phase.

We now show the connection between a local minimum m of G and a metastable state. According to Penrose-Lebowitz, one realizes this state by modifying the joint distribution (1.1). We denote by 1_T the indicator function of a subset T of $\mathbb{R}^n$. We define the new joint distribution

$$\frac{1}{Z_{n,a}(\beta)} 1_{\left\{\frac{\sum x_j}{n} \in [m-a, m+a]\right\}}(x_1, \ldots, x_n) \exp\left(\beta \frac{(\sum x_j)^2}{2n}\right) \prod d\rho(x_j), \tag{1.12}$$

where $a>0$ is given and $Z_{n,a}(\beta)$ is the new normalizing constant. Instead of allowing the spins to take on arbitrary real values in the support of ρ, (1.12) restricts the magnetization per site to be in an a-neighborhood of m. Integration with respect to the distribution (1.12) corresponds to conditioning.

Theorem 2.3 states the analogue of (1.10) for m a local minimum of G: conditional upon $n^{-1}S_n$ being near m,

$$\frac{S_n(\beta)}{n} \xrightarrow{\mathscr{D}} \delta(w-m). \tag{1.13}$$

Thus, under the constraint that the (random) magnetization per site be near m for each $n \in \{1, 2, \ldots\}$, the state in the thermodynamic limit will have magnetization per site precisely equal to m. This is consistent with the identification between a local minimum of G and a metastable state. We also study the fluctuations of S_n in a metastable state corresponding to a local minimum m of G. We prove that conditional upon $n^{-1}S_n$ being near m,

$$\frac{S_n(\beta)-nm}{n^{1-1/2k(m)}} \xrightarrow{\mathscr{D}} F_{k(m), \lambda(m), \beta}, \tag{1.14}$$

where the latter measure is the same as in (1.11). In order to handle a spinodal state, which corresponds to a point of inflection of G, say at m, we replace the indicator function in (1.12) by the indicator function either of the set $\{n^{-1}\sum x_j \in [m, m+a]\}$ or of the set $\{n^{-1}\sum x_j \in [m-a, m]\}$ according to whether $\lambda(m)$ is positive or negative. Analogues of (1.13)–(1.14) are then valid.

We illustrate these limit theorems by finding examples of measures ρ which satisfy the various hypotheses. Let ρ be any measure in $\mathscr{B}$ such that for (say) $\beta = 1$, G_ρ has a unique global minimum at the origin of type $k \geq 2$. The existence of such measures was proved in [4]. Then for $\beta \in (0, 1)$, G_ρ will have a unique global minimum at the origin, but of type 1, and so $S_n(\beta)$ satisfies a weak law of large numbers and a central limit theorem. For $\beta = 1$, the weak law of large numbers still holds; however, concerning fluctuations, $S_n(\beta)$ must be scaled by $n^{1/c}$, $c = 2k/(2k-1) < 2$, for a non-trivial limit to exist; this limit is non-Gaussian. Concerning the situation for $\beta > 1$, we single out a class of measures which are of some physical significance. Assume that ρ is any symmetric measure in $\mathscr{B}$ which satisfies the

Griffiths-Hurst-Sherman inequality [6], [5] ($d^3\phi_\rho(s)/ds^3 \leq 0$ for all $s \geq 0$, where ϕ_ρ is defined in (1.5)) and which has variance one. An example is the symmetric Bernoulli measure $\frac{1}{2}[\delta(x-1)+\delta(x+1)]$. One may then show that for $\beta \in (0,1]$ G_ρ has the properties mentioned at the beginning of this paragraph. Furthermore, for $\beta > 1$, G_ρ will have exactly two global minima, each of type 1, at points $\pm m$, $m = m(\beta) > 0$. (Thus, $\beta_c = 1$ for the corresponding Curie-Weiss model.) For this same ρ and any h real, let $d\rho_h(x)$ be the probability measure proportional to $\exp(hx)\,d\rho(x)$. The parameter h corresponds to an external magnetic field acting at each site of the ferromagnet. One may prove that for fixed $\beta > 1$ there exists a number $\tilde{h} > 0$ such that G_{ρ_h} has two global minima for $h = 0$, a global minimum and a local minimum for $h \in (-\tilde{h}, \tilde{h})$, a global minimum and a point of inflection for $h = \pm\tilde{h}$, and a single global minimum for $|h| > \tilde{h}$. In Theorem 2.6 and in the discussion which follows that theorem, we state more general results concerning the existence of measures ρ such that G_ρ has prescribed extremal properties.

II. Statement of Results

Our first theorem, proved as Theorem 3.8 of [4], is included for reference.

Theorem 2.1. *Given $\rho \in \mathscr{B}$, let $m_1, \ldots, m_\theta$ denote the set of global minima of maximal type k^* of G_ρ. Define*

$$\bar{b}(m_i) = \lambda(m_i)^{-1/2k^*}, \qquad b(m_i) = \bar{b}(m_i) \Big/ \sum_{j=1}^{\theta} \bar{b}(m_j),$$

$$d\tau(w) = \sum_{j=1}^{\theta} b(m_j)\,\delta(w - m_j).$$

Then

$$\frac{S_n}{n} \xrightarrow{\mathscr{D}} \tau. \tag{2.1}$$

Remark. This was stated in [4] for $\beta = 1$.

Our next theorem was proved in [4] for the special case that G_ρ has a unique global minimum.

Theorem 2.2. *Given $\rho \in \mathscr{B}$, let m be one of the global minima of maximal type k^* of G_ρ. Then*

$$\frac{S_n - nm}{n^{1-1/2k^*}} \xrightarrow{\mathscr{D}} b(m)\,F_{k^*,\lambda(m),\beta},$$

where $F_{k,\lambda,\beta}$ is defined by

$$dF_{k,\lambda,\beta} = \begin{cases} \dfrac{\exp(-w^2/2\sigma^2)\,dw}{\int \exp(-w^2/2\sigma^2)\,dw} & \text{if } k = 1, \\[2ex] \dfrac{\exp(-\lambda w^{2k}/(2k)!)\,dw}{\int \exp(-\lambda w^{2k}/(2k)!)\,dw} & \text{if } k > 1. \end{cases} \tag{2.2}$$

Here, $\sigma^2=\lambda^{-1}-\beta^{-1}$ *so that for* $\lambda=\lambda(m)$ *as in* (1.9), $\sigma^2=([\phi''(\beta m)]^{-1}-\beta)^{-1}$. *If* m *is not a global minimum of* G_ρ *of maximal type, then* $(S_n-nm)/n^{1-c^{-1}}\to 0$ *for any* $c>1$.

Remark. Concerning the form of (2.2) for the case $k=1$, recall (1.9). Also note that if m is a non-unique global minimum of maximal type, then the limiting distribution of $(S_n-nm)/n^{1-1/2k^*}$ is defective.

Theorems 2.3–2.5 are analogues of Theorems 2.1–2.2 obtained by conditioning. For a random variable Q and an event A, we write $P(Q\in dw|A)$ to denote the measure $\mu(dw)$ defined by $\mu(\Gamma)=P(Q\in\Gamma|A)$ for Borel sets Γ in $\mathbb{R}$. Given F a probability distribution on $\mathbb{R}$, we write $(Q_n|A_n)\xrightarrow{\mathscr{L}}F$ to mean $P(Q_n\in dw|A_n)$ converges weakly to $dF(w)$.

Theorem 2.3. *Given* $\rho\in\mathscr{B}$, *for any* $m\in C_\rho$ *there exists a number* $A=A(m)>0$ *such that the following holds. If* m *is a local or a global minimum of* G_ρ, *then for any* $a\in(0,A)$

$$\left(\frac{S_n}{n}\,\middle|\,\frac{S_n}{n}\in[m-a,m+a]\right)\xrightarrow{\mathscr{L}}\delta(w-m). \tag{2.3}$$

If m *is a point of inflection of* G_ρ *and* $\lambda(m)>0$, *then for any* $a\in(0,A)$,

$$\left(\frac{S_n}{n}\,\middle|\,\frac{S_n}{n}\in[m,m+a]\right)\xrightarrow{\mathscr{L}}\delta(w-m). \tag{2.4}$$

If $\lambda(m)<0$, *then we condition on* $n^{-1}S_n\in[m-a,m]$ *and an analogous result holds.*

Theorem 2.4. *Given* $\rho\in\mathscr{B}$, *choose* $m\in C_\rho$ *a global or a local minimum of* G_ρ. *Then for any* $a\in(0,A)$, A *as above,*

$$\left(\frac{S_n-nm}{n^{1-1/2k}}\,\middle|\,\frac{S_n}{n}\in[m-a,m+a]\right)\xrightarrow{\mathscr{L}}F_{k(m),\lambda(m),\beta}, \tag{2.5}$$

where $F_{k,\lambda,\beta}$ *is given by* (2.2).

Theorem 2.5. *Suppose* $\rho\in\mathscr{B}$ *is chosen so that* G_ρ *has a point of inflection* m *with* $\lambda(m)>0$. *Then for any* $a\in(0,A)$, A *as above,*

$$\left(\frac{S_n-nm}{n^{1-1/2k(m)}}\,\middle|\,\frac{S_n}{n}\in[m,m+a]\right)\xrightarrow{\mathscr{L}}(F_{k(m),\lambda(m)}|[0,\infty)), \tag{2.6}$$

where for $k\geqq 3/2$ *and* $\lambda>0$, $(F_{k,\lambda}|[0,\infty))$ *is defined by*

$$d(F_{k,\lambda}|[0,\infty))=\frac{1_{[w\geq 0]}(w)\exp(-\lambda w^{2k}/(2k)!)\,dw}{\int_0^\infty\exp(-\lambda w^{2k}/(2k)!)\,dw}.$$

If $\lambda(m)<0$, *then we condition upon* $n^{-1}S_n\in[m-a,m]$ *and an analogous result holds with the limit distribution supported on* $(-\infty,0]$.

We next discuss the existence of measures $\rho\in\mathscr{B}$ such that G_ρ has prescribed extremal points. The case of global minima is covered by the next theorem, the proof of which will appear elsewhere.

Theorem 2.6. *Let η be a positive integer; $m_1, \ldots, m_\eta$, η distinct real numbers; $k_1, \ldots, k_\eta$, η positive integers. There exists a unique discrete measure $\bar{\rho} \in \mathscr{B}$ which is supported on $(k_1 + \ldots + k_\eta)$ points such that $G_{\bar{\rho}}$ has a global minimum at each m_j of type k_j, $j = 1, \ldots, \eta$. If $\rho \in \mathscr{B}$ is distinct from $\bar{\rho}$ and G_ρ has a global minimum at each m_j of type k_j, $j = 1, \ldots, \eta$, then ρ is not supported on fewer than $(k_1 + \ldots + k_\eta + 1)$ points and $\lambda_\rho(m_j) < \lambda_{\bar{\rho}}(m_j)$, $j = 1, \ldots, \eta$.*

An analogous result covering local minima and points of inflection is lacking. On the other hand, it is not difficult to generate examples. Concerning local minima, suppose that in Theorem 2.6 $m_1 < m_2 < \ldots < m_\eta$. Given h real, let $d\bar{\rho}_h(x)$ be the probability measure proportional to $\exp(hx) \cdot d\bar{\rho}(x)$. By continuity, $G_{\bar{\rho}_h}$ has local minima near m_j, $j = 1, 2, \ldots, \eta - 1$, provided h is positive and sufficiently small. We are also able to construct $\rho \in \mathscr{B}$ such that G_ρ has countably infinitely many local minima. The existence of $\rho \in \mathscr{B}$ such that G_ρ has a point of inflection was pointed out at the end of Sect. 1.

III. Proofs

We first prove Theorem 2.4 in the case $k(m) > 1$; the case $k(m) = 1$ is handled similarly. We then outline the proof of Theorem 2.5, pointing out the relevant differences from the proof of Theorem 2.4. Theorem 2.3 follows from Theorems 2.4 and 2.5; it could also be obtained directly by using similar arguments. Finally we prove Theorem 2.2. The main technical tools for proving Theorem 2.4 are a Gaussian transform and a transfer principle applied to the characteristic function of

$$P\left(\frac{S_n - nm}{n^{1-1/2k(m)}} \in dx \,\middle|\, \frac{S_n}{n} \in [m-a, m+a]\right).$$

The Gaussian transform simplifies the characteristic function at the expense of introducing a new variable w; the transfer principle transfers the restriction on $n^{-1}S_n$ to a restriction on w. Dominated convergence does the rest.

To ease the notation, we set $\gamma = 1/2k(m)$ and $\beta = 1$. General β can be handled analogously. Given $k(m) > 1$, we must find $A > 0$ such that for each r real and any $a \in (0, A)$

$$\frac{\displaystyle\int_{\left|\frac{\sum x_j}{n} - m\right| \leq a} \exp\left(ir\left(\frac{\sum x_j - nm}{n^{1-\gamma}}\right)\right) \exp\left(\frac{(\sum x_j)^2}{2n}\right) \prod d\rho(x_j)}{\displaystyle\int_{\left|\frac{\sum x_j}{n} - m\right| \leq a} \exp\left(\frac{(\sum x_j)^2}{2n}\right) \prod d\rho(x_j)} \tag{3.1}$$

tends, as $n \to \infty$, to

$$\frac{\int \exp(irw) \exp(-\lambda w^{2k}/(2k)!)\, dw}{\int \exp(-\lambda w^{2k}/(2k)!)\, dw},$$

where $k=k(m)$ and $\lambda=\lambda(m)$. Defining

$$d\rho_m(x)=\exp(mx)\,d\rho(x)/\int\exp(mx)\,d\rho(x),$$

we rewrite (3.1) as

$$\frac{\displaystyle\int_{\left|\frac{\sum x_j}{n}-m\right|\leq a}\exp\left(ir\left(\frac{\sum x_j-nm}{n^{1-\gamma}}\right)+\frac{1}{2}\left(\frac{\sum x_j-nm}{\sqrt{n}}\right)^2\right)\prod d\rho_m(x_j)}{\displaystyle\int_{\left|\frac{\sum x_j}{n}-m\right|\leq a}\exp\left(\frac{1}{2}\left(\frac{\sum x_j-nm}{\sqrt{n}}\right)^2\right)\prod d\rho_m(x_j)}$$

$$=\frac{\displaystyle\int_{|u|\leq a}\exp\left(irn^{\gamma}u+\frac{n}{2}u^2\right)dv_n(u)}{\displaystyle\int_{|u|\leq a}\exp\left(\frac{n}{2}u^2\right)dv_n(u)}\,; \tag{3.2}$$

v_n in (3.2) denotes the distribution of the random variable

$$U_n=\left(\sum_{j=1}^{n}x_j-nm\right)\Big/n\quad\text{on}\quad\left(\mathbb{R}^n,\prod_{j=1}^{n}d\rho_m(x_j)\right).$$

The Gaussian transform replaces $\exp\left(\frac{n}{2}u^2\right)$ in the numerator and denominator of (3.2) by

$$(n/2\pi)^{\frac{1}{2}}\int\exp(-nw^2/2)\exp(nwu)\,dw.$$

After we cancel the terms $(n/2\pi)^{\frac{1}{2}}$ and make the change of variable $w+\frac{ir}{n^{1-\gamma}}\to w$, the right-hand side of (3.2) becomes

$$\frac{\exp\left(\frac{r^2}{2n^{1-2\gamma}}\right)\int\exp(irn^{\gamma}w)\exp(-nw^2/2)\displaystyle\int_{|u|\leq a}\exp(nwu)\,dv_n\,dw}{\int\exp(-nw^2/2)\displaystyle\int_{|u|\leq a}\exp(nwu)\,dv_n\,dw}\,. \tag{3.3}$$

The change of variables is justified by the analyticity of the integrands in (3.3) as functions of w complex and the rapid decrease of these integrands to zero as $|\mathrm{Re}\,w|\to\infty$, $|\mathrm{Im}\,w|\leq|r|\,n^{\gamma-1}$. Since $k(m)>1$, we have that $1-2\gamma>0$ and thus $\exp(r^2/2n^{1-2\gamma})\to 1$ as $n\to\infty$ for each r real. Hence, we may neglect this factor in (3.3) for the rest of the proof. (If $k(m)=1$, then this factor contributes to the limit of (3.1) which leads to the special form of (2.2) for the case $k=1$.)

Transfer Principle. There exists $\hat{B}>0$ depending only on ρ such that for each $B\in(0,\hat{B})$ and for each $a\in(0,B/2)$ and each r real, there exists $\delta=\delta(a,B)>0$ such that as $n\to\infty$

$$\int \exp(irn^{\gamma}w)\exp\left(-\frac{n}{2}w^2\right)\int_{|u|\leq a}\exp(nwu)\,d\nu_n\,dw$$
$$=\int_{|w|\leq B}\exp(irn^{\gamma}w)\exp\left(-\frac{n}{2}w^2\right)\int\exp(nwu)\,d\nu_n\,dw+O(e^{-n\delta}). \tag{3.4}$$

We prove the transfer principle after we have completed the rest of the proof of the theorem. We accomplish the latter by finding a positive number $\bar{B}\leqq\hat{B}$ such that for each r real

$$\int_{|w|\leq \bar{B}n^{\gamma}}\exp(irw)\exp\left(-n^{1-2\gamma}\frac{w^2}{2}\right)\int\exp(n^{1-\gamma}wu)\,d\nu_n\,dw$$
$$\to\int\exp(irw)\exp(-\lambda w^{2k}/(2k)!)\,dw. \tag{3.5}$$

Once we have found $\bar{B}$, we set A in Theorem 2.4 equal to $\bar{B}/2$. The theorem now follows. If $a\in(0,\bar{B}/2)$, then at the price of an exponentially small error, the quotient of integrals in (3.3) can be replaced by a quotient of integrals as on the right-hand side of (3.4) with B replaced by $\bar{B}$ (the numerator with r as given, the denominator with $r=0$); the change of variables $n^{\gamma}w\to w$ and (3.5) do the rest. The proof of (3.5) is essentially the same as the proof of (3.30) in [4], but to keep our presentation self-contained, we give the details.

We use the previously defined function $\phi(y)=\phi_{\rho}(y)$, which for $\beta=1$ is given by the formula

$$\phi_{\rho}(y)=\ln\int\exp(yx)\,d\rho(x)=\frac{y^2}{2}-G_{\rho}(y). \tag{3.6}$$

Since m is a minimum of G, we have

$$\phi'(m)=m. \tag{3.7}$$

We now express in terms of ϕ the function of w multiplying $\exp(irw)$ on the left-hand side of (3.5):

$$\exp\left(-n^{1-2\gamma}\frac{w^2}{2}\right)\int\exp(n^{1-\gamma}wu)\,d\nu_n(u)$$
$$=\frac{\exp\left(-n^{1-2\gamma}\frac{w^2}{2}\right)\int\exp\left(\frac{w}{n^{\gamma}}\left(\sum x_j-nm\right)\right)\exp\left(m\sum x_j\right)\prod d\rho(x_j)}{\int\exp\left(m\sum x_j\right)\prod d\rho(x_j)}$$
$$=\exp\left(-n^{1-2\gamma}\frac{w^2}{2}\right)\exp\left(n\left\{\phi\left(m+\frac{w}{n^{\gamma}}\right)-\phi(m)-m\frac{w}{n^{\gamma}}\right\}\right)$$
$$=\exp\left(-n\left\{\frac{w^2}{2n^{2\gamma}}-\left(\phi\left(m+\frac{w}{n^{\gamma}}\right)-\phi(m)-\phi'(m)\frac{w}{n^{\gamma}}\right)\right\}\right). \tag{3.8}$$

To obtain the last equation, we used (3.7). By (1.8) and (3.6), we write

$$n\left(\phi\left(m+\frac{w}{n^{\gamma}}\right)-\phi(m)-\phi'(m)\frac{w}{n^{\gamma}}\right)$$
$$=\frac{n}{2}\left(\frac{w}{n^{\gamma}}\right)^2-n\frac{\lambda}{(2k)!}\left(\frac{w}{n^{\gamma}}\right)^{2k}+nO\left(\left(\frac{w}{n^{\gamma}}\right)^{2k+1}\right)$$
$$=\frac{n}{2}\frac{w^2}{n^{2\gamma}}-\frac{\lambda}{(2k)!}w^{2k}+O\left(\frac{w^{2k+1}}{n^{\gamma}}\right).$$

Hence, for each w real, the last expression in (3.8) tends to $\exp(-\lambda w^{2k}/(2k)!)$ as $n\to\infty$. Let ε be any number which satisfies $\varepsilon\in(0,\lambda/(2k)!)$. We may find a number $\tilde{B}>0$ such that

$$n\left(\phi\left(m+\frac{w}{n^{\gamma}}\right)-\phi(m)-\phi'(m)\frac{w}{n^{\gamma}}\right)\leqq\frac{nw^2}{n^{2\gamma}}-\left(\frac{\lambda}{(2k)!}-\varepsilon\right)w^{2k}$$

whenever $\left|\frac{w}{n^{\gamma}}\right|\leqq\tilde{B}$. Hence the last expression in (3.8) is bounded by

$$\exp\left(-\left(\frac{\lambda}{(2k)!}-\varepsilon\right)w^{2k}\right)\qquad\text{whenever } |w/n^{\gamma}|\leqq\tilde{B}.$$

Setting $\bar{B}=\min(\hat{B},\tilde{B})$, we obtain (3.5) by the dominated convergence theorem.

Proof of Transfer Principle. We shall find $\hat{B}>0$ such that for each $B\in(0,\hat{B})$ and each $a\in(0,B/2)$, there exists $\delta=\delta(a,B)$ such that

$$\int_{|w|>B}\exp\left(-\frac{n}{2}w^2\right)\int_{|u|\leqq a}\exp(nwu)\,dv_n\,dw=O(e^{-n\delta})\tag{3.9}$$

and

$$\int_{|w|\leqq B}\exp\left(-\frac{n}{2}w^2\right)\int_{|u|>a}\exp(nwu)\,dv_n\,dw=O(e^{-n\delta})\tag{3.10}$$

as $n\to\infty$. The proof of (3.9) is easy. For any $B>0$, any $a\in(0,B/2)$, the left-hand side of (3.9) is bounded by

$$2\int_B^{\infty}\exp\left(-n\left(\frac{w^2}{2}-aw\right)\right)dw\leqq2\int_B^{\infty}\exp\left(-nw\left(\frac{B}{2}-a\right)\right)dw=O(e^{-n\delta_1}),\tag{3.11}$$

with $\delta_1=B\left(\frac{B}{2}-a\right)$. For (3.10), we have to work harder. It is convenient to introduce the function $\phi^*=\phi^*_\rho$, known as the Legendre transformation of the function ϕ:

$$\phi^*_\rho(v)=\sup\{vy-\phi_\rho(y):\ y\text{ real}\},\qquad v\text{ real}.\tag{3.12}$$

The following lemma is proved later. For an open interval J, we denote by $\bar{J}$ its closure and by J^c the complement of the closure. We define $e^{-\infty}=0$.

Lemma. *The function ϕ^* is convex, finite, and smooth (i.e., real analytic) on a certain open (possibly unbounded) interval J containing m, $\phi^* = +\infty$ on $\bar{J}^c$, and $(\phi^*)'$ is strictly increasing on J. For any $u > 0$,*

$$\text{Prob}\{U_n > u\} \leqq \exp(-n\{\phi^*(m+u) - \phi^*(m) - (\phi^*)'(m)u\}). \tag{3.13}$$

There exists a number $u_0 > 0$ such that for all $u \in (0, u_0)$

$$(\phi^*)'(m+u) - (\phi^*)'(m) = u + \xi(u) \quad \text{with } \xi(u) > 0. \tag{3.14}$$

We return to the proof of (3.10). The left-hand side of (3.10) is bounded by

$$2B \sup_{|w| \leqq B} \int_{|u| > a} \exp\left(-n\left(\frac{w^2}{2} - wu\right)\right) d\nu_n(u). \tag{3.15}$$

The integral breaks up into one over (a, ∞) and another over $(-\infty, -a)$. We work with the first; the second is handled similarly. Integrating by parts, we have

$$\begin{aligned}
&\sup_{|w| \leqq B} \int_a^\infty \exp\left(-n\left(\frac{w^2}{2} - wu\right)\right) d\nu_n(u) \\
&\quad \leqq \sup_{|w| \leqq B} \exp\left(-n\left(\frac{w^2}{2} - wa\right)\right) \text{Prob}\{U_n > a\} \\
&\qquad + nB \sup_{|w| \leqq B} \int_a^\infty \exp\left(-n\left(\frac{w^2}{2} - wu\right)\right) \text{Prob}\{U_n > u\}\, du.
\end{aligned} \tag{3.16}$$

Using (3.13), we next bound $\text{Prob}\{U_n > u\}$, where $u \geqq a$. We claim that

$$\phi^*(m+u) - \phi^*(m) - (\phi^*)'(m)u \geqq \begin{cases} u^2/2 + \theta_1, & \text{for } a \leqq u \leqq u_0, \\ u\theta_2, & \text{for } u > u_0, \end{cases} \tag{3.17}$$

where $\theta_1 = \int_0^a \xi(t)\,dt > 0$ and $\theta_2 = \xi(u_0/2)/2 > 0$. We prove (3.17) for $m+u$ in the interval J. If $J = \mathbb{R}$, then this takes care of (3.17). If J is a proper subset of $\mathbb{R}$, then (3.17) will hold for all $m+u$ in $\bar{J}$ by the convexity of ϕ^* and for all $m+u$ in $\bar{J}^c$ since for these values $\phi^*(m+u) = +\infty$. If $a \leqq u \leqq u_0$, then by (3.14)

$$\begin{aligned}
&\phi^*(m+u) - \phi^*(m) - (\phi^*)'(m)u \\
&\quad = \int_0^u ((\phi^*)'(m+t) - (\phi^*)'(m))\,dt \\
&\quad = \int_0^u (t + \xi(t))\,dt = u^2/2 + \int_0^u \xi(t)\,dt \geqq u^2/2 + \theta_1.
\end{aligned}$$

This is the first line of (3.17). If $u > u_0$, then the monotone increase of $(\phi^*)'$ together with (3.14) imply that

$$(\phi^*)'(m+t) - (\phi^*)'(m) \geqq (\phi^*)'\left(m + \frac{u_0}{2}\right) - (\phi^*)'(m) \geqq \xi(u_0/2) \quad \text{for } u \geqq t \geqq u_0/2.$$

Thus, if $u \geqq u_0$, then

$$\int_0^u ((\phi^*)'(m+t)-(\phi^*)'(m))\,dt$$

$$\geqq \int_{u_0/2}^{u} ((\phi^*)'(m+t)-(\phi^*)'(m))\,dt$$

$$\geqq (u-u_0/2)\,\xi(u_0/2) \geqq u\theta_2 .$$

This is the second line of (3.17). Now pick $\hat{B}$ to satisfy $0<\hat{B}<\theta_2$ and take any $B\in(0,\hat{B})$. Using (3.13) and (3.17), we may bound the last term in (3.16) by

$$nB \sup_{|w|\leqq B} \int_a^{u_0} \exp\left(-n\frac{w^2}{2}+nwu-n\frac{u^2}{2}-n\theta_1\right) du$$

$$+nB \sup_{|w|\leqq B} \int_{u_0}^{\infty} \exp\left(-n\frac{w^2}{2}+nwu-nu\theta_2\right) du$$

$$=O(n\exp(-n\theta_1))+O(\exp\{-nu_0(\theta_2-B)\})$$

$$=O(e^{-n\delta_2}),$$

where $\delta_2=\min(\theta_1/2, u_0(\theta_2-B))$. The bound $O(n\exp(-n\theta_1))$ follows from the inequality $(-nw^2/2+nwu-nu^2/2)\leqq 0$ for all u, w real. The term in (3.16) involving $\mathrm{Prob}\{U_n>a\}$ is handled similarly. We have thus proved (3.9)-(3.10) with $\delta=\min(\delta_1,\delta_2)$; δ_1 is defined after (3.11). The proof of the transfer principle will be complete once we have proved the lemma.

Proof of Lemma. As the supremum of a family of affine functions of v, ϕ^* is convex. The function ϕ' is strictly monotone increasing since

$$\phi''(y)=\int\left(y-\frac{\int y\exp(yx)\,d\rho(x)}{\int \exp(yx)\,d\rho(x)}\right)^2 \frac{\exp(yx)\,d\rho(x)}{\int\exp(yx)\,d\rho(x)}>0$$

for all y real. We denote by $(\phi')^{-1}$ the inverse function of ϕ'. By (3.12), $|\phi^*(v)|<\infty$ if $v=\phi'(\tilde{y})$ for some $\tilde{y}$ real (which is then unique) and for such v

$$\phi^*(v)=v\tilde{y}-\phi(\tilde{y}), \quad (\phi^*)'(v)=\tilde{y}=(\phi')^{-1}(v). \tag{3.18}$$

This shows that ϕ^* is smooth (real analytic). Since by (3.7) $m=\phi'(m)$, we conclude that $\phi^*(m)<\infty$. By calculus, one can prove that $\phi^*(v)=+\infty$ for v in the complement of the closure of the range of ϕ'. Thus, the interval J in the lemma is the range of the function ϕ'. This proves the first sentence of the lemma.

Let μ by any measure in $\mathscr{B}$. The proof of (3.13) starts from the inequality

$$\int_{\frac{\Sigma x_j}{n}>v} \prod_{j=1}^{n} d\mu(x_j) \leqq \exp(-n\phi_\mu^*(v)) \quad \text{whenever } v>\int x\,d\mu(x), \tag{3.19}$$

which is a consequence of Čebyšev's inequality. Indeed, the left-hand side is bounded by $\exp[-n(vy-\phi_\mu(y))]$ for any $y>0$ and thus by

$$\exp[-n\cdot\sup\{vy-\phi_\mu(y): y>0\}].$$

This equals the right-hand side of (3.19) since whenever $v>\int x\,d\mu(x)$, the supremum in (3.12) (with $\rho=\mu$) is taken on for $y>0$. We return to the proof of (3.13). Because

$$\text{Prob}\{U_n>u\}=\int_{\frac{\Sigma x_j}{n}>m+u}\prod d\rho_m(x_j),$$

$$\int x\,d\rho_m(x)=\frac{\int x\,e^{mx}\,d\rho(x)}{\int e^{mx}\,d\rho(x)}=\phi'(m)=m,$$

(3.19) implies that

$$\text{Prob}\{U_n>u\}\leqq\exp(-n\phi^*_{\rho_m}(m+u)).$$

Now (3.13) follows since

$$\begin{aligned}
&\phi^*_{\rho_m}(m+u)\\
&\quad=\sup_y\{(m+u)y-\ln\int\exp((y+m)x)\,d\rho(x)+\ln\int\exp(mx)\,d\rho(x)\}\\
&\quad=\sup_y\{(m+u)(y+m)-(m+u)m\\
&\qquad-\ln\int\exp((y+m)x)\,d\rho(x)+\ln\int\exp(mx)\,d\rho(x)\}\\
&\quad=-m^2+\phi(m)-um+\phi^*(m+u)\\
&\quad=\phi^*(m+u)-\phi^*(m)-(\phi^*)'(m)u.
\end{aligned}$$

The third equality follows from the definition of ϕ, the change of variables $(y+m)\to y$, and the definition of ϕ^*. Concerning the fourth equality, (3.7) implies $(\phi')^{-1}(m)=m$, so by (3.18) $\phi^*(m)=m^2-\phi(m)$ and $(\phi^*)'(m)=m$.

To prove (3.14), we notice that since m is a minimum of the function $y^2/2-\phi(y)$, there exists $u_0>0$ such that $y>\phi'(y)$ for any $y\in(m,m+u_0)$. Thus, for such y, $(\phi^*)'(y)>y$ or $(\phi^*)'(m+u)>m+u$ for any $u\in(0,u_0)$. Since $(\phi^*)'(m)=m$, (3.14) is proved. This completes the proof of the lemma and thus of Theorem 2.4 for $k(m)\geqq 2$.

Proof of Theorem 2.5. The proof of Theorem 2.5 is essentially identical to that of Theorem 2.4 except that a different version of the transfer principle is used. We suppose that $\lambda(m)>0$ and use previously defined notation. We propose to show that for appropriate $B>0$ and $a\in(0,B/2)$,

$$\begin{aligned}
&\int\exp(irn^{\gamma}w)\exp\left(-\frac{n}{2}w^2\right)\int_{u\in[0,a]}\exp(nwu)\,dv_n(u)\,dw\\
&\quad=\int_{w\in[0,B]}\exp(irn^{\gamma}w)\exp\left(-\frac{n}{2}w^2\right)\int\exp(nwu)\,dv_n(u)\,dw+O(n^{-1}).
\end{aligned}\tag{3.20}$$

This would suffice to obtain the desired result since then

$$n^{\gamma}\int \exp(irn^{\gamma}w)\exp\left(-\frac{n}{2}w^2\right)\int_{u\in[0,a]}\exp(nwu)\,d\nu_n(u)\,dw$$

$$=\int_{w\in[0,Bn^{\gamma}]}\exp(irw)\exp\left(-n^{1-2\gamma}\frac{w^2}{2}\right)\int\exp(n^{1-\gamma}wu)\,d\nu_n(u)\,dw+O(n^{\gamma-\frac{1}{2}})$$

$$\to\int_0^{\infty} e^{irw}\exp(-\lambda(m)\,w^{2k(m)}/(2k(m))!)\,dw.$$

The above limit is based on the fact that since $k(m)\geqq 3/2$, $\gamma-\frac{1}{2}=1/2k(m)-\frac{1}{2}<0$; its proof is the same as that of (3.5) given above. To prove (3.20), we define

$$dH(w,u)=\exp\left(-\frac{n}{2}w^2\right)\exp(nwu)\,d\nu_n(u)\,dw$$

and then obtain the following estimates (for some $\bar{\delta}=\bar{\delta}(a,B)>0$):

$$\int_{w>B}\int_{u\in[0,a]} dH(w,u)=O(e^{-n\bar{\delta}}), \tag{3.21}$$

$$\int_{w<0}\int_{u\in[0,a]} dH(w,u)=O(n^{-\frac{1}{2}}), \tag{3.22}$$

$$\int_{w\in[0,B]}\int_{u>a} dH(w,u)=O(e^{-n\bar{\delta}}), \tag{3.23}$$

$$\int_{w\in[0,B]}\int_{u<0} dH(w,u)=O(n^{-\frac{1}{2}}). \tag{3.24}$$

The estimates (3.21) and (3.23) are derived in an identical fashion as (3.9) and (3.10), respectively, while (3.22) and (3.24) follow from the weak convergence limit

$$n^{\frac{1}{2}}1_{\{wu\leq 0\}}\,dH(n^{-\frac{1}{2}}w,n^{-\frac{1}{2}}u)$$
$$\to 1_{\{wu\leq 0\}}\exp\left(-\frac{w^2}{2}+wu-\frac{u^2}{2}\right)dw\,du. \tag{3.25}$$

The limit (3.25) is a consequence of the central limit theorem applied to the sequence of measures $d\nu_n(u/\sqrt{n})$ and the fact that

$$\int(x-m)^2\,d\rho_m(x)=\phi_\rho''(m)=1$$

since $G_\rho''(m)=1-\phi_\rho''(m)=0$. This completes the proof of Theorem 2.5.

Proof of Theorem 2.2. We define $Y_n=(S_n-nm)/n^{\alpha}$ for α some positive number, denote by $N_n=N_n(a)$ the event $\{n^{-1}S_n\in[m-a,m+a]\}$, and use the fact that for any Borel set $\Gamma\subset\mathbb{R}^1$,

$$P\{Y_n\in\Gamma\}=P\{Y_n\in\Gamma\,|\,N_n\}\,P\{N_n\}+P\{Y_n\in\Gamma\,|\,N_n^c\}\,P\{N_n^c\}. \tag{3.26}$$

If Γ is *bounded* and $\alpha<1$, then there is a finite positive constant C so that $\{Y_n\in\Gamma\}\subset\{|n^{-1}S_n-m|<Cn^{\alpha-1}\}$; the latter set is disjoint from N_n^c for all large n

so that $P\{Y_n \in \Gamma | N_n^c\} \to 0$. In the case where m is not a global minimum of maximal type, we let $\alpha = 1/c$ and choose a sufficiently small so that $\tau([m-a, m+a]) = 0$ (where τ is defined in Theorem 2.1). It follows from (2.1) that $P\{N_n\} \to 0$ so that by (3.26), $P\{Y_n \in \Gamma\} \to 0$ for any bounded Γ; thus $(S_n - nm)/n^{1-\alpha} \to 0$ as desired. In the case where m is a global minimum of maximal type k^*, we let $\alpha = 1 - 1/2k^*$ and choose a in accordance with Theorems 2.3 and 2.4. It follows from (2.1) that $P\{N_n\} \to b(m)$. Thus by Theorem 2.4 and (3.26) we have for any bounded Γ that

$$P\{Y_n \in \Gamma\} \to b(m) \int_\Gamma d F_{k^*, \lambda(m), \beta}.$$

This completes the proof of the theorem.

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